Notes on the Capital Asset Pricing Model  
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The goal of the CAPM is to explain the equilibrium risk premium on all assets. These notes cover the material in B-K-M Chapters 8-9.

CAPM B-K-M Chapter 9 Section 9.1 (you don’t need to study 9.2 or 9.3—Extensions of the CAPM—but, you need to study the Black-Treynor model B-K-M 27.4; see the Syllabus)

**CAPM equilibrium conditions**

\[
\bar{A} = \left( \frac{E_{r_m} - r_f}{\sigma_m} \right); \text{ utility maximization}
\]

\[
E_{r_c} = r_f + \left( \frac{E_{r_m} - r_f}{\sigma_m} \right) \sigma_c; \text{ CML}
\]

\[
E_{r_j} - r_f = \beta_j [E_{r_m} - r_f]; \text{ SML}
\]

\[
\beta_j \equiv \frac{\text{cov}(r_j, r_m)}{\sigma_m^2}
\]

here \( r_m \) denotes the market portfolio.

**Chapter 8: Individual Behavior**  
 Portfolio Theory and the Mutual Fund Separation theorem

Individuals choose quantities given prices. In this case the investor chooses (1) the weight\(^1\), \( w \), in the risky portfolio, and (2) the fraction of their wealth, \( y^j \), to put in the portfolio of risky assets.

Define the risky portfolio as

\[
r_p = (1 - w)r_1 + wr_2
\]

where \( r_1 \) and \( r_2 \) are random returns that are normally distributed.

Then the efficient frontier of risky assets is:

\[
E_{r_p}(w) = \max_w (E_{r_p} \mid \sigma_p = \text{constant}) \tag{2.1}
\]

\(^1\) This generalizes to many risky assets—the investor chooses a weight for each risky asset.
is the maximum expected return on the portfolio for a given level of risk. (In the case of 2 assets varying \( w \) from zero to one traces out the frontier—PS 5. In the case of many risky asset one must do constrained maximization on the computer—see B-K-M 8.5)

Computing the efficient frontier requires only data on the distribution of returns, means, variances, and covariances.

**Mutual Fund Separation B-K-M 8.6**

The precise simple results of the CAPM rest on the mutual fund separation theorem. The mutual fund separation theorem sits on three critical assumptions (B-K-M 9.1 give six assumptions, but they condense to these three)

1. The utility function—each investor cares only about the mean and variance of their portfolio
   \[
   U^i = E_r - \frac{1}{2} A^i \sigma^2
   \]
   Investors differ only in their tolerance for risk, \( A^i \).

2. Markets are perfect and complete (this means all assets are traded—eg, your car).
3. Homogenous expectations—all investors agree on the distribution of asset returns.

**Result**

Every risk averse investor chooses the same portfolio of risky assets, ie, the same weights, \( w^p \).

**Derivation**

 Investors choose the weights that maximize the slope (“price of risk”) of the CAL,

\[
S(w^p) \equiv \max_w \left( \frac{E(r_p(w)) - r_f}{\sigma_p(w)} \right)
\]  

(2.2)

See B-K-M equation 8.7 and PS5.

**This follows from the assumption that investors are identical except for their risk tolerance, \( A^i \).**

**Investor tastes affect only the allocation of wealth between “the” portfolio of risky assets and the risk free asset.**

Each investor, \( i \), chooses the fraction of her wealth, \( y^i \), to put in the risky portfolio according to her risk tolerance, \( A^i \) (PS 4)

\[
y^i = \frac{E(r_p(w^p)) - r_f}{A^i \sigma(w^p)^2}
\]  

(2.3)
Investors differ in the fraction of wealth they invest in the risky portfolio, but they all hold the same risky portfolio.

There is only one mutual fund of risky assets.

The CAPM Market Equilibrium

Prices are endogenous in equilibrium. Prices adjust until excess demands equal zero. In the CAPM model, and in reality, the quantity of equities is (almost) fixed and, in reality and the CAPM model, the net quantity of debt equals zero.

Equilibrium Restrictions

In the portfolio choice model investors choose the quantity of risk they want to bear. Equilibrium conditions mean the quantity of risk demanded equals the quantity supplied.

Assume all investors, \( i \), have the same wealth \( W \) (normalized to one, \( W = 1 \)) and that there are \( I \) investors.

a. First let’s do borrowing equals lending restriction—this means that,

\[
\sum_{i=1}^{I} (1 - y^i) = 0
\]

in equilibrium, every borrower, \( y^i > 1 \), borrows from a lender, \( y^i < 1 \).

b. Second let’s do the demand for risky assets equals the supply restriction—this means that,

\[
\sum_{i} y^i = I
\]

all risky assets are held. Or, in equilibrium, the “average” investor has all her wealth in the market portfolio,

\[
\bar{y} \equiv \frac{1}{I} \sum_{i} y^i = 1
\]

1. Market Equilibrium

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\(^2\) Verify that \( W \neq 1 \) gives the same equilibrium conditions. Furthermore, all the results go through for an unequal distribution of wealth, \( W' \). Either case just makes the algebra messier.
The Capital Market Line summarizes an equilibrium condition for the CAPM,

$$E r^i = r_f + \left( \frac{E r_m - r_f}{\sigma_m} \right) \sigma^i$$

$$\sigma^i = y^i \sigma_m$$

In equilibrium all investors choose a combined portfolio that falls on the CML.

Demand for the market portfolio (mutual fund of risky assets) by an individual is,

$$y^i = \frac{1}{A^i} \left( \frac{E r_m - r_f}{\sigma_m^2} \right); i = 1, \ldots, I$$  \hspace{1cm} (2.4)

Imposing the Equilibrium Conditions

$$\frac{1}{I} \sum_{i=1}^{I} y^i = \bar{y} = 1 \Rightarrow \frac{1}{I} \sum_{i=1}^{I} y^i A^i \equiv \bar{A} = \frac{E r_m - r_f}{\sigma_m^2}$$  \hspace{1cm} (2.5)

demand = supply => average coef risk aversion = price risk

Rearranging equation (2.5) shows that the average investor’s indifference curve is tangent to the capital market line the utility maximizing point,

$$\bar{A} \sigma_m = \frac{E r_m - r_f}{\sigma_m}$$  \hspace{1cm} (2.6)

This also holds for each investor—equation 2.4.

**Disequilibrium:** Consider a representative investor (the average investor) and suppose \(y > 1\). Then the investor wants to buy more of the market portfolio and borrow.

Recall that,

$$1 + E r_i = \frac{E (S_{t+1} + d_{t+1})}{S_t}$$

$$1 + r_f = \frac{1}{B_t}$$

where \(S\) is the stock price and \(B < I\) the price of a discount bond. Trying to buy more of the market portfolio raises the price, \(S_m\), and lowers the expected return, \(E r_m\), and trying
to borrow (sell bonds) lowers the price, $B$, and raises the risk free rate. In equilibrium, $ar{y} = 1$.

2. **Securities Market Line (SML)**

Use Elmo’s representation of the equilibrium condition that the risk adjusted expected excess return on all assets is the same,

$$\frac{E_{r_j} - r_f}{\text{cov}(r_j, r_m)} = \frac{E_{r_k} - r_f}{\text{cov}(r_k, r_m)} = \frac{E_{r_m} - r_f}{\text{cov}(r_m, r_m)} = \frac{E_{r_l} - r_f}{\text{cov}(r_l, r_m)} = \frac{E_{r_m} - r_f}{\sigma_m^2}$$  \hspace{2cm} (2.7)

where the risk adjustment is the covariance of the asset’s return with the market portfolio.

The traditional way to write this equilibrium condition is the securities market line,

$$E_{r_j} - r_f = \beta_j (E_{r_m} - r_f)$$  \hspace{2cm} (2.8)

$$\beta_j \equiv \frac{\text{cov}(r_j, r_m)}{\sigma_m^2}$$

The SML uses the asset’s Beta to adjust for risk,

$$\frac{E_{r_j} - r_f}{\beta_j} = \frac{E_{r_m} - r_f}{1} = \frac{E_{r_l} - r_f}{\beta_l}$$  \hspace{2cm} (2.9)

Beta is the covariance normalized by the variance of the market return.

The equilibrium condition that all risk adjusted expected excess returns are equal is intuitive. What’s not intuitive is that the covariance of the asset’s return with the return on the market portfolio (or its Beta) is the correct measure of the asset’s risk. After all investors care about expected return and risk,

$$U = E_{r_c} - \frac{1}{2} A' \sigma_c^2$$  \hspace{2cm} (2.10)

where risk is measured by the variance of their combined portfolio,

$$r_c = rf + y^i (r_m - rf)$$

The variance of the combined portfolio is,

$$\sigma_c^2 = (y^i)^2 \sigma_m^2$$

the variance of the market portfolio times the (squared) fraction of the investor’s wealth in the market portfolio.
The covariance is the correct measure of an asset’s risk, because it measures the asset’s contribution to the risk (variance) of the market portfolio. Consider the simple case where the market portfolio only has two assets

\[ r_m = (1 - w)r_1 + wr_2 \]

then, the variance of the market portfolio can be written as (PS5)

\[ \sigma_m^2 = (1 - w)\text{cov}(r_1, r_m) + w\text{cov}(r_2, r_m) \]  \hspace{1cm} (2.11)

as the weighted average of the covariances of the elements of the portfolio (B-K-M, C9 do the case with n assets).