Notes on Random Walks, Mean Reversion and Efficient Markets  
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Elmo says the key to understanding the implications of mean-reversion is in understanding the implications of predictability for stock pricing. Predictability implies that a trading strategy, buy low—sell high, that earns expected return greater than the long run average return. But Elmo realizes the contradiction—only in Lake Owbegone can everyone be above the average. If everyone follows that strategy, then equilibrium requires that prices change to eliminate the expected excess return. In an efficient market there are no expected excess returns. The random walk model has no expected excess returns—in the jargon of finance, returns are unpredictable in the random walk model.

Statistical Descriptions

Let’s start with statistical descriptions of the time-series properties of stock prices. Here are alternative descriptions:

1. **Mean Reversion**

\[
\ln(S_{t+1} + d_{t+1}) = \ln S_t + \bar{\alpha} - \lambda((\ln(S_t + d_t) - \ln S_{t-1}) - \bar{\alpha}) + u_{t+1}, \quad 0<\lambda<1, u \sim N(0, \sigma^2)
\]

or in returns, 
\[
r_{t+1} = \bar{\alpha} - \lambda(r_t - \bar{\alpha}) + u_{t+1}; \quad r_{t+1} \equiv (\ln(S_{t+1} + d_{t+1}) - \ln S_t)
\]

The notation here is \(S\) denotes the stock price, \(d\) flow distributions (usually dividends), and \(\bar{\alpha}\) the long run average (logarithmic) return.\(^1\)

2. **Random Walk**

\[
\ln(S_{t+1} + d_{t+1}) = \ln S_t + \bar{\alpha} + e_{t+1}; \quad e \sim N(0, \sigma^2)
\]

or in returns, 
\[
r_{t+1} = \bar{\alpha} + e_{t+1}
\]

Predictability

Now let’s analyze what the equations say about predictability. We will use returns because it is easier to see the implications for stock valuation in returns.

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\(^1\) The mean reversion description can be generalized to include many lagged values and nonlinear functions.
Split the right hand-side of the equations into predictable and unpredictable components. Then, when you take expectations—the forecast for next period—only the predictable part remains.

Of course, reflects Elmo, “If the prediction error weren’t zero, then it would be predictable and it would be in the predictable portion”.

So let’s start with,

Random Walk

\[ r_{t+1} = \bar{r} + e_{t+1} \]

Now consider the return forecast (the return for the next period expected today.)

\[ E_t(r_{t+1}) = E_t(\bar{r} + e_{t+1}) \]

Here \( E_t \) means the expectation at time \( t \). By definition (statistics and English) the expected value of the forecast error for next period is zero,

\[ E_t(e_{t+1}) = 0 \]

So the expected return next period is the predictable portion, ie, the long run average return, \( \bar{r} \).

Now let’s look at the mean-reversion model,

Mean-reversion

\[ r_{t+1} = \bar{r} - \lambda (r_t - \bar{r}) + u_{t+1} \]

The mean-reversion model includes a correction term, \( -\lambda (r_t - \bar{r}) \), that depends on today’s return. If today’s return is above the long run average return, \( r_t > \bar{r} \), then the correction term forces next period’s return down.

Consider the return forecast (the return for the next period expected today.)
The expected value of the forecast error for next period is zero,

\[ E_t(u_{t+1}) = 0 \]

So the expected return next period is the predictable portion, ie, the long run average return, \( \bar{r} - \lambda (r_t - \bar{r}) \), minus the correction factor.

**Summary:**

**Random Walk:** The best prediction of next period’s return is the long run average return. The usual jargon in finance is that returns are unpredictable in the random walk model. More precisely, deviations from the long run average return are unpredictable in the random walk model.

**Mean-Reversion:** The best prediction of next period’s return is the long run average return plus a correction factor that depends on the deviation of the current return from the long-run average.

**Implications for Asset Pricing**

Many processes have predictable components that depend on their current level. At high tide the water depth is above the average and the best prediction is the water depth will decrease. Temperatures are above the annual average in the summer and they will return to the average.

Predictability in asset returns, however, implies an opportunity to make expected excess returns. In equilibrium, in an efficient market there are no expected excess returns.

Here’s the argument. Take expected return from the mean-reversion model,

\[ E_t(r_{t+1}) = E_t(\bar{r} - \lambda (r_t - \bar{r}) + u_{t+1}) \]
\[ = \bar{r} - \lambda (r_t - \bar{r}) \]

**Individual decision rule:** If the expected excess return is positive buy, if it is negative sell.

**Market equilibrium:** If everyone follows this rule, then the current price would change until the expected excess return is zero in equilibrium, ie, until, \( r_t = \bar{r} \).
Predictable excess returns imply that unexploited expected excess returns exist, or that markets are inefficient.

The random walk model is consistent with an efficient market. In the random walk model

1. agents form an expectation of the excess return for next period,

\[ E_t r_{t+1} - \bar{r} = E_t (\ln(\frac{S_{t+1} + d_{t+1}}{S_t}) - \ln S_t) - \bar{r} \]

2. agents follow the decision rule that says buy if the excess return is positive, sell if it is negative

3. if everyone follows the rule, then the price adjusts until the expected excess return is zero in equilibrium, ie, until \( E_t r_{t+1} - \bar{r} \).

In the random walk model no expected excess returns exist.