1 Games of complete information

1.1 Formal definitions

Definition 1.1. A game (of complete information) $G$ consists of

(i) A set $N$ of players;
(ii) A set $A_i$ of actions for each player $i \in N$;
(iii) A payoff function $u_i : \times_{j \in N} A_j \to \mathbb{R}$ for each player $i \in N$.

Definition 1.2. Consider a vector $a = (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n)$ of actions. Then the notation $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ refers to the vector of actions for every player except $i$.

The notation $(a'_i, a_{-i}) = (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots, a_n)$ refers to the vector where $i$’s action $a_i$ is replaced with $a'_i$ and all other players’ actions remain the same.

When there are two players with finite actions, we sometimes write the game as a matrix. For example, if $A_1 = \{a^1, a^2, a^3\}$ and $B_2 = \{b^1, b^2, b^3\}$, we would write the game as:

$$
\begin{array}{ccc}
  a^1 & b^1 & b^2 & b^3 \\
  a^2 & u_1(a^1, b^1), u_2(a^1, b^1) & u_1(a^1, b^2), u_2(a^1, b^2) & u_1(a^1, b^3), u_2(a^1, b^3) \\
  a^3 & u_1(a^2, b^1), u_2(a^2, b^1) & u_1(a^2, b^2), u_2(a^2, b^2) & u_1(a^2, b^3), u_2(a^2, b^3) \\
  a^3 & u_1(a^3, b^1), u_2(a^3, b^1) & u_1(a^3, b^2), u_2(a^3, b^2) & u_1(a^3, b^3), u_2(a^3, b^3) \\
\end{array}
$$

The rows are player 1’s actions, the columns are player 2’s actions, the entries are the two players’ utilities.

Definition 1.3. A Nash equilibrium of a game $G$ is a vector of actions $a^* = (a^*_1, \ldots, a^*_n)$ such that, for all $i \in N$ and for all $a'_i \in A_i$:

$$
u_i(a^*) \geq u_i(a'_i, a^{*,-i}).$$

Example 1.4.

$$
\begin{array}{cc}
  & L & R \\
  T & 100, 100 & 0, 0 \\
  B & 0, 0 & 0, 0 \\
\end{array}
$$

There are two Nash equilibria: $(T, L)$ and $(B, R)$.  

1
Example 1.5 (Bach or Stravinsky).

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>S</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

There are two Nash equilibria: (B, B) and (S, S).

**Definition 1.6.** An action $a_i \in A_i$ *weakly dominates* $a'_i$ if, for all $a_{-i}$:

$$u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}),$$

and there exists $a_{-i}$ in $A_{-i}$ such that:

$$u_i(a_i, a_{-i}) > u_i(a'_i, a_{-i}).$$

An action $a_i$ is *weakly dominant* if it weakly dominates all other actions $a'_i \neq a_i$.

**Example 1.7** (Prisoner’s Dilemma).

<table>
<thead>
<tr>
<th></th>
<th>Silence</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silence</td>
<td>-1, -1</td>
<td>-3, 0</td>
</tr>
<tr>
<td>Confess</td>
<td>0, -3</td>
<td>-2, -2</td>
</tr>
</tbody>
</table>

The only Nash equilibrium is (Confess, Confess). Confess also weakly dominates Silence for both players.

**Example 1.8** (Rock-Paper-Scissors).

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>P</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>0, 0</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
<tr>
<td>P</td>
<td>1, -1</td>
<td>0, 0</td>
<td>-1, 1</td>
</tr>
<tr>
<td>S</td>
<td>-1, 1</td>
<td>1, -1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

There is no Nash equilibrium.

1.2 Example: Quantity competition

Suppose that the demand function in the market is determined by

$$p(q) = 100 - q$$

where $q$ is total quantity. The cost for producing $q$ units is $q^2$. For a monopolist, its profits for producing $q$ units is

$$u(q) = (100 - q)q - q^2.$$
To find its optimal quantity $q^*$, take the necessary first order condition:

$$\frac{\partial u}{\partial q} = 100 - 2q^* - 2q^* = 0$$

$$4q^* = 100$$

$$q^* = 25$$

Then 25 units are produced by the monopolist and sold at the price 75.

Now suppose an identical competing firm enters the market. The market demand is defined by

$$p(q_1, q_2) = 100 - q_1 - q_2.$$ 

The cost of producing $q_i$ units is still $q_i^2$. The profit for firm $i$ is

$$u_i(q_i, q_{-i}) = q_i(100 - q_1 - q_2) - q_i^2.$$ 

This can be considered a game where $\mathcal{N} = \{1, 2\}$, $\mathcal{A} = \mathbb{R}_+$, and $u_i$ is specified as above. The first order condition for firm 1, given the quantity $q_2^*$ is produced by firm 2, is

$$\frac{\partial u_1}{\partial q_1} = 100 - 2q_1^* - q_2^* - 2q_2^* = 0.$$ 

This becomes:

$$q_1^* = \frac{100 - q_2^*}{4} \quad (A)$$

Symmetrically, exchanging the subscripts for firm 1 and 2, the first order condition for firm 2’s optimal quantity, given quantity $q_1$ by firm 1 is:

$$q_2^* = \frac{100 - q_1^*}{4} \quad (B)$$

Equations (A) and (B) define a system of two linear equations with two unknowns, whose solution is

$$q_1^* = q_2^* = 20.$$ 

Therefore, the Nash equilibrium in this duopoly game is (20, 20). Then a total of 40 units are produced for the market, which are sold at the price 60.

### 1.3 Example: First price auction with known values

There are $1, \ldots, N$ players with values $v_1 > v_2 > \cdots > v_N > 0$ for some object. Each submits a bid for the object in a first-price auction. The player who submits the highest bid wins the object and pays her bid. In the case of a tie, the lowest-indexed player, i.e. the one with the higher value, wins the object.
We can model this as a game:

- \( \mathcal{N} = \{1, \ldots, N\} \).
- \( \mathcal{A} = \mathbb{R}_+ \).
- \( u_i(a_i) = \begin{cases} v_i - a_i & \text{if } i = \min \{ j : a_j \geq a_k, \forall k \in \mathcal{N} \} \\ 0 & \text{otherwise} \end{cases} \)

**Proposition 1.9.** The action vector \( \mathbf{a} = (v_1, v_2, \ldots, v_N) \) where each player bids her value for the object is not a Nash equilibrium.

**Proof.** Suppose bidder 1 changes her bid to \( v_2 \). Then she would win the object at price \( v_2 \) (since she would be the lowest-indexed player bidding the highest bid). This would yield her a payoff of \( v_1 - v_2 \), which would strictly larger than her payoff 0 from bidding her value \( v_1 \). More formally:

\[
    u_1(v_2, a_{-1}) = v_1 - v_2 > 0 = u_1(\mathbf{a}).
\]

**Proposition 1.10.** An action profile \( \mathbf{a}^* \) is a Nash equilibrium if and only if

\[
a_1^* = \max_{j > 1} a_j^*
\]

for some \( a_1^* \in [v_2, v_1] \).

**Proof.** We first show that all action profiles of the suggested form are Nash equilibria; this is the “only if” part. So, fix some \( \mathbf{a}^* \) with \( a_1^* = \max_{j > 1} a_j^* \) and \( a_1^* \in [v_2, v_1] \). We need to show that for every player \( i = 1, \ldots, N \), \( u_i(\mathbf{a}^*) \geq u_i(a'_i, \mathbf{a}_{-i}^*) \) for all bids \( a' \in \mathbb{R}_+ \). So, either \( i = 1 \) or \( i = 2, \ldots, N \).

- Case 1: \( i = 1 \). Consider an alternative bid \( a'_1 \) for the first player. Either \( a'_1 \geq a_1^* \) or \( a'_1 < a_1^* \).
  - Case 1a: \( a'_1 \geq a_1^* \). Then the first player would just pay more for the same object, which would leave her worse off. Formally:
    \[
    u_1(a'_1, a_{-1}^*) = v_1 - a'_1 \leq v_1 - a_1^* = u_1(\mathbf{a}^*).\]
  - Case 1b: \( a'_1 < a_1^* \). Then \( a'_1 < a_2^* \), in which case the first player would no longer win the object, since she would be outbid by the second player’s bid of \( a_2^* \). Then she has a payoff of zero, which is no better than her positive payoff in equilibrium. Formally:
    \[
    u_1(a'_1, a_{-1}^*) = 0 \quad \text{since 1 does not win the object given } (a'_1, a_{-1})
    \leq v_1 - v_1
    \leq v_1 - a_1^* \quad \text{since } a_1^* \leq v_1
    = u_1(\mathbf{a}^*) \quad \text{since 1 wins the object given } \mathbf{a}^*.
    \]
• Case 2: \( i = 2, \ldots, N \). Consider an alternative bid \( a'_i \) for player \( i \). Either \( a'_i > a^*_i \) or \( a'_i \leq a^*_i \).

  - Case 2a: \( a'_i > a^*_i \). Then she would end up winning the object at more than her value, which would give her a negative payoff, which is worse than the zero payoff she gets in equilibrium. Formally:

    \[
    u_i(a'_i, a^*_{-i}) = v_i - a'_i \quad \text{since } i \text{ wins the object given } (a'_i, a^*_{-i})
    \]

    \[
    < v_i - a^*_i \quad \text{since } a'_i > a^*_i
    \]

    \[
    \leq v_i - v_2 \quad \text{since } a^*_i \geq v_2
    \]

    \[
    < v_i - v_i \quad \text{since } v_2 \geq v_i
    \]

    \[
    = 0
    \]

    \[
    = u_i(a^*) \quad \text{since } i \text{ does not win the object given } a^*
    \]

  - Case 2b: \( a'_i \leq a^*_i \). Then player \( i \) would still not win the object, which does not change her payoff and would not make her better off:

    \[
    u(a'_i, a^*_{-i}) = 0 = u(a^*).
    \]

The next step is to show that if \( a^* \) is a Nash equilibrium, it must have this form; this is the “if” part. So, suppose \( a^* \) is a Nash equilibrium. First, we want to show that \( a^*_1 \geq a^*_j \) for all \( j \), i.e. that player 1 submits the highest bid (or, to be precise, one of the highest bids) and wins the object. We will prove this claim by contradiction. So, suppose not, i.e. there exists some \( j \) such that \( a^*_j > a^*_1 \). Then the winner of the object cannot be player 1, but is some \( k \neq 1 \). Either \( a^*_k \geq v_1 \) or \( a^*_k < v_1 \). We show in either case, some player can make herself strictly better off by playing a different strategy, which contradicts the assumption \( a^* \) is a Nash equilibrium.

Case 1: \( a^*_k \geq v_1 \). Then player \( k \) is paying more than her value for the object and would be strictly better off by bidding zero:

    \[
    u_k(0, a^*_{-k}) = 0
    \]

    \[
    > v_k - v_1 \quad \text{since } v_1 > v_k
    \]

    \[
    \geq v_k - a^*_k \quad \text{since } a^*_k \geq v_1
    \]

    \[
    = u_k(a^*) \quad \text{since } k \text{ wins the object given } a^*
    \]

But this means \( k \) has a strictly better action \( a'_k = 0 \), which contradicts our assumption that \( a^* \) is a Nash equilibrium.

Case 2: \( a^*_k < v_1 \). Then player 1 can bid \( a'_1 = a^*_k \) and win the object and pay less than her value,
yielding a strictly positive payoff larger than her equilibrium payoff:

\[
\begin{align*}
    u_k(a'_1, a^*_2) &= v_1 - a'_1 & \text{since } a'_1 > a^*_k, \text{ so 1 wins the object} \\
    &= v_1 - a^*_k & \text{since } a'_1 = a^*_k \\
    > v_1 - v_1 & \text{since } a^*_k < v_1 \\
    = 0 & \\
    = u_1(a^*) & \text{since 1 does not win the object given } a^*
\end{align*}
\]

But this means that player 1 has a strictly better action \( a'_1 \in (v_1, a^*_k) \), which also contradicts our assumption that \( a^* \) is a Nash equilibrium.

Second, we want to show that \( \max_{j > 1} a^*_j = a^*_1 \). Again, we do this by contradiction. Let \( k \) be one of the bidders whose bid is maximal among the \( j > 1 \) bidders, so \( a^*_k \geq a^*_j \) for all \( j > 1 \). So, suppose \( a^*_k \neq a^*_1 \). Since we just proved \( a^*_1 \leq a^*_k \), this leaves \( a^*_k < a^*_1 \) as the only option. But, if this is the case, then player one could bid \( a'_1 = a^*_k \) and win the object at a lower price:

\[
\begin{align*}
    u_1(a'_1, a^*_2) &= v_1 - a'_1 \\
    &= v_1 - a^*_2 & \text{since } a'_1 = a^*_2 \\
    > v_1 - a^*_1 & \text{since } a^*_2 < a^*_1 \\
    = u_1(a^*)
\end{align*}
\]

Corollary 1.11. In the first price auction described above, player 1 wins the object at a price between \( v_2 \) and \( v_1 \).