Economics 209A
Theory and Application of Non-Cooperative Games
(Fall 2013)

Strategic games of complete information
OR 2-4, FT 1 and 2
Formalities

**A strategic game** A finite set $N$ of players, and for each player $i \in N$

- a non-empty set $A_i$ of actions

- a preference relation $\succeq_i$ on the set $A = \times_{j \in N} A_j$ of possible outcomes.

We will denote a strategic game by

$$\langle N, (A_i), (\succeq_i) \rangle$$

or by

$$\langle N, (A_i), (u_i) \rangle$$

when $\succeq_i$ can be represented by a utility function $u_i : A \to \mathbb{R}$. 
A two-player finite strategic game can be described conveniently in a bi-matrix.

For example, a $2 \times 2$ game

$$\begin{array}{c|cc}
L & T & B \\
\hline
A_1, A_2 & A_1, A_2 & B_1, B_2 \\
C_1, C_2 & C_1, C_2 & D_1, D_2 \\
\end{array}$$
Best response

For any list of strategies $a_{-i} \in A_{-i}$

$$B_i(a_{-i}) = \{a_i \in A_i : (a_{-i}, a_i) \succeq_i (a_{-i}, a'_i) \ \forall a'_i \in A_i\}$$

is the set of players $i$’s best actions given $a_{-i}$.

Strategy $a_i$ is $i$’s best response to $a_{-i}$ if it is the optimal choice when $i$ conjectures that others will play $a_{-i}$. 
Nash equilibrium

Nash equilibrium ($NE$) is a steady state of the play of a strategic game.

A $NE$ of a strategic game $\langle N, (A_i), (\succeq_i) \rangle$ is a profile $a^* \in A$ of actions s.t.

$$(a^*_{-i}, a^*_i) \succeq_i (a^*_{-i}, a_i)$$

$\forall i \in N$ and $\forall a_i \in A_i$, or equivalently,

$$a^*_i \in B_i(a^*_{-i})$$

$\forall i \in N$.

In words, no player has a profitable deviation given the actions of the other players.
Examples

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Existence of Nash equilibrium

Let the set-valued function $B : A \rightarrow A$ defined by

$$B(a) = \times_{i \in N} B_i(a_{-i})$$

and rewrite the equilibrium condition

$$a_i^* \in B_i(a_{-i}^*) \ \forall i \in N$$

in vector form as follows

$$a^* \in B(a^*)$$

Kakutani’s fixed point theorem gives conditions on $B$ under which $\exists a^*$ such that $a^* \in B(a^*)$. 
Kakutani’s fixed point theorem

Let $X \subseteq \mathbb{R}^n$ be non-empty compact (closed and bounded) and convex set and $f : X \to X$ be a set-valued function for which

- the set $f(x)$ is non-empty and convex $\forall x \in X$.

- the graph of $f$ is closed

\[ y \in f(x) \text{ for any } \{x_n\} \text{ and } \{y_n\} \text{ such that} \]
\[ y_n \in f(x_n) \forall n \text{ and } x_n \to x \text{ and } y_n \to y. \]

Then, $\exists x^* \in X \text{ s.t. } x^* \in f(x^*)$. 
Necessity of conditions in Kakutani’s theorem

- $X$ is compact

  \[ X = \mathbb{R}^1 \text{ and } f(x) = x + 1 \]

- $X$ is convex

  \[ X = \{ x \in \mathbb{R}^2 : \|x\| = 1 \} \text{ and } f \text{ is } 90^\circ \text{ clock-wise rotation.} \]
– $f(x)$ is convex for any $x \in X$

$$X = [0, 1] \text{ and }$$

$$f(x) = \begin{cases} 
1 & \text{if } x < \frac{1}{2}, \\
\{0, 1\} & \text{if } x = \frac{1}{2}, \\
\{0\} & \text{if } x > \frac{1}{2}. 
\end{cases}$$

– $f$ has a closed graph

$$X = [0, 1] \text{ and }$$

$$f(x) = \begin{cases} 
1 & \text{if } x < 1, \\
0 & \text{if } x = 1. 
\end{cases}$$
A strategic game $\langle N, (A_i), (\succeq_i) \rangle$ has a NE if for all $i \in N$

- $A_i$ is non-empty, compact and convex.

- $\succeq_i$ is continuous and quasi-concave on $A_i$.

$B$ has a fixed point by Kakutani:

- $B_i(a_{-i}) \neq \emptyset$ ($A_i$ is compact and $\succeq_i$ is continuous).

- $B_i(a_{-i})$ is convex ($\succeq_i$ is quasi-concave on $A_i$).

- $B$ has a closed graph ($\succeq_i$ is continuous).
Randomization

Recall that a strategic game is a triple $\langle N, (A_i), (\succeq_i) \rangle$ where

- $N$ is a finite set of players, and for each player $i \in N$
- a non-empty set $A_i$ of actions
- a preference relation $\succeq_i$ on the set $A = \times_{j \in N} A_j$ of possible outcomes.

or a triple $\langle N, (A_i), (u_i) \rangle$ when $\succeq_i$ can be represented by a utility function $u_i : A \to \mathbb{R}$. 
Suppose that,

- each player $i$ can **randomize** among all her strategies so choices are not deterministic, and

- player $i$’s preferences over lotteries on $A$ can be represented by $vNM$ expected utility function.

Then, we need to add these specifications to the primitives of the model of strategic game $\langle N, (A_i), (\succeq_i) \rangle$. 
A mixed strategy of player $i$ is $\alpha_i \in \Delta(A_i)$ where $\Delta(A_i)$ is the set of all probability distributions over $A_i$.

- A profile $(\alpha_i)_{i \in N}$ of mixed strategies induces a probability distribution over the set $A$.

- Assuming independence, the probability of an action profile (outcome) $a$ is then

$$\prod_{i \in N} \alpha_i(a_i).$$
A \textit{vNM} utility function

\[ U_i : \times_{j \in N} \Delta(A_j) \rightarrow \mathbb{R} \]

represents player \( i \)'s preferences over the set of lotteries over \( A \).

For any mixed strategy profile \( \alpha = (\alpha_j)_{j \in N} \in \times_{j \in N} \Delta(A_j) \)

\[ U_i(\alpha) = \sum_{a \in A} \left( \prod_{j \in N} \alpha_j(a_j) \right) u_i(a) \]

which is linear in \( \alpha \).

The mixed extension of a the strategic game \( \langle N, (A_i), (u_i) \rangle \) is the strategic game \( \langle N, (\Delta(A_i)), (U_i) \rangle \).


Existence of mixed strategy Nash equilibrium

Every finite (action sets) strategic game has a mixed strategy $NE$.

- The set of player $i$’s mixed strategies $\Delta(A_i)$

\[
\{(p_k)_{k=1}^{m_i} : \sum_{k=1}^{m_i} p_k = 1 \text{ and } p_k \geq 0 \quad \forall k\}
\]

where $m_i$ is the number of $a_i \in A_i$ (pure strategies) is non empty, convex and compact.

- $vNM$ expected utility is linear probabilities so $U_i$ is quasi-concave and continuous.

Therefore, the mixed extension has a $NE$ by Kakutani.
Two results on mixed strategy Nash equilibrium

Let $G = \langle N, (A_i), (u_i) \rangle$ be a strategic game and $G' = \langle N, (\Delta(A_i)), (U_i) \rangle$ be its mixed extension.

[1] If $a \in NE(G)$ then $a \in NE(G').$

[2] $\alpha \in NE(G')$ if and only if

$$U_i(\alpha_{-i}, a_i) \geq U_i(\alpha_{-i}, a'_i)$$

for all $a'_i$ and all $\alpha_i(a_i) > 0.$
Interpretation of mixed strategy Nash equilibrium

Since she is indifferent among all strategies in the support, why should a player choose her $NE$ mixed strategy?

[1] Mixed strategies as objects of choice

[2] Mixed strategy $NE$ as a steady state


Strictly competitive game

A strategic game $\langle\{1, 2\}, (A_i), (\succeq_i)\rangle$ is strictly competitive if for any $a \in A$ and $b \in A$ we have $a \succeq_1 b$ if and only if $b \succeq_2 a$.

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If $(x^*, y^*)$ is a NE of a strictly competitive game then

$$u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y).$$
Maxminimization

A max min mixed strategy of player $i$ is a mixed strategy that solves the problem

$$\max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i})$$

A player’s payoff in $\alpha^* \in NE(G)$ is at least her max min payoff:

$$U_i(\alpha^*) \geq U_i(\alpha_i, \alpha^*_{-i}) \geq \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}).$$

Thus,

$$U_i(\alpha^*) \geq \max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i})$$

since the above holds for all $\alpha_i \in \Delta(A_i)$. 
Two min-max results

\[ \max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \leq \min_{\alpha_{-i} \in \Delta A_{-i}} \max_{\alpha_i \in \Delta A_i} U_i(\alpha_i, \alpha_{-i}) \]

For every \( \alpha' \)

\[ \min_{\alpha_{-i}} U_i(\alpha'_i, \alpha_{-i}) \leq U_i(\alpha'_i, \alpha'_{-i}) \]

and thus

\[ \min_{\alpha_{-i}} U_i(\alpha'_i, \alpha_{-i}) \leq \max_{\alpha_i} U_i(\alpha_i, \alpha'_{-i}) \]

However, since the above holds for every \( \alpha'_i \) and \( \alpha'_{-i} \) it must hold for the “best” and “worst” such choices

\[ \max_{\alpha_i} \min_{\alpha_{-i}} U_i(\alpha_i, \alpha_{-i}) \leq \min_{\alpha_{-i}} \max_{\alpha_i} U_i(\alpha_i, \alpha_{-i}). \]
[2] In a zero-sum game

\[
\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) = U_1(\alpha^*)
\]

\[\iff\]

Since \(\alpha^* \in NE(G)\)

\[U_1(\alpha^*) = \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha^*_2) \geq \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)\]

and since \(U_1 = -U_2\) at the same time

\[U_1(\alpha^*) = \min_{\alpha_2 \in \Delta A_2} U_1(\alpha^*_1, \alpha_2) \leq \max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2)\]

Hence,

\[\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) \geq \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)\]

which together with [1] gives the desired conclusion.
Let $\alpha_1^{\text{max}}$ be player 1's $\max \min$ strategy and $\alpha_2^{\text{min}}$ be player 2's $\min \max$ strategy. Then,

$$\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1^{\text{max}}, \alpha_2) \leq U_1(\alpha_1^{\text{max}}, \alpha_2) \quad \forall \alpha_2 \in \Delta A_2$$

and

$$\min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) = \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2^{\text{min}}) \geq U_1(\alpha_1, \alpha_2^{\text{min}}) \quad \forall \alpha_1 \in \Delta A_1$$
But

$$\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) = U_1(\alpha_1^{\text{max}}, \alpha_2^{\text{min}})$$

implies that

$$U_1(\alpha_1, \alpha_2^{\text{min}}) \leq U_1(\alpha_1^{\text{max}}, \alpha_2^{\text{min}}) \leq U_1(\alpha_1^{\text{max}}, \alpha_2)$$

$$\forall \alpha_2 \in \Delta A_2 \text{ and } \forall \alpha_1 \in \Delta A_1.$$ 

Hence, \((\alpha_1^{\text{max}}, \alpha_2^{\text{min}})\) is an equilibrium.
Interchangeability

If $\alpha$ and $\alpha'$ are $NE$ in a zero-sum game, then so are $(\alpha_1, \alpha'_2)$ and $(\alpha'_1, \alpha_2)$.

- Since $\alpha$ and $\alpha'$ are equilibria

$$U_1(\alpha_1, \alpha_2) \geq U_1(\alpha'_1, \alpha_2) \text{ and } U_2(\alpha'_1, \alpha'_2) \geq U_2(\alpha'_1, \alpha_2),$$

and because $U_1 = -U_2$

$$U_1(\alpha'_1, \alpha'_2) \leq U_1(\alpha'_1, \alpha_2).$$

Therefore,

$$U_1(\alpha_1, \alpha_2) \geq U_1(\alpha'_1, \alpha_2) \geq U_1(\alpha'_1, \alpha'_2). \quad (1)$$

and similar analysis gives that

$$U_1(\alpha_1, \alpha_2) \leq U_1(\alpha'_1, \alpha'_2) \leq U_1(\alpha'_1, \alpha_2). \quad (2)$$
– (1) and (2) yield
\[ U_1(\alpha_1, \alpha_2) = U_1(\alpha'_1, \alpha_2) = U_1(\alpha_1, \alpha'_2) = U_1(\alpha'_1, \alpha'_2) \]

– Since \( \alpha \) is an equilibrium
\[ U_2(\alpha_1, \alpha''_2) \leq U_2(\alpha_1, \alpha_2) = U_2(\alpha_1, \alpha'_2) \]
for any \( \alpha''_2 \in \Delta A_2 \), and since \( \alpha' \) is an equilibrium
\[ U_1(\alpha''_1, \alpha'_2) \leq U_1(\alpha'_1, \alpha'_2) = U_1(\alpha_1, \alpha'_2) \]
for any \( \alpha''_1 \in \Delta A_1 \). Therefore, \( (\alpha_1, \alpha'_2) \) is an equilibrium and similarly also \( (\alpha_1, \alpha'_2) \).
Evolutionary stability

A single population of players. Players interact with each other pair-wise and randomly matched.

Players are assigned modes of behavior (mutation). Utility measures each player’s ability to survive.

ε of players consists of mutants taking action $a$ while others take action $a^*$. 
Evolutionary stable strategy \((ESS)\)

Consider a payoff symmetric game \(G = \langle \{1, 2\}, (A, A), (u_i) \rangle\) where \(u_1(a) = u_2(a')\) when \(a'\) is obtained from \(a\) by exchanging \(a_1\) and \(a_2\).

\(a^* \in A\) is \(ESS\) if and only if for any \(a \in A\), \(a \neq a^*\) and \(\varepsilon > 0\) sufficiently small

\[
(1 - \varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) > (1 - \varepsilon)u(a, a^*) + \varepsilon u(a, a)
\]

which is satisfied if and only if for any \(a \neq a^*\) either

\[
u(a^*, a^*) > u(a, a^*)
\]

or

\[
u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) > u(a, a)
\]
Three results on $ESS$

[1] If $a^*$ is an $ESS$ then $(a^*, a^*)$ is a $NE$.

Suppose not. Then, there exists a strategy $a \in A$ such that

$$u(a, a^*) > u(a^*, a^*).$$

But, for $\varepsilon$ small enough

$$(1 - \varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) < (1 - \varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

and thus $a^*$ is not an $ESS$. 
[2] If \((a^*, a^*)\) is a strict \(NE\) \((u(a^*, a^*) > u(a, a^*)\) for all \(a \in A\)) then \(a^*\) is an \(ESS\).

Suppose \(a^*\) is not an \(ESS\). Then either

\[ u(a^*, a^*) \leq u(a, a^*) \]

or

\[ u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) \leq u(a, a). \]

so \((a^*, a^*)\) can be a \(NE\) but not a strict \(NE\).
[3] A 2 × 2 game $G = \langle \{1, 2\}, (A, A), (u_i) \rangle$ where $u_i(a) \neq u_i(a')$ for any $a, a'$ has a mixed strategy which is $ESS$ (OR 51.1)

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If $w > y$ or $z > x$ then $(a, a)$ or $(a', a')$ are strict $NE$, and thus $a$ or $a'$ are $ESS$.

If $w < y$ and $z < x$ then there is a unique symmetric mixed strategy $NE (\alpha^*, \alpha^*)$ where

$$\alpha^*(a) = \frac{(z - x)}{(w - y + z - x)}$$

and $u(\alpha^*, \alpha) > u(\alpha, \alpha)$ for any $\alpha \neq \alpha^*$. 
Equivalence and symmetry

Payoff Equivalence

Two games

\[ G = \langle N, (A_i), (u_i) \rangle \quad \text{and} \quad G' = \langle N, (A'_i), (u'_i) \rangle \]

are payoff equivalent if for each player \( i \), \( A'_i \) may be relabelled such that \( A'_i = A_i \), and \( \exists (x_i, y_i) \) with \( y_i > 0 \) such that

\[ u'_i(a') = x_i + y_i u_i(a) \]

when \( A'_i = A_i \).
Examples

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Two results on payoff equivalence

[1] $G$ and $G'$ have the same mixed strategy $NE$ if they are payoff equivalent.

[2] $G$ and $G'$ have the same pure strategy $NE$ under any strictly increasing monotonic transformations on $u_i$.

Why?

[1] Affine transformations on $u_i$: the preference ordering of each $i$ over $\Delta A$ is the same as over $\Delta A'$.

[2] Monotonic transformations on $u_i$: the preference ordering of each $i$ over $A$ is the same as over $A'$. 
A game

\[ G = \langle N, (A_i), (u_i) \rangle \]

is payoff symmetric if it is payoff equivalent to a game

\[ G' = \langle N, (A'_i), (u'_i) \rangle \]

in which for any two players, \( i \) and \( j \), \( A'_i = A'_j \), and if \( i \) and \( j \) exchange strategies, they also exchange payoffs

\[
u'_i(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_N) =
\]

\[= u'_j(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_N)\]

Note: all the games above are payoff symmetric. \( MP \) is not payoff symmetric.
Strategic equivalence

Two games

\[ G = \langle N, (A_i), (u_i) \rangle \text{ and } G' = \langle N, (A'_i), (u'_i) \rangle \]

are strategically equivalent if for each player \( i \),

- \( A'_i \) may be relabelled such that \( A'_i = A_i \), and
- \( B'_i(a') = B_i(a) \) when \( A'_i = A_i \).
Also, the two games

\[ G = \langle N, (A_i), (u_i) \rangle \quad \text{and} \quad G' = \langle N, (A'_i), (u'_i) \rangle \]

are strategically equivalent if for each player \( i \), \( \exists y_i > 0 \) and a function \( f_i : A_{-i} \to \mathbb{R} \) such that

\[ u'_i(a) = f_i(a_{-i}) + y_i u_i(a) \]

for all \( a \in A \).
Proof.

Fix $\alpha_{-i}$ and let $f_i(\alpha_{-i}) \equiv \sum_{a_{-i} \in A_{-i}} \alpha_{-i}(a_{-i})f_i(a_{-i})$. Then,

$$u_i'(a_i, \alpha_{-i}) = \sum_{a_{-i} \in A_{-i}} \alpha_{-i}(a_{-i})u_i'(a_i, a_{-i})$$

$$= \sum_{a_{-i} \in A_{-i}} \alpha_{-i}(a_{-i})(f_i(a_{-i}) + y_iu_i(a_i, a_{-i}))$$

$$= f_i(\alpha_{-i}) + y_i \sum_{a_{-i} \in A_{-i}} \alpha_{-i}(a_{-i})u_i(a_i, a_{-i})$$

$$= f_i(\alpha_{-i}) + y_iu_i(a_i, \alpha_{-i})$$

Thus, $a_i \in B_i(\alpha_{-i})$ if $f a_i \in B_i'(\alpha_{-i})$ and thus $G$ and $G'$ have the same set of $NE$. The converse is also true if no player has any dominated strategies.
A game

\[ G = \langle N, (A_i), (u_i) \rangle \]

is strategically symmetric if it is strategically equivalent to a game

\[ G' = \langle N, (A'_i), (u'_i) \rangle \]

in which for any two players, \( i \) and \( j \), \( A'_i = A'_j \), and \( B_i(\alpha_{-i}) = B_j(\alpha_{-j}) \) for all \( \alpha \in \Delta A \) with \( \alpha_i = \alpha_j \).

A mixed strategy \( NE \) of a strategically symmetric game in which each player players the same strategy is called symmetric \( NE \).
More results on payoff and strategic equivalence and symmetry

1. Payoff equivalence implies strategic equivalence, but not the converse.

2. Strategic equivalence does not preserve Pareto dominance relation.

3. A $2 \times 2$ strategically symmetric game is (generically) strategically equivalent to either $PD$, $BoS$ or $PC$.

4. Any strategically symmetric game with finite action set has a symmetric $NE$.

5. Generically, any symmetric game has an odd number of symmetric $NE$. 