Microeconomics III

Nash equilibrium II
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Randomization

Recall that a strategic game is a triple \( \langle N, (A_i), (\succeq_i) \rangle \) where

- \( N \) is a finite set of players, and for each player \( i \in N \)
- a non-empty set \( A_i \) of actions
- a preference relation \( \succeq_i \) on the set \( A = \times_{j \in N} A_j \) of possible outcomes.

or a triple \( \langle N, (A_i), (u_i) \rangle \) when \( \succeq_i \) can be represented by a utility function \( u_i : A \rightarrow \mathbb{R} \).
Suppose that,

– each player $i$ can randomize among all her strategies so choices are not deterministic, and

– player $i$’s preferences over lotteries on $A$ can be represented by $vNM$ expected utility function.

Then, we need to add these specifications to the primitives of the model of strategic game $\langle N, (A_i), (\succ_i) \rangle$. 
A mixed strategy of player $i$ is $\alpha_i \in \Delta(A_i)$ where $\Delta(A_i)$ is the set of all probability distributions over $A_i$.

- A profile $(\alpha_i)_{i \in N}$ of mixed strategies induces a probability distribution over the set $A$.

- Assuming independence, the probability of an action profile (outcome) $a$ is then

$$\prod_{i \in N} \alpha_i(a_i).$$
A vNM utility function

\[ U_i : \times_{j \in N} \Delta(A_j) \to \mathbb{R} \]
represents player \( i \)'s preferences over the set of lotteries over \( A \).

The mixed extension of a the strategic game \( \langle N, (A_i), (u_i) \rangle \) is the strategic game

\[ \langle N, (\Delta(A_i)), (U_i) \rangle. \]
Preferences toward risk

The standard model of decisions under risk (known probabilities) is based on von Neumann and Morgenstern Expected Utility Theory.

Consider a set of lotteries, or gambles, (outcomes and probabilities). A fundamental axiom about preferences toward risk is independence:

For any lotteries \( x, y, z \) and \( 0 < \alpha < 1 \)

\[ x \succ y \text{ implies } \alpha x + (1 - \alpha)z \succ \alpha y + (1 - \alpha)z. \]
Expected Utility Theory has some very convenient properties for analyzing choice under uncertainty.

To clarify, we will consider the utility that a consumer gets from her or his income.

More precisely, from the consumption bundle that the consumer’s income can buy.
Behavioral economics

Allais (1953) I

Choose between the two gambles:

\[ A := \begin{array}{c} \cdot.33 \rightarrow \$25,000 \\ \cdot.66 \rightarrow \$24,000 \\ \cdot.01 \rightarrow \$0 \end{array} \quad B := \begin{array}{c} 1 \rightarrow \$24,000 \end{array} \]
Allais (1953) II

Choose between the two gambles:

\[
\begin{align*}
C & := \begin{array}{c}
\text{.33} \\
\text{.67}
\end{array} \quad \text{\$25,000} \\
D & := \begin{array}{c}
\text{.34} \\
\text{.66}
\end{array} \quad \text{\$24,000}
\end{align*}
\]
Two results on mixed strategy Nash equilibrium

Let $G = \langle N, (A_i), (u_i) \rangle$ be a strategic game and $G' = \langle N, (\Delta(A_i)), (U_i) \rangle$ be its mixed extension.

[1] If $a \in NE(G)$ then $a \in NE(G').$

[2] $\alpha \in NE(G')$ if and only if

$$U_i(\alpha_{-i}, a_i) \geq U_i(\alpha_{-i}, a_i')$$

for all $a_i'$ and all $\alpha_i(a_i) > 0.$
Proof: If $a \in NE(G)$ then

$$u_i(a_{-i}, a_i) \geq u_i(a_{-i}, a'_i) \quad \forall i \in N \text{ and } \forall a'_i \in A_i.$$ 

Then, by the linearity of $U_i$ in $\alpha_i$

$$U_i(a_{-i}, a_i) \geq U_i(a_{-i}, \alpha_i) \quad \forall i \in N \text{ and } \forall \alpha_i \in \Delta(A_i)$$

and thus $a \in NE(G')$. 
Proof: Let $\alpha \in NE(G')$

Suppose that $\exists a_i \in A_i$ such that $\alpha_i(a_i) > 0$ and

$$U_i(\alpha_{-i}, a'_i) \geq U_i(\alpha_{-i}, a_i)$$

for some $a'_i \neq a_i$.

Then, player $i$ can increase her payoff by transferring probability from $a_i$ to $a'_i$ so $\alpha$ is not a $NE$.

This implies that $U_i(\alpha_{-i}, a_i) = U_i(\alpha_{-i}, a'_i)$ for all $a_i, a'_i$ in the support of $\alpha$. 

Evolutionary stability

A single population of players. Players interact with each other pair-wise and randomly matched.

Players are assigned modes of behavior (mutation). Utility measures each player’s ability to survive.

$\varepsilon$ of players consists of mutants taking action $a$ while others take action $a^*$. 
Evolutionary stable strategy (ESS)

Consider a payoff symmetric game $G = \langle \{1, 2\}, (A, A), (u_i) \rangle$ where $u_1(a) = u_2(a')$ when $a'$ is obtained from $a$ by exchanging $a_1$ and $a_2$.

$a^* \in A$ is ESS if for any $a \in A$, $a \neq a^*$ and $\varepsilon > 0$ sufficiently small

$$(1 - \varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) > (1 - \varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

which is satisfied if for any $a \neq a^*$ either

$$u(a^*, a^*) > u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) > u(a, a)$$
Three results on $ESS$

[1] If $a^*$ is an $ESS$ then $(a^*, a^*)$ is a $NE$.

Suppose not. Then, there exists a strategy $a \in A$ such that

$$u(a, a^*) > u(a^*, a^*).$$

But, for $\varepsilon$ small enough

$$(1 - \varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) < (1 - \varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

and thus $a^*$ is not an $ESS$. 
[2] If \((a^*, a^*)\) is a strict \(NE\) \(u(a^*, a^*) > u(a, a^*)\) for all \(a \in A\) then \(a^*\) is an \(ESS\).

Suppose \(a^*\) is not an \(ESS\). Then either

\[ u(a^*, a^*) \leq u(a, a^*) \]

or

\[ u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) \leq u(a, a). \]

so \((a^*, a^*)\) can be a \(NE\) but not a strict \(NE\).
[3] A $2 \times 2$ game $G = \langle \{1, 2\}, (A, A), (u_i) \rangle$ where $u_i(a) \neq u_i(a')$ for any $a, a'$ has a mixed strategy which is ESS

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<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>a'</th>
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<td>w, w</td>
<td>x, y</td>
</tr>
<tr>
<td>a'</td>
<td>y, x</td>
<td>z, z</td>
</tr>
</tbody>
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If $w > y$ or $z > x$ then $(a, a)$ or $(a', a')$ are strict NE, and thus $a$ or $a'$ are ESS.

If $w < y$ and $z < x$ then there is a unique symmetric mixed strategy NE $(\alpha^*, \alpha^*)$ where

$$\alpha^*(a) = (z - x)/(w - y + z - x)$$

and $u(\alpha^*, \alpha) > u(\alpha, \alpha)$ for any $\alpha \neq \alpha^*$. 
Strictly competitive games

A strategic game $\langle\{1, 2\}, (A_i), (\succeq_i)\rangle$ is strictly competitive if for any $a \in A$ and $b \in A$ we have $a \succeq_1 b$ if and only if $b \succeq_2 a$.

\[
\begin{array}{c|cc}
 & L & R \\
\hline
T & A, -A & B, -B \\
B & C, -C & D, -D \\
\end{array}
\]
Maxminimization

A max min mixed strategy of player \( i \) is a mixed strategy that solves the problem

\[
\max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i})
\]

A player’s payoff in \( \alpha^* \in NE(G) \) is at least her max min payoff:

\[
U_i(\alpha^*) \geq U_i(\alpha_i, \alpha^*_i) \geq \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i})
\]

and thus

\[
U_i(\alpha^*) \geq \max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i})
\]

since the above holds for all \( \alpha_i \in \triangle(A_i) \).
Two min-max results

\[ \max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \leq \min_{\alpha_{-i} \in \Delta A_{-i}} \max_{\alpha_i \in \Delta A_i} U_i(\alpha_i, \alpha_{-i}) \]

For every \( \alpha' \)

\[ \min_{\alpha_{-i}} U_i(\alpha', \alpha_{-i}) \leq U_i(\alpha', \alpha'_{-i}) \]

and thus

\[ \min_{\alpha_{-i}} U_i(\alpha', \alpha_{-i}) \leq \max_{\alpha_i} U_i(\alpha_i, \alpha'_{-i}) \]

However, since the above holds for every \( \alpha' \) and \( \alpha'_{-i} \) it must hold for the “best” and “worst” such choices

\[ \max \min_{\alpha_i} U_i(\alpha_i, \alpha_{-i}) \leq \min \max_{\alpha_i} U_i(\alpha_i, \alpha_{-i}). \]
[2] In a zero-sum game

\[
\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) = U_1(\alpha^*)
\]

\[
\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)
\]

\[
\min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) = \max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2)
\]

\[
\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) \geq \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)
\]

\[
\min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) \leq \max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2)
\]

Hence, \[
\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) \geq \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)
\]

which together with [1] gives the desired conclusion.
Let $\alpha_1^{\text{max}}$ be player 1's max min strategy and $\alpha_2^{\text{min}}$ be player 2's min max strategy. Then,

$$\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1^{\text{max}}, \alpha_2) \leq U_1(\alpha_1^{\text{max}}, \alpha_2) \quad \forall \alpha_2 \in \Delta A_2$$

and

$$\min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) = \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2^{\text{min}}) \geq U_1(\alpha_1, \alpha_2^{\text{min}}) \quad \forall \alpha_1 \in \Delta A_1$$
But

\[
\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) = U_1(\alpha_1^{\max}, \alpha_2^{\min})
\]

implies that

\[
U_1(\alpha_1, \alpha_2^{\min}) \leq U_1(\alpha_1^{\max}, \alpha_2^{\min}) \leq U_1(\alpha_1^{\max}, \alpha_2)
\]

\forall \alpha_2 \in \Delta A_2 \text{ and } \forall \alpha_1 \in \Delta A_1.

Hence, \((\alpha_1^{\max}, \alpha_2^{\min})\) is an equilibrium.
Interchangeability

If $\alpha$ and $\alpha'$ are NE in a zero-sum game, then so are $(\alpha_1, \alpha'_2)$ and $(\alpha'_1, \alpha_2)$.

- Since $\alpha$ and $\alpha'$ are equilibria

$$U_1(\alpha_1, \alpha_2) \geq U_1(\alpha'_1, \alpha_2) \text{ and } U_2(\alpha'_1, \alpha'_2) \geq U_2(\alpha'_1, \alpha_2),$$

and because $U_1 = -U_2$

$$U_1(\alpha'_1, \alpha'_2) \leq U_1(\alpha'_1, \alpha_2).$$

Therefore,

$$U_1(\alpha_1, \alpha_2) \geq U_1(\alpha'_1, \alpha_2) \geq U_1(\alpha'_1, \alpha'_2). \quad (1)$$

and similar analysis gives that

$$U_1(\alpha_1, \alpha_2) \leq U_1(\alpha_1, \alpha'_2) \leq U_1(\alpha'_1, \alpha'_2). \quad (2)$$
(1) and (2) yield

\[ U_1(\alpha_1, \alpha_2) = U_1(\alpha'_1, \alpha_2) = U_1(\alpha_1, \alpha'_2) = U_1(\alpha'_1, \alpha'_2) \]

Since \( \alpha \) is an equilibrium

\[ U_2(\alpha_1, \alpha''_2) \leq U_2(\alpha_1, \alpha_2) = U_2(\alpha_1, \alpha'_2) \]

for any \( \alpha''_2 \in \Delta A_2 \), and since \( \alpha' \) is an equilibrium

\[ U_1(\alpha''_1, \alpha'_2) \leq U_1(\alpha'_1, \alpha'_2) = U_1(\alpha_1, \alpha'_2) \]

for any \( \alpha''_1 \in \Delta A_1 \). Therefore, \( (\alpha_1, \alpha'_2) \) is an equilibrium and similarly also \( (\alpha_1, \alpha'_2) \).