Microeconomics III

Bargaining I
The strategic approach
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The strategic approach

The players bargain over a pie of size 1.

An agreement is a pair \((x_1, x_2)\) where \(x_i\) is player \(i\)’s share of the pie. The set of possible agreements is

\[
X = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 = 1\}
\]

Player \(i\) prefers \(x \in X\) to \(y \in X\) if and only if \(x_i > y_i\).
The bargaining protocol

The players can take actions only at times in the (infinite) set \( T = \{0, 1, 2, \ldots\} \). In each \( t \in T \) player \( i \), proposes an agreement \( x \in X \) and \( j \neq i \) either accepts (\( Y \)) or rejects (\( N \)).

If \( x \) is accepted (\( Y \)) then the bargaining ends and \( x \) is implemented. If \( x \) is rejected (\( N \)) then the play passes to period \( t + 1 \) in which \( j \) proposes an agreement.

At all times players have perfect information. Every path in which all offers are rejected is denoted as disagreement (\( D \)). The only asymmetry is that player 1 is the first to make an offer.
Preferences

Time preferences (toward agreements at different points in time) are the driving force of the model.

A bargaining game of alternating offers is

– an extensive game of perfect information with the structure given above, and

– player $i$’s preference ordering $\lesssim_i$ over $(X \times T) \cup \{D\}$ is complete and transitive.

Preferences over $X \times T$ are represented by $\delta_i^t u_i(x_i)$ for any $0 < \delta_i < 1$ where $u_i$ is increasing and concave.
Assumptions on preferences

A1 Disagreement is the worst outcome

For any \((x, t) \in X \times T\),

\[(x, t) \succ_i D\]

for each \(i\).

A2 Pie is desirable

– For any \(t \in T, x \in X\) and \(y \in X\)

\[(x, t) \succ_i (y, t)\] if and only if \(x_i > y_i\).
A3  **Time is valuable**

For any $t \in T$, $s \in T$ and $x \in X$

$$(x, t) \succ_i (x, s) \text{ if } t < s$$

and with strict preferences if $x_i > 0$.

A4  **Preference ordering is continuous**

Let $\{(x_n, t)\}_{n=1}^{\infty}$ and $\{(y_n, s)\}_{n=1}^{\infty}$ be members of $X \times T$ for which

$$\lim_{n \to \infty} x_n = x \text{ and } \lim_{n \to \infty} y_n = y.$$ 

Then, $(x, t) \succ_i (y, s)$ whenever $(x_n, t) \succ_i (y_n, s)$ for all $n$. 
A2-A4 imply that for any outcome \((x, t)\) either there is a unique \(y \in X\) such that

\[(y, 0) \sim_i (x, t)\]

or

\[(y, 0) \succ_i (x, t)\]

for every \(y \in X\).

Note \(\succsim_i\) satisfies A2-A4 \textit{iff} it can be represented by a continuous function

\[U_i : [0, 1] \times T \to \mathbb{R}\]

that is increasing (decreasing) in the first (second) argument.
A5  Stationarity

For any $t \in T$, $x \in X$ and $y \in X$

$$(x, t) \succeq_i (y, t + 1) \text{ if and only if } (x, 0) \succeq_i (y, 1).$$

If $\succeq_i$ satisfies A2-A5 then for every $\delta \in (0, 1)$ there exists a continuous increasing function $u_i : [0, 1] \to \mathbb{R}$ (not necessarily concave) such that

$$U_i(x_i, t) = \delta_t^i u_i(x_i).$$
Present value

Define $v_i : [0, 1] \times T \rightarrow [0, 1]$ for $i = 1, 2$ as follows

$$v_i(x_i, t) = \begin{cases} y_i & \text{if } (y, 0) \sim_i (x, t) \\ 0 & \text{if } (y, 0) \succ_i (x, t) \end{cases}$$

for all $y \in X$.

We call $v_i(x_i, t)$ player $i$’s present value of $(x, t)$ and note that

$$(y, t) \succ_i (x, s) \text{ whenever } v_i(y_i, t) > v_i(x_i, s).$$
If \( \preceq_i \) satisfies \textbf{A2-A4}, then for any \( t \in T \) \( v_i(\cdot, t) \) is continuous, non decreasing and increasing whenever \( v_i(x_i, t) > 0 \).

Further, \( v_i(x_i, t) \leq x_i \) for every \( (x, t) \in X \times T \) and with strict whenever \( x_i > 0 \) and \( t \geq 1 \).

With \textbf{A5}, we also have that

\[
v_i(v_i(x_i, 1), 1) = v_i(x_i, 2)
\]

for any \( x \in X \).
Delay

A6 Increasing loss to delay

\[ x_i - v_i(x_i, 1) \] is an increasing function of \( x_i \).

If \( u_i \) is differentiable then under A6 in any representation \( \delta_i u_i(x_i) \) of \( \succ_i \)

\[ \delta_i u'_i(x_i) < u'_i(v_i(x_i, 1)) \]

whenever \( v_i(x_i, 1) > 0 \).

This assumption is weaker than concavity of \( u_i \) which implies

\[ u'_i(x_i) < u'_i(v_i(x_i, 1)). \]
The single crossing property of present values

If \( \preceq_i \) for each \( i \) satisfies A2-A6, then there exist a unique pair \((x^*, y^*) \in X \times X\) such that

\[
y_1^* = v_1(x_1^*, 1) \quad \text{and} \quad x_2^* = v_2(y_2^*, 1).
\]

- For every \( x \in X \), let \( \psi(x) \) be the agreement for which

\[
\psi_1(x) = v_1(x_1, 1)
\]

and define \( H : X \to \mathbb{R} \) by

\[
H(x) = x_2 - v_2(\psi_2(x), 1).
\]
– The pair of agreements \( x \) and \( y = \psi(x) \) satisfies also \( x_2 = v_2(\psi_2(x), 1) \) if \( H(x) = 0. \)

– Note that \( H(0, 1) \geq 0 \) and \( H(1, 0) \leq 0 \), \( H \) is a continuous function, and

\[
H(x) = [v_1(x_1, 1) - x_1] +
+ [1 - v_1(x_1, 1) - v_2(1 - v_1(x_1, 1), 1)].
\]

– Since \( v_1(x_1, 1) \) is non decreasing in \( x_1 \), and both terms are decreasing in \( x_1 \), \( H \) has a unique zero by \textbf{A6}. 
Examples

[1] For every \((x, t) \in X \times T\)

\[ U_i(x_i, t) = \delta_i^t x_i \]

where \(\delta_i \in (0, 1)\), and \(U_i(D) = 0\).

[2] For every \((x, t) \in X \times T\)

\[ U_i(x_i, t) = x_i - c_i t \]

where \(c_i > 0\), and \(U_i(D) = -\infty\) (constant cost of delay).

Although \textbf{A6} is violated, when \(c_1 \neq c_2\) there is a unique pair \((x, y) \in X \times X\) such that \(y_1 = v_1(x_1, 1)\) and \(x_2 = v_2(y_2, 1)\).
Strategies

Let $X^t$ be the set of all sequences $\{x^0, \ldots, x^{t-1}\}$ of members of $X$.

A strategy of player 1 (2) is a sequence of functions

$$\sigma = \{\sigma^t\}_{t=0}^\infty$$

such that $\sigma^t : X^t \rightarrow X$ if $t$ is even (odd), and $\sigma^t : X^{t+1} \rightarrow \{Y, N\}$ if $t$ is odd (even).

The way of representing a player’s strategy in closely related to the notion of automation.
Nash equilibrium

For any $\bar{x} \in X$, the outcome $(\bar{x}, 0)$ is a $NE$ when players’ preference satisfy $A1$-$A6$.

To see this, consider the stationary strategy profile

<table>
<thead>
<tr>
<th>Player 1</th>
<th>proposes</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>accepts</td>
<td>$x_1 \geq \bar{x}_1$</td>
</tr>
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</table>

<table>
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<tr>
<th>Player 2</th>
<th>proposes</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>accepts</td>
<td>$x_2 \geq \bar{x}_2$</td>
</tr>
</tbody>
</table>

This is an example for a pair of one-state automata.

The set of outcomes generated in the Nash equilibrium includes also delays (agreements in period 1 or later).
Subgame perfect equilibrium

Any bargaining game of alternating offers in which players’ preferences satisfy A1-A6 has a unique SPE which is the solution of the following equations

\[ y_1^* = v_1(x_1^*, 1) \text{ and } x_2^* = v_2(y_2^*, 1). \]

Note that if \( y_1^* > 0 \) and \( x_2^* > 0 \) then

\[ (y_1^*, 0) \sim_1 (x_1^*, 1) \text{ and } (x_2^*, 0) \sim_2 (y_2^*, 1). \]
The equilibrium strategy profile is given by

<table>
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<tr>
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</tr>
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<tbody>
<tr>
<td></td>
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<th>Player 2</th>
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</tr>
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<tbody>
<tr>
<td></td>
<td>accepts</td>
<td>( x_2 \geq x_2^* )</td>
</tr>
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</table>

The unique outcome is that player 1 proposes \( x^* \) in period 0 and player 2 accepts.
Step 1 \((x^*, y^*)\) is a SPE

Player 1:

- proposing \(x^*\) at \(t^*\) leads to an outcome \((x^*, t^*)\). Any other strategy generates either

\[
(x, t) \text{ where } x_1 \leq x_1^* \text{ and } t \geq t^*
\]

or

\[
(y^*, t) \text{ where } t \geq t^* + 1
\]

or \(D\).

- Since \(x_1^* > y_1^*\) it follows from A1-A3 that \((x^*, t^*)\) is a best response.
Player 2:

- accepting $x^*$ at $t^*$ leads to an outcome $(x^*, t^*)$. Any other strategy generates either

\[
(y, t) \text{ where } y_2 \leq y^*_2 \text{ and } t \geq t^* + 1
\]

or

\[
(x^*, t) \text{ where } t \geq t^*
\]

or $D$. 
– By A1-A3 and A5

\[(x^*, t^*) \preceq_2 (y^*, t^* + 1)\]

and thus accepting \(x^*\) at \(t^*\), which leads to the outcome \((x^*, t^*)\), is a best response.

Note that similar arguments apply to a subgame starting with an offer of player 2.
Step 2 \((x^*, y^*)\) is the unique \(SPE\)

Let \(G_i\) be a subgame starting with an offer of player \(i\) and define

\[
M_i = \sup \{v_i(x_i, t) : (x, t) \in SPE(G_i)\},
\]

and

\[
m_i = \inf \{v_i(x_i, t) : (x, t) \in SPE(G_i)\}.
\]

It is suffices to show that

\[
M_1 = m_1 = x_1^* \text{ and } M_2 = m_2 = y_2^*.
\]
First, note that in any SPE the first offer is accepted because
\[ v_1(y_1^*, 1) \leq y_1^* < x_1^*. \]
Thus, after a rejection, the present value for player 1 is less than \( x_1^* \).

Then, it remains to show that
\[ m_2 \geq 1 - v_1(M_1, 1) \tag{1} \]
and
\[ M_1 \leq 1 - v_2(m_2, 1). \tag{2} \]
1 implies that the pair \((M_1, 1 - m_2)\) lies below the line
\[ y_1 = v_1(x_1, 1) \]
and 2 implies that the pair \((M_1, 1 - m_2)\) lies to the left the line
\[ x_2 = v_2(y_2, 1). \]

Thus,
\[ M_1 = x_1^* \text{ and } m_2 = y_2^*, \]
and with the role of the players reversed, the same argument show that
\[ M_2 = y_2^* \text{ and } m_1 = x_1^*. \]
With constant discount rates the equilibrium condition implies that

\[ y_1^* = \delta_1 x_1^* \text{ and } x_2^* = \delta_2 y_2^* \]

so that

\[ x^* = \left( \frac{1 - \delta_2}{1 - \delta_1 \delta_2}, \frac{\delta_2(1 - \delta_1)}{1 - \delta_1 \delta_2} \right) \text{ and } y^* = \left( \frac{\delta_1(1 - \delta_2)}{1 - \delta_1 \delta_2}, \frac{1 - \delta_1}{1 - \delta_1 \delta_2} \right). \]
Thus, if \( \delta_1 = \delta_2 = \delta \) (\( v_1 = v_2 \)) then

\[
x^* = \left( \frac{1}{1 + \delta}, \frac{\delta}{1 + \delta} \right) \quad \text{and} \quad y^* = \left( \frac{\delta}{1 + \delta}, \frac{1}{1 + \delta} \right)
\]

so player 1 obtains more than half of the pie.

But, shrinking the length of a period by considering a sequence of games indexed by \( \Delta \) in which \( u_i = \delta_i^\Delta x_i \) we have

\[
\lim_{\Delta \to 0} x^*(\Delta) = \lim_{\Delta \to 0} y^*(\Delta) = \left( \frac{\log \delta_2}{\log \delta_1 + \log \delta_2}, \frac{\log \delta_1}{\log \delta_1 + \log \delta_2} \right).
\]