Problem Set 1
Due: Monday, September 13, 2004

1. Answer both parts of the following question:
   a) Let \( \{X_t\} \) be a two-state Markov chain with transition matrix
   \[
   A = \begin{bmatrix}
   0 & 0.5 \\
   1 & 0.5 
   \end{bmatrix}.
   \]
   and marginal probability vector \( p_t \). Find the unique initial probability vector \( p_0 = \pi \) that yields a stationary process. For arbitrary initial \( p_0 \), find an explicit solution for \( p_t \) and show that \( p_t \) tends to \( \pi \) as \( t \) tends to infinity.

   b) Let \( \{X_t\} \) be a continuous-state discrete-time Markov chain taking values on the unit interval. Suppose the conditional distribution of \( X_t \) given \( X_{t-1} = y \) is uniform on the interval \((1-y, 1)\). Show that the process is stationary if the density of \( X_0 \) is \( 2x \) for \( x \) in the unit interval.

2. Suppose for all integer \( t \) the time series \( \{y_t\} \) was generated as
   \[
y_t = \mu_t + \eta_t, \quad \mu_t = \mu_{t-1} + \epsilon_t
   \]
   where \( \eta_t \) and \( \epsilon_t \) are independent white noise processes with variances \( \sigma^2_\eta \) and \( \sigma^2_\epsilon \).
   a) Derive the Wold representation for \( \Delta y_t \). I.e., find \( c \)’s such that \( \Delta y_t = \sum_j c_j u_{t-j} \) for some white noise process \( u_t \).
   b) Provide consistent method-of-moments estimators for \( \sigma^2_\eta \) and \( \sigma^2_\epsilon \).

3. Compute the autocorrelation function and the partial autocorrelation function of the AR(2) process
   \[
y_t = 0.8y_{t-2} + \epsilon_t
   \]
   where \( \epsilon_t \) is white noise with mean zero and variance \( \sigma^2 \).

4. Let \( \{y_t\} \) be a stationary AR(p) process generated as \( A(L)y_t = \epsilon_t \), where \( \{\epsilon_t\} \) is a mean zero white-noise sequence with unit variance and \( A(L) = 1 - a_1 L - \cdots - a_p L^p \). Suppose the process has the moving average representation \( y_t = C(L)\epsilon_t \) where \( C(L) = 1 + c_1 L + c_2 L^2 + \cdots \). Show that \( E\epsilon_t \epsilon_{t-s} = c_s \) and hence the moving average coefficients satisfy \( A(L)c_s = 0 \) when \( s > 0 \). How does this result differ from the Yule-Walker equation?

5. Consider the stationary ARMA(1,1) process generated by
   \[
y_t = \alpha y_{t-1} + \epsilon_t + \beta \epsilon_{t-1}
   \]
   where the \( \epsilon \)'s are white noise with mean zero and variance one. Assuming that both \( \alpha \) and \( \beta \) are positive and less than one, show that the autocovariances \( \gamma_r = E\epsilon_t \epsilon_{t-r} \) are monotonically decreasing in \( |r| \).

6. Let \( \{y_t\} \) be the stationary solution of the difference equation
   \[
y_t = \alpha y_{t-1} + \epsilon_t
   \]
   where \( |\alpha| > 1 \) and \( \{\epsilon_t\} \) is white noise with mean zero and variance \( \sigma^2 \). Define \( \beta \equiv 1/\alpha \).
   a) Find the moving average representation of \( y_t \) by forward iteration of the equation \( y_t = \beta y_{t+1} - \beta \epsilon_{t+1} \).
   b) Show that \( \eta_t \equiv y_t - \beta y_{t-1} \) is white noise. Find its variance.
   c) Argue that \( y_t \) is also the solution of the stable AR difference equation \( y_t = \beta y_{t-1} + \eta_t \).