1. Suppose the observed time series \( y_1, \ldots, y_T \) was generated as
\[
y_t = \alpha y_{t-1} + u_t, \quad u_t = \rho u_{t-1} + \epsilon_t \quad t = 1, \ldots, T
\]  
where \( y_0 = u_0 = 0 \). Assume that \(|\rho| < 1\) and the \( \epsilon_t \) are unobserved i.i.d. errors with zero mean and unit variance.

(a) Show that this model can be rewritten as
\[
\Delta y_t = \alpha y_{t-1} + b \Delta y_{t-1} + \epsilon_t \quad t = 1, \ldots, T.
\]

What are the initial conditions? Express \( a \) and \( b \) as functions of \( \alpha \) and \( \rho \).

(b) Suppose you estimate \( a \) and \( b \) in (2) by a least-squares regression using the final \( T-2 \) observations. Using the fact that \( \hat{a} - a \) can be written as \( y_0^{-1} M \epsilon / y_0^{-1} M y_0 \) for some idempotent matrix \( M \), find a limiting stochastic integral representation for \( T \hat{a} \) when \( \alpha = 1 \). Explain how you could construct a test of \( \alpha = 1 \) based on a modified version of \( \hat{a} \). Find the limiting representation for the t-ratio \( a \sqrt{y_0^{-1} M y_0} \) when \( \alpha = 1 \) and use it to form a test.

(c) Let \( \tilde{\alpha} \) be the estimate of \( \alpha \) obtained by applying least squares directly to (1), ignoring the fact that \( u_t \) is autocorrelated. Find the limiting representations for \( T(\tilde{\alpha} - 1) \) and the t-ratio \( (\tilde{\alpha} - 1) \sqrt{y_0^{-1} y_0} \) when \( \alpha = 1 \). Explain how these two statistics could be modified to obtain test statistics with the correct asymptotic size.

(d) Relate your tests to the Dickey-Fuller and Perron-Phillips tests.

2. Suppose the observed time series \( y_1, \ldots, y_T \) was generated as
\[
y_t = \beta + u_t, \quad u_t = \alpha u_{t-1} + \epsilon_t \quad t = 1, \ldots, T
\]
where \( u_0 = 0 \) and the \( \epsilon_t \) are unobserved i.i.d. errors with zero mean and unit variance.

(a) Show that this model can be rewritten as
\[
\Delta y_t = \alpha y_{t-1} + b \Delta y_{t-1} + \epsilon_t \quad t = 1, \ldots, T.
\]

(b) Show that the least-squares estimator of \( \alpha \) from (4) can be written as
\[
\hat{\alpha} = \alpha + \frac{y_0' M \epsilon}{y_0' M y_0}
\]
for some idempotent matrix \( M \). What is \( M \) in this case? Show that the distribution of \( \hat{\alpha} \) does not depend on \( \beta \). Find the limiting stochastic integral representation for \( T(\hat{\alpha} - \alpha) \) when \( \alpha = 1 \).

(c) Assuming normality, find equations that define the ML estimates of \( \alpha \) and \( \beta \). Is the MLE for \( \alpha \) the same as the OLS estimator of part (b)?
3. Consider the model

\[ y_t = \alpha y_{t-1} + u_t, \quad u_t = \varepsilon_t + \theta \varepsilon_{t-1} \]

where the \( \varepsilon_t \) are unobserved i.i.d. \( \text{N}(0,1) \) and \( |\theta| < 1 \). Let \( a \) be the OLS estimate of \( \alpha \) obtained by regressing \( y_t \) on \( y_{t-1} \) for observations \( t = 2, \ldots, T \).

(a) What is the probability limit of \( a \) if \( |\alpha| < 1 \)? If \( |\alpha| = 1 \)?

(b) Show that \( \sqrt{T}(a - \text{plim} a) \) is asymptotically normal when \( |\alpha| < 1 \). [You need not do the tedious calculation to find the asymptotic variance.] Find the limiting distribution of the estimator when \( |\alpha| = 1 \).

(c) Consider now using an instrumental variable estimation method with \( y_{t-2} \) as an instrument. After appropriate standardization, find the limiting distribution of this estimator when \( |\alpha| < 1 \). When \( |\alpha| = 1 \).

(d) Show how your results in part (c) can be used to construct a unit root test in the presence of MA residuals.

4. Suppose the observed time series \( y_1, \ldots, y_T \) was generated as

\[ y_t = \alpha y_{t-1} + \varepsilon_t \]

where \( y_0 = 0 \) and the \( \varepsilon_t \) are unobserved i.i.d. \( \text{N}(0,1) \) innovations.

(a) Write the log likelihood function in terms of the new parameter \( \gamma = T(1 - \alpha) \)

(b) Show that \( T^{-2} \sum_{t=2}^{T} y_{t-1}^2 \) and \( T^{-1} \sum_{t=2}^{T} y_{t-1} \Delta y_t \) are sufficient statistics for \( \gamma \).

(c) Squaring the identity \( y_t = y_{t-1} + \Delta y_t \), show that

\[
\frac{2}{T} \sum_{t=2}^{T} y_{t-1} \Delta y_t = \frac{y_T^2}{T} - 1 + o_p(1)
\]

when \( \gamma \) stays constant as \( T \) tends to infinity.

(d) Show that the LM test of \( \gamma = 0 \) (based on the score evaluated at the null) depends asymptotically only on the last observation \( y_T \).