The Optimal Use of Government Purchases for Macroeconomic Stabilization

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Abstract

This paper extends Samuelson's theory of optimal government purchases by accounting for the contribution of government purchases to macroeconomic stabilization. Using a matching model of the macroeconomy, we derive a sufficient-statistics formula for optimal government purchases. The formula implies that the deviation of optimal government purchases from the Samuelson level is proportional to the elasticity of substitution between government and personal consumption times the government-purchases multiplier times the deviation of the unemployment rate from its efficient level. Hence, with a positive multiplier, optimal government purchases are above the Samuelson level when unemployment is inefficiently high and below it when unemployment is inefficiently low. We calibrate the formula to US data. A first implication is that US government purchases are optimal with a small multiplier of 0.04; if the multiplier is larger, US government purchases are not countercyclical enough. Another implication is that optimal government purchases should increase during recessions. With a multiplier of 0.5 the optimal government purchases-output ratio increases from 16.6% to 20.0% when the unemployment rate rises from the US average of 5.9% to 9%. With multipliers higher than 0.5 the optimal ratio increases less because fewer government purchases are required to fill the unemployment gap: with a multiplier of 2 the optimal ratio only increases from 16.6% to 17.6%; this is the same increase as with a multiplier of 0.07.

Keywords: government purchases, business cycles, multiplier, unemployment, matching

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1. Introduction

In the United States the Full Employment and Balanced Growth Act of 1978 imparts the responsibility of achieving full employment to the Board of Governors of the Federal Reserve System and to the government. In practice, however, it is the Federal Reserve that has been in charge of macroeconomic stabilization. This reliance on the Federal Reserve reflects the consensus among policymakers and academics that monetary policy is the policy most adapted to stabilization. But the stabilization achieved through monetary policy alone remains imperfect. Of course, at the zero lower bound on nominal interest rates, monetary policy is severely constrained—that is what happens after 2009 in Figure 1.[1] But that is not all: as Figure 1 shows, monetary policy was not subject to the zero lower bound in the 1991 and 2001 recessions, yet stabilization was only partial. Furthermore, local economies in a monetary union—countries in the eurozone or US States—cannot use monetary policy and must rely on other tools for stabilization.

In this paper, we explore how government purchases can be used to improve stabilization. To that end, we embed the standard theory of optimal government purchases, developed by Samuelson [1954], within a matching model of the macroeconomy.[2] Samuelson showed that in a competitive, efficient model, the optimal provision of government consumption satisfies a simple formula: the marginal rate of substitution between government and personal consumption equals the marginal rate of transformation between government and personal consumption—one in our model.

But a matching model is not necessarily efficient. Our model builds on the matching framework from Michaillat and Saez [2015*a*]. There is one matching market where households sell labor services to other households and the government. In equilibrium there is some unemployment: sellers are unable to sell all the labor services that they could produce. The unemployment rate may not be efficient: the unemployment rate is inefficiently low when too many resources are devoted to purchasing labor services, and it is inefficiently high when too much of the economy's productive capacity is idle.

[1] Krugman [1998] and Eggertsson and Woodford [2003] explain how the effectiveness of monetary policy is restricted by the existence of a zero lower bound on nominal interest rates.

[2] A small literature analyzes optimal government purchases in disequilibrium models [for example, Drèze, 1985; Mankiw and Weinzierl, 2011; Roberts, 1982]. Since our model of unemployment is smoother than the disequilibrium model (see the discussion in Michaillat and Saez [2015*a*]), it provides nondegenerate policy trade-offs that can be resolved with optimal formulas expressed in estimable statistics.

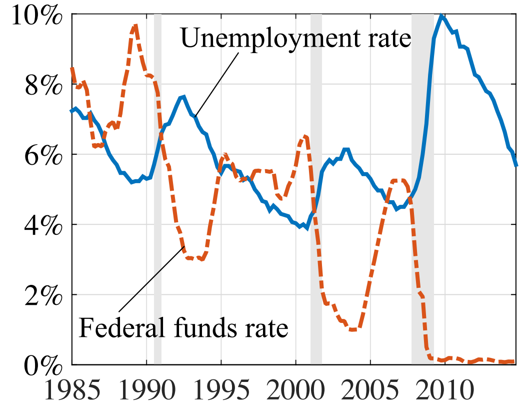

Figure 1: Unemployment and Monetary Policy in the United States, 1985–2014

Notes: The unemployment rate is the quarterly average of the seasonally adjusted monthly unemployment rate constructed by the Bureau of Labor Statistics (BLS) from the Current Population Survey (CPS). The federal funds rate is the quarterly average of the daily effective federal funds rate set by the Board of Governors of the Federal Reserve System. The shaded areas represent the recessions identified by the National Bureau of Economic Research (NBER).

When the unemployment rate is inefficient and government purchases influence unemployment, government purchases have an effect on welfare that is unaccounted for in Samuelson's theory. Hence, our formula for optimal government purchases adds to the Samuelson formula a correction term that measures the effect of government purchases on welfare through their influence on unemployment. The formula implies that optimal government purchases are above the Samuelson level if and only if government purchases bring unemployment closer to its efficient level; this occurs if unemployment is inefficiently high and government purchases lower unemployment, or if unemployment is inefficiently low and government purchases raise unemployment.

We express our formula for optimal government purchases in terms of estimable sufficient statistics.[3] By virtue of being expressed with sufficient statistics, the formula applies broadly, irrespective of the specification of the utility function, aggregate demand, and price mechanism. We derive our results with exogenous labor supply, lump-sum taxation, and a representative household. The formula we derive is robust to introducing heterogeneous households, endogenous labor supply, and distortionary income taxation, paralleling the robustness of the Samuelson formula in public economics [Kaplow, 1996].

The formula shows that the deviation of optimal government purchases from the Samuelson

[3] Chetty [2009] discusses the advantages of using sufficient statistics. The new dynamic public finance literature has also recently strived to express optimal policy formulas with estimable statistics [Golosov and Tsyvinski, 2015].

level is proportional to the elasticity of substitution between government and personal consumption times the government-purchases multiplier times the deviation of the unemployment rate from its efficient level. The elasticity of substitution determines how the marginal rate of substitution between government and personal consumption depends on the ratio between government and personal consumption. The multiplier determines the effect of government purchases on unemployment. The deviation of unemployment from its efficient level determines the effect of unemployment on welfare.

By showing how optimal government purchases in recessions depend on the multiplier and the marginal social value of the purchases (measured by the elasticity of substitution), our formula contributes to the policy discussions about the design of stimulus packages. A voluminous empirical literature estimates multipliers to describe the effects of government purchases on output and other variables.[4] Even though the empirical literature abstracts from welfare considerations, stimulus advocates believe in large multipliers and argue that government purchases can help fill the output gap in recessions [Romer and Bernstein, 2009]; conversely, stimulus skeptics believe in small, even negative, multipliers and argue that more government purchases could be detrimental [Barro and Redlick, 2011]. Our formula proposes a formal connection between the estimates of the government-purchases multiplier and the welfare-maximizing policy. Stimulus skeptics also warn that additional government spending could be wasteful. Our formula shows how the marginal social value of government purchases, measured by the elasticity of substitution, influences the size of the welfare-maximizing policy.

We calibrate the formula to US data and use it to address several policy questions. First, we find that actual US government purchases, which are mildly countercyclical, are optimal under a minuscule multiplier of 0.04. If the actual multiplier is larger than 0.04, US government purchases are not countercyclical enough.

Second, we find that the formula implies significant increases in government purchases during recessions, even for small multipliers. With a multiplier of 0.1 the optimal government purchases-output ratio increases from 16.6% to 18.0% when the unemployment rate rises from the US average

[4]In the literature estimating multipliers on aggregate US data, a few representative studies include Rotemberg and Woodford [1992], Ramey and Shapiro [1998], Blanchard and Perotti [2002], Galí, Lopez-Salido and Valles [2007], Mountford and Uhlig [2009], Hall [2009], Auerbach and Gorodnichenko [2012], Barro and Redlick [2011], and Ramey [2011*b*]. See Ramey [2011*a*] and Parker [2011] for excellent surveys.

of 5.9% to a high level of 9%; with a multiplier of 0.5 the optimal ratio increases more, from 16.6% to 20.0%. With multipliers higher than 0.5, however, the optimal government purchases-output ratio increases less: with a multiplier of 2 the optimal ratio only increases from 16.6% to 17.6%; this is the same increase as with a multiplier of 0.07. In fact, we prove that the relation between the multiplier and the increase of the optimal government purchases-output ratio for a given increase in unemployment rate has a hump shape. For small multipliers, the optimal amount of government purchases is determined by the crowding out of personal consumption by government consumption; a higher multiplier means less crowding out and thus higher optimal government purchases. For large multipliers, it is optimal to fill the unemployment gap; a higher multiplier means that fewer government purchases are required to fill this gap.

Our analysis is carried out without fleshing out a complete model of how government purchases affect unemployment and output because this is not needed to derive our sufficient statistics formulas which apply to a broad range of models. As an illustration, we use the simple dynamic model with an aggregate demand coming out of utility for wealth developed in Michaillat and Saez [2015*b*] can be used to calibrate any government purchase multiplier. If the interest rate is not fully flexible, shocks will create inefficient fluctuations in unemployment and the government purchase multiplier will be positive. We use this model to show that our formulas are good approximations to fully optimal policies.

2. A Macroeconomic Model of Unemployment and Government Purchases

This section proposes a macroeconomic model of unemployment and government purchases, building on the matching framework from Michaillat and Saez [2015*a*]. The model is dynamic and set in continuous time. The model is generic in that we do not place much structure on the utility function, aggregate demand, price mechanism, and tax system. The components of the model that we introduce are sufficient to define a feasible allocation and describe the mathematical structure of an equilibrium. These are the only elements on which the optimal policy analysis relies. By maintaining this level of generality, we are able to show in Section 3 that our sufficient-statistics formula for optimal government purchases applies to a broad range of models. We provide a specific model building on Michaillat and Saez [2015*b*] as an example in Section 5.

The economy consists of a government and a measure 1 of identical households. Households are self-employed, producing and selling services on a matching market.[5] Each household has a productive capacity normalized to 1; the productive capacity indicates the maximum amount of services that a household could sell at any point in time. At time t, the household sells $C(t)$ services to other households and $G(t)$ services to the government. The household's output is

$$Y(t) = C(t) + G(t).$$

The matching process prevents households from selling their entire capacity so $Y(t) < 1$.

The services are sold through long-term relationships. The idle capacity of the household at time t therefore is $1 - Y(t)$. Since some of the capacity of the household is idle, some household members are unemployed. The rate of unemployment, defined as the share of workers who are idle, is $u(t) = 1 - Y(t)$, where $Y(t)$ is the aggregate output of services.

To purchase labor services, households and government advertise $v(t)$ vacancies at time t. The rate h at which new long-term relationships are formed is given by a Cobb-Douglas matching function: $h(t) = \omega \cdot (1 - Y(t))^{\eta} \cdot v(t)^{1-\eta}$, where $1 - Y(t)$ is aggregate idle capacity, $v(t)$ is aggregate number of vacancies, $\omega > 0$ governs the efficacy of matching, and $\eta \in (0,1)$ is the elasticity of the matching function with respect to idle capacity.[6]

The market tightness x is defined by $x(t) = v(t)/(1 - Y(t))$. The market tightness is the ratio of the two arguments in the matching function: aggregate vacancies and aggregate idle capacity. With constant returns to scale in matching, the tightness determines the rates at which sellers and buyers enter into new long-term trading relationships. At time t, each of the $1 - Y(t)$ units of idle productive capacity is sold at rate $f(x(t)) = h(t)/(1 - Y(t)) = \omega \cdot x(t)^{1-\eta}$ and each of the $v(t)$ vacancies is filled at rate $q(x(t)) = h(t)/v(t) = \omega \cdot x(t)^{-\eta}$. The selling rate $f(x)$ is increasing in x and the buying rate $q(x)$ is decreasing in x; hence, when the tightness is higher, it is easier to sell services but harder to buy them.

[5] We assume that households cannot consume their own labor services. To simplify the analysis, we abstract from firms and assume that all production directly takes place within households. Michaillat and Saez [2015*a*] show how the model can be modified to include firms hiring workers on a labor market and selling their production on a product market.

[6] The empirical evidence summarized by Petrongolo and Pissarides [2001] indicates that a Cobb-Douglas specification for the matching function fits the data well.

Long-term relationships separate at rate $s > 0$. Accordingly, output is a state variable with law of motion $\dot{Y}(t) = f(x(t)) \cdot (1 - Y(t)) - s \cdot Y(t)$. The term $f(x(t)) \cdot (1 - Y(t))$ is the number of new relationships formed at t. The term $s \cdot Y(t)$ is the number of existing relationships separated at t. However, in practice, because the transitional dynamics of output are fast, output rapidly adjusts to its steady-state level where market flows are balanced.[7] Throughout the paper, we therefore simplify the analysis by modeling output as a jump variable equal to its steady-state value defined by $f(x(t)) \cdot (1 - Y(t)) = s \cdot Y(t)$. With this simplification, output becomes a function of market tightness defined by

$$Y(x) = \frac{f(x)}{f(x) + s}. \tag{1}$$

The function $Y(x)$ is in $[0,1]$, increasing on $[0, +\infty)$, with $Y(0) = 0$ and $\lim_{x \to +\infty} Y(x) = 1$. By definition, output is directly related to the unemployment rate: $Y = 1 - u$. Hence, the simplification also implies that the unemployment rate is function of market tightness defined by

$$u(x) = \frac{s}{s + f(x)}. \tag{2}$$

The function $u(x)$ is in $[0,1]$, decreasing on $[0, +\infty)$, with $u(0) = 1$ and $\lim_{x \to +\infty} u(x) = 0$. Intuitively, when the market tightness is higher, it is easier to sell services so output is higher and the unemployment rate is lower. The elasticity of $Y(x)$ is $(1 - \eta) \cdot u(x)$ and that of $u(x)$ is $-(1 - \eta) \cdot (1 - u(x))$.

Advertising vacancies is costly. Posting one vacancy costs $\rho > 0$ services per unit time. Hence, a total of $\rho \cdot v(t)$ services are spent at time t on filling vacancies. These services represent the resources devoted by households and government to matching with appropriate providers of services. Since these resources devoted to matching do not enter households' utility function, we define two concepts of consumption. We refer to the quantities $C(t)$ and $G(t)$ purchased by households and government as *gross personal consumption* and *gross government consumption*. Following common usage, *government consumption* designates the consumption by households of services purchased by the government. We define the *net personal consumption* $c(t) < C(t)$ and *net gov-*

[7] Hall [2005] and Shimer [2012] establish this property for the employment rate, which is proportional to output in our model. Appendix A confirms this result over a longer time period.

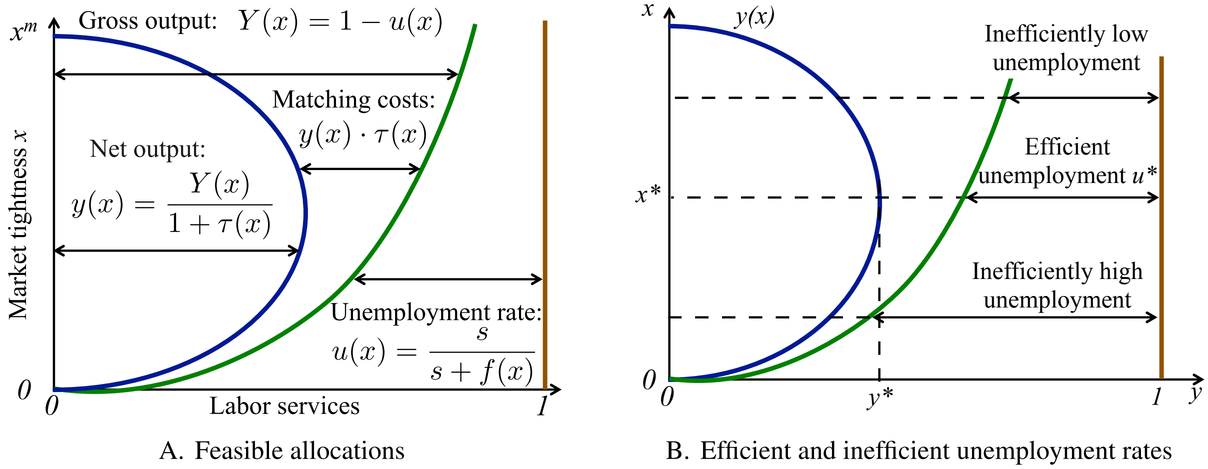

Figure 2: The Market for Labor Services

ernment consumption $g(t) < G(t)$ as the gross consumptions net of the services used for matching. We also refer to $Y(t) = C(t) + G(t)$ as *gross output* and $y(t) = c(t) + g(t)$ as *net output*.

As market flows are balanced, $s \cdot Y(t) = v(t) \cdot q(x(t))$. Hence, $y(t) = Y(t) - \rho \cdot v(t) = Y(t) - \rho \cdot s \cdot Y(t)/q(x(t))$, which implies that $Y(t) = [1 + \tau(x(t))] \cdot y(t)$ where we define

$$\tau(x) = \frac{\rho \cdot s}{q(x) - \rho \cdot s}.$$

Of course we also have $C(t) = [1 + \tau(x(t))] \cdot c(t)$ and $G(t) = [1 + \tau(x(t))] \cdot g(t)$. Hence, enjoying one service requires to purchase $1 + \tau$ services—one service that enters the utility function plus τ services for matching. The matching wedge $\tau(x)$ is positive and increasing on $[0, x^m)$, where $x^m \in (0, +\infty)$ is defined by $q(x^m) = \rho \cdot s$. In addition, $\lim_{x \to x^m} \tau(x) = +\infty$. Intuitively, when the market tightness is higher, it is more difficult to match with a seller so the matching wedge is higher. The elasticity of $\tau(x)$ is $\eta \cdot (1 + \tau(x))$.

The concepts of gross consumption and gross output correspond to the quantities measured in national accounts.[8] Indeed, gross output is proportional to employment in our model, and part of employment measured in national accounts is used to create matches—for instance, human resource workers, placement agency workers, procurement workers, buyers—even though the services they provide are used for matching and do not enter households' utility.

[8] In the US National Income and Product Accounts (NIPA), $C(t)$ is "personal consumption expenditures" and $G(t)$ "government consumption expenditures".

It is useful to write net output as a function of market tightness:

$$y(x) = \frac{1-u(x)}{1+\tau(x)}. \tag{3}$$

This function $y(x)$ plays a central role in the analysis because it gives the amount of services that can be allocated between net personal consumption and net government consumption for a given tightness. The expression (3) shows that net consumption is below the productive capacity (normalized to 1) because some services are not sold in equilibrium ($u(x) > 0$) and some services are used for matching instead of net consumption ($\tau(x) > 0$). The function $y(x)$ is defined on $[0, x^m]$, positive, with $y(0) = 0$ and $y(x^m) = 0$. The elasticity of $y(x)$ is $(1-\eta)\cdot u(x) - \eta \cdot \tau(x)$. Hence, the elasticity of $y(x)$ is $1-\eta$ at $x = 0$, and it is $-\infty$ at $x = x^m$, and it is strictly decreasing in x. Therefore, the function $y(x)$ is strictly increasing for $x \leq x^*$, strictly decreasing for $x \geq x^*$, where the tightness x^* is defined by

$$(1-\eta)\cdot u(x^*) = \eta \cdot \tau(x^*). \tag{4}$$

The function $y(x)$ is therefore maximized at $x = x^*$. Since x^* maximizes net output, we refer to it as the efficient tightness. The efficient tightness is the tightness underlying the condition of Hosios [1990] for efficiency in a matching model. The efficient unemployment rate is $u^* \equiv u(x^*)$.

Figure 2 summarizes the model. Panel A depicts how net output, gross output, and unemployment rate depend on market tightness in feasible allocations. Panel B depicts the function $y(x)$, the efficient tightness x^*, the efficient unemployment rate u^*, and situations in which tightness is inefficiently high and unemployment is inefficiently low ($x > x^*$, $u < u^*$), and situations in which tightness is inefficiently low and unemployment is inefficiently high ($x < x^*$, $u > u^*$). When unemployment is inefficiently high, too much of the economy's productive capacity is idle, and a marginal decrease in unemployment increases net output. When unemployment is inefficiently low, too many resources are devoted to purchasing labor services, and a further decrease in unemployment reduces net output.[9]

[9] In our model gross output, Y, is proportional to the employment rate, $1-u$; hence, when output is 1 percent below trend, the employment rate is 1 percent below trend and the unemployment rate is slightly less than 1 percentage point above trend. This property seemingly contradicts Okun's law; Okun [1963] found that in US data for 1954–1962, output was 3 percent below trend when the unemployment rate was 1 percentage point above trend. The relationship

Next, we assume that the government's budget is balanced at all times using a lump-sum tax $T(t) = G(t)$ levied on households.[10] We also assume that the government sets $g(t)$ as a function of the other variables at time t and parameters. In that case, the dynamical system describing the equilibrium of the model only has jump variables and no state variables. We assume that the equilibrium system is a source.[11] Hence, the equilibrium converges immediately to its steady-state value from any initial condition. Since transitional dynamics are immediate, an equilibrium is completely characterized by its steady state.

Finally, we define a feasible allocation and an equilibrium. We give a static definition as the equilibrium converges immediately to its steady state:

DEFINITION 1. *A* feasible allocation *is a net personal consumption $c \in [0,1]$, a net government consumption, $g \in [0,1]$, and net output $y \in [0,1]$, and a market tightness $x \in [0,+\infty)$ that satisfy $y = y(x)$ and $c = y - g$. The function $y(x)$ is defined by* (3).

DEFINITION 2. *An* equilibrium function *is a mapping from a net government consumption g to a feasible allocation $[c,g,y,x]$. Given that y and c are functions of x and g in a feasible allocation, the equilibrium function is summarized by a mapping from a net government consumption g to a tightness x.*

In the model, an equilibrium is just a value of the equilibrium function. In practice the equilibrium function $x(g)$ arises from the household's optimal consumption choice and the price mechanism. The function $x(g)$ can describe efficient prices, bargained prices, or rigid prices. To provide a concrete example, Section 5 describes the function $x(g)$ in a specific model.

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between output gap and unemployment gap has evolved over time, however. In Appendix B, we estimate Okun's law for the entire 1951–2014 period and for the recent 1994–2014 period. We find that when the unemployment rate is 1 percentage point above trend, output is 1.8 percent below trend in the 1951–2014 period and 1.3 percent below trend in the 1994–2014 period.

[10] When households are Ricardian—in the sense that they do not view government bonds as net wealth because such bonds need to be repaid with taxes later on—financing government purchases with debt is economically equivalent to maintaining budget balance using a lump-sum tax [Barro, 1974]. Hence, our analysis would remain valid if the government financed government purchases with debt and households were Ricardian. Michaillat and Saez [2015*b*] analyze debt policy when individuals are not necessarily Ricardian.

[11] Without this assumption, the model would suffer from dynamic indeterminacy, making the welfare analysis impossible. The complete model we analyze in Section 5 satisfies the source system assumption.

3. Sufficient-Statistics Formulas for Optimal Government Purchases

The representative household derives instantaneous utility $\mathscr{U}(c,g)$ from net personal consumption c and net government consumption g. The function $\mathscr{U}$ is increasing in its two arguments, concave, and homothetic. The marginal rate of substitution between government consumption and personal consumption is

$$MRS_{gc} \equiv \frac{\partial \mathscr{U}/\partial g}{\partial \mathscr{U}/\partial c}.$$

Since $\mathscr{U}$ is homothetic, the marginal rate of substitution is a decreasing function of $g/c = G/C$.[12]

Since the equilibrium immediately converges to its steady state, the welfare of an equilibrium is $\mathscr{U}(c,g)$. In a feasible allocation, net personal consumption is given by $c = y(x) - g$, so welfare can be written as $\mathscr{U}(y(x)-g,g)$. Given an equilibrium function $x(g)$, the government chooses g to maximize welfare $\mathscr{U}(y(x(g))-g,g)$. We assume that the welfare function $g \mapsto \mathscr{U}(y(x(g))-g,g)$ is well behaved: it admits a unique extremum and the extremum is a maximum.[13] Under this assumption, first-order conditions are not only necessary but also sufficient to describe the optimum of the government's problem.

In this section we derive several sufficient-statistics formulas giving the optimal level of government purchases. These formulas are adapted to answer different questions. The first formula relates the optimal level of government purchases to the level given by the Samuelson formula. The first formula is exact but it is not expressed with statistics that can be estimated in the data; the second and third formula are approximate but they are expressed with estimable statistics, which makes them appropriate for practical policy applications. The second formula relates the deviation of

[12]By homothetic, we mean that the utility can be written as $\mathscr{U}(c,g) = \mathscr{W}(w(c,g)))$ where the function $\mathscr{W}$ is increasing and the function w is increasing in its two arguments, concave, and homogeneous of degree 1. Since w is homogeneous of degree 1, its derivatives $\partial w/\partial c$ and $\partial w/\partial g$ are homogeneous of degree 0. Combining these properties, and we have

$$MRS_{gc} = \frac{\frac{\partial w}{\partial g}(c,g)\cdot \mathscr{W}'(w(c,g)))}{\frac{\partial w}{\partial c}(c,g)\cdot \mathscr{W}'(w(c,g)))} = \frac{\frac{\partial w}{\partial g}\left(1,\frac{g}{c}\right)}{\frac{\partial w}{\partial c}\left(\frac{c}{g},1\right)}.$$

We see that MRS_{gc} is a function of g/c only. As w is concave, $\partial w/\partial g$ is decreasing in its second argument while $\partial w/\partial c$ is decreasing in its first argument; hence, MRS_{gc} is a decreasing function of g/c.

[13]We showed that $x \mapsto y(x)$ has a unique extremum and this extremum is a maximum. We assumed that $\mathscr{U}$ is concave. Therefore, we need $g \mapsto x(g)$ to be well behaved in order for $g \mapsto \mathscr{U}(y(x(g))-g,g)$ to satisfy the assumption.

optimal government purchases from the Samuelson level to the government-purchases multiplier, the elasticity of substitution between government and personal consumption, and the deviation of actual market tightness from efficient market tightness. The second formula is helpful to assess actual government purchases, but it only defines optimal government purchases implicitly, so it does not say how government purchases should be adjusted after a macroeconomic shock. Thus, we propose a third formula that expresses optimal government purchases as an explicit function of the change in unemployment rate observed after the shock, the government-purchases multiplier, and the elasticity of substitution between government and personal consumption.

3.1. An Exact Formula

Taking the first-order condition of the government's problem, we obtain

$$0 = \frac{\partial \mathcal{U}}{\partial g} - \frac{\partial \mathcal{U}}{\partial c} + \frac{\partial \mathcal{U}}{\partial c} \cdot y'(x) \cdot x'(g).$$

Reshuffling the terms in the optimality condition and dividing the condition by $\partial \mathcal{U} / \partial c$ yields the formula for optimal government purchases:

Proposition 1. *Optimal government purchases satisfy*

$$1 = MRS_{gc} + y'(x) \cdot x'(g). \tag{5}$$

As is standard in optimal tax formulas, formula (5) characterizes the optimal level of government purchases implicitly. If the formula holds, then g maximizes welfare. If the right-hand side of (5) is above 1, a marginal increase in g raises welfare; conversely, if the right-hand side is below 1, a marginal increase in g reduces welfare. Although the formula is only implicit, it is useful because it transparently shows the economic forces at play.

Formula (5) is the formula of Samuelson [1954] plus a correction term. The Samuelson formula is $1 = MRS_{gc}$; it requires that the marginal utilities from personal consumption and government consumption are equal. With homothetic preferences, MRS_{gc} decreases with G/C, so the Samuelson formula determines a unique ratio G/C denoted by $(G/C)^*$. As $G/Y = (G/C)/(G/C+1)$, it also defines a unique government purchases-output ratio G/Y denoted by $(G/Y)^*$. The correction

Table 1: Optimal Government Purchases-Output Ratio Compared to Samuelson Ratio

	Effect of net government consumption on unemployment		
Unemployment rate	$du/dg > 0$	$du/dg = 0$	$du/dg < 0$
Inefficiently high	lower	same	higher
Efficient	same	same	same
Inefficiently low	higher	same	lower

Notes: The government purchases-output ratio in the theory of Samuelson [1954] is given by $1 = MRS_{gc}$. Formula (5) implies that compared to the Samuelson ratio, the optimal government purchases-output ratio is higher if $y'(x) \cdot x'(g) > 0$, same if $y'(x) \cdot x'(g) = 0$, and lower if $y'(x) \cdot x'(g) < 0$. By definition, the unemployment rate is inefficiently high when $y'(x) > 0$, inefficiently low when $y'(x) < 0$, and efficient when $y'(x) = 0$. Last, $du/dg = u'(x) \cdot x'(g)$ where $u(x)$ is given by (2). Since $u'(x) > 0$, du/dg and $x'(g)$ have the same sign.

term is the product of the effect of government purchases on tightness, $x'(g)$, and the effect of tightness on net output, $y'(x)$. The correction term measures dy/dg; it is positive if and only if more government purchases yield higher net output. Given the relation between net output, tightness, and unemployment rate (Figure 2), the correction term is positive when government purchases bring the unemployment rate toward its efficient level.

Our formula gives general conditions for the optimal level of government purchases to be above or below the Samuelson level. The formula indicates that the government purchases-output ratio should be above the Samuelson ratio if the correction term is positive, and below the Samuelson ratio if the correction term is negative. If the correction term is zero, the optimal government purchases-output ratio satisfies the Samuelson formula.

There are two situations in which the correction term is zero and the optimal government purchases-output ratio is given by the Samuelson formula. The first is when the unemployment rate is efficient ($y'(x) = 0$). In that case, the marginal effect of government purchases on unemployment has no first-order effect on welfare and the principles of Samuelson's theory apply. The second is when government purchases have no effect on tightness and thus on the unemployment rate ($x'(g) = 0$). In that case, the model is isomorphic to Samuelson's framework.

In all other situations, the correction term is nonzero and the optimal government purchases-output ratio departs from the Samuelson ratio. The formula implies that the optimal government purchases-output ratio is above the Samuelson ratio if and only if government purchases bring unemployment closer to its efficient level. This occurs either if the unemployment rate is inefficiently high and government purchases lower it, or if the unemployment rate is inefficiently low and government purchases raise it. Table 1 summarizes all the possibilities.

The results described here are closely related to those obtained by Farhi and Werning [2012]. Farhi and Werning study the optimal use government purchases for stabilization in a fiscal union.[14] In their model, government purchases of nongraded goods increase output with a multiplier of 1. They find that optimal government purchases are given by the Samuelson formula plus a correction term equal to a labor wedge that measures of the state of the business cycle. Consistent with our analysis, they find that government purchases should be provided above the Samuelson level in recessions and below the Samuelson level in booms.

The results are also consistent with those obtained by others in new Keynesian models. Woodford [2011] notes that away from the zero lower bound, monetary policy perfectly stabilizes the economy; hence, government purchases are not needed for stabilization, and they should follow the Samuelson formula. We obtain the same result: when unemployment is efficient, government purchases are given by the Samuelson formula. Werning [2012] describes the optimal use of government purchases in a liquidity trap. Like us, he finds that the optimal level of government purchases is the Samuelson level plus a correction term arising from stabilization motives.

Furthermore, formula (5) can be used to recover the results on government purchases obtained in the Keynesian regime of disequilibrium models pioneered by Barro and Grossman [1971]. The correction term in (5) can be written as dy/dg. In a disequilibrium model, there are no matching costs so $y = Y$ and $g = G$ and the correction term is equal to the standard multiplier dY/dG. In the Keynesian regime, personal consumption is fixed because it is determined by aggregate demand and the above-market-clearing price; hence, there is no crowding out of personal consumption by government consumption and $dY/dG = 1$. On the other hand when the product market clears, crowding out is one-for-one and $dY/dG = 0$. We assume that there is some value for government purchases such that $MRS_{gc} > 0$. Since $MRS_{gc} + dY/dG > 1$ as long as the output gap is not closed, our formula implies that additional government purchases raise welfare in the Keynesian regime and that it is optimal to use government purchases to fill entirely the output gap.

Finally, the structure of the formula—a standard formula from public economics plus a cor-

[14] See also Galí and Monacelli [2008] for an analysis of optimal monetary and fiscal policy in a currency union.

rection term capturing stabilization motives—is similar to the structure of the formula for optimal unemployment insurance derived by Landais, Michaillat and Saez [2010] in a matching model, and to the structure of the formulas for optimal macroprudential policies derived by Farhi and Werning [2013] in models with price rigidities.

3.2. An Implicit Formula in Estimable Statistics

Formula (5) is useful to describe the economic forces at play, but it is difficult to use it for practical policy applications because it is not expressed with estimable statistics. The correction term in the formula can be written as a multiplier dy/dg. But this multiplier is not directly estimable in the data because it gives the effect of net government consumption on net output whereas the data measure gross government consumption and gross output. To adapt formula (5) for policy applications, we re-express it with estimable statistics. The main task is to express dy/dg as a function of the government-purchases multiplier dY/dG estimated by macroeconomists in aggregate data, and other estimable statistics. We first obtain an exact formula in Lemma 1 and then provide a much simpler approximation in Proposition 2.

LEMMA 1. *Optimal government purchases satisfy*

$$1 = MRS_{gc} + \left(1 - \frac{\eta}{1-\eta} \cdot \frac{\tau(x)}{u(x)}\right) \cdot \frac{dY}{dG} \cdot \left(1 - \frac{\eta}{1-\eta} \cdot \frac{\tau(x)}{u(x)} \cdot \frac{G}{Y} \cdot \frac{dY}{dG}\right)^{-1}. \tag{6}$$

All the statistics are defined above and listed in Table 2.

Proof. First, note that

$$\frac{d\ln(y)}{d\ln(g)} = \frac{d\ln(y)}{d\ln(x)} \cdot \frac{d\ln(x)}{d\ln(G)} \cdot \frac{d\ln(G)}{d\ln(g)}.$$

Next, as the elasticity of $Y(x)$ is $(1-\eta)\cdot u$, we find that

$$\frac{d\ln(x)}{d\ln(G)} = \frac{1}{(1-\eta)\cdot u} \cdot \frac{d\ln(Y)}{d\ln(G)}.$$

Last, using $G=(1+\tau(x))\cdot g$ and as the elasticity of $1+\tau(x)$ is $\eta\cdot\tau$, we find that

$$\frac{d\ln(G)}{d\ln(g)}=1+\eta\cdot\tau\cdot\frac{d\ln(x)}{d\ln(G)}\cdot\frac{d\ln(G)}{d\ln(g)}.$$

Combining this equation with the expression for $d\ln(x)/d\ln(G)$ obtained above, we get

$$\frac{d\ln(G)}{d\ln(g)}=\left(1-\frac{\eta}{1-\eta}\cdot\frac{\tau}{u}\cdot\frac{d\ln(Y)}{d\ln(G)}\right)^{-1}.$$

Combining all these results and as the elasticity of $y(x)$ is $(1-\eta)\cdot u-\eta\cdot\tau$, we obtain

$$\frac{dy}{dg}=\left(1-\frac{\eta}{1-\eta}\cdot\frac{\tau}{u}\right)\cdot\frac{dY}{dG}\cdot\left(1-\frac{\eta}{1-\eta}\cdot\frac{\tau}{u}\cdot\frac{d\ln(Y)}{d\ln(G)}\right)^{-1}.$$

Bringing all the elements together, we obtain (6). □

Next, relying on first-order approximations, we obtain the following formula:

PROPOSITION 2. *Optimal government purchases approximately satisfy*

$$\frac{G/C-(G/C)^*}{(G/C)^*}\approx-\varepsilon\cdot m\cdot\frac{x-x^*}{x^*}, \tag{7}$$

where

$$\varepsilon\equiv-\frac{d\ln(MRS_{gc})}{d\ln(G/C)} \tag{8}$$

is the elasticity of substitution between government and personal consumption,

$$m\equiv\frac{dY/dG}{1-(G/Y)\cdot(dY/dG)} \tag{9}$$

is an increasing function of the government-purchases multiplier dY/dG, and $(G/C)^*$ and x^* are defined above and listed in Table 2. The statistics ε and m are evaluated at $[(G/C)^*,x^*]$. The approximation is valid up to a remainder that is $O((x-x^*)^2+(G/C-(G/C)^*)^2)$.

The rigorous proof is presented in Appendix C but the heuristic derivation of (6) is simple. First, by definition of the marginal rate of substitution, we have $MRS_{gc}(G/C)-MRS_{gc}((G/C)^*)\approx$

Table 2: The Sufficient Statistics Used in the Formulas for Optimal Government Purchases

Notation	Description	Definition
$u(x)$	unemployment rate	$u(x) = \frac{s}{f(x)+s}$
$\tau(x)$	matching wedge	$\tau(x) = \frac{s\cdot\rho}{q(x)-s\cdot\rho}$
$1-\eta$	elasticity of the selling rate	$1-\eta = \frac{d\ln(f(x))}{d\ln(x)}$
a	decreasing function of $1-\eta$	$a = \frac{(G/Y)\cdot(1-G/Y)}{(1-\eta)\cdot u}$
$y'(x)$	marginal effect of tightness on net output	$\frac{d\ln(y)}{d\ln(x)} = (1-\eta)\cdot u(x) - \eta\cdot\tau(x)$
x^*	efficient tightness	$y'(x^*) = 0$
u^*	efficient unemployment rate	$u^* = u(x^*)$
MRS_{gc}	marginal rate of substitution between government and personal consumption	$MRS_{gc} = \frac{\partial\mathscr{U}/\partial g}{\partial\mathscr{U}/\partial c}$
$(G/C)^*$	Samuelson ratio of government consumption to personal consumption	$MRS_{gc}((G/C)^*) = 1$
$(G/Y)^*$	Samuelson ratio of government consumption to output	$(G/Y)^* = \frac{(G/C)^*}{1+(G/C)^*}$
ε	elasticity of substitution between government and personal consumption	$\varepsilon = -\frac{d\ln(MRS_{gc})}{d\ln(G/C)}$
$x'(g)$	marginal effect of net government consumption on equilibrium tightness	
dY/dG	government-purchases multiplier	
m	increasing function of dY/dG	$m = \frac{dY/dG}{1-(G/Y)\cdot(dY/dG)}$

$(-1/\varepsilon)\cdot(G/C-(G/C)^*)/(G/C)^*$. As $MRS_{gc}((G/C)^*) = 1$, we have $1-MRS_{gc} \approx (1/\varepsilon)\cdot(G/C-(G/C)^*)/(G/C)^*$ so we only need to show that the last term in equation (6), which is a product of three terms, is approximately equal to $-m\cdot(x-x^*)/x^*$. Second, at $x = x^*$, from (4) we have $(\eta/(1-\eta))\cdot(\tau/u) = 1$. Hence, the product of the last two terms of the three term product in equation (6) is approximately m when x is close to x^*. $(\eta/(1-\eta))\cdot(\tau/u) = 1$ at $x = x^*$ leads to the first-order expansion $1-(\eta/(1-\eta))\cdot(\tau(x)/u(x)) \approx \zeta\cdot(x-x^*)$ where ζ is the derivative of $-[\eta/(1-\eta)]\cdot\tau(x)/u(x)$ with respect to x evaluated at x^*. We show in Appendix C that $\alpha = -1/x^*$ which is the only non obvious step that completes the proof.[15]

Like formula (5), formula (7) is implicit because its right-hand-side is endogenous to the policy.

[15]This derivative result is consistent with the result in Lemma 2 that $[\eta/(1-\eta)]\cdot\tau(x)/u(x) \approx x/x^*$ for all x.

If the right-hand side of (7) is higher than the left-hand side, a marginal increase in G would raise welfare; conversely, if the right-hand side is lower than the left-hand side, a marginal increase in G would reduce welfare. Although the formula is only implicit, it is useful to assess actual policy, as showed in Section 4.

Formula (7) shows that the elasticity of substitution between government and personal consumption is critical to determine the optimal level of government purchases. The elasticity plays an important role because it determines how quickly the marginal value of government purchases relative to that of personal consumption fades when government purchases increase. The role of this elasticity has been largely neglected in previous work.

Consider the case with $\varepsilon = 0$.[16] This would be a situation in which we need a certain number of bridges for an economy of a given size, but beyond that number, additional bridges have zero value ("bridges to nowhere"). The formula says that with $\varepsilon = 0$ the ratio G/C should stay at the Samuelson ratio $(G/C)^*$, irrespective of the level of unemployment. Increasing G/C beyond $(G/C)^*$ is never optimal because government consumption g is useless at the margin for G/C beyond $(G/C)^*$ and personal consumption c is always crowded out by g.[17]

Consider next the case with $\varepsilon \to +\infty$.[18] This would be a situation in which the services provided by the government substitute exactly for the services purchased by individuals on the market. The formula says that with $\varepsilon \to +\infty$ government purchases should completely fill the tightness gap such that $x = x^*$, even if government purchases crowd out personal consumption. Government purchases should be used to maximize net output, or equivalently bring tightness to x^*, because only net output matters for welfare—-the composition of output does not.

In reality, government purchases probably have some value at the margin, without being perfect substitute for personal consumption; that is, $\varepsilon > 0$ but $\varepsilon < +\infty$. In that case, the ratio G/C and tightness x generally departs from the Samuelson ratio $(G/C)^*$ and from the efficient tightness x^*.

Formula (7) also shows that the welfare-maximizing level of government purchases depends on the government-purchases multiplier, confirming an intuition that macroeconomists have had for

[16]The Leontief utility function $\mathscr{U}(c,g) = \min\{(1-\gamma)\cdot c, \gamma\cdot g\}$ has $\varepsilon = 0$. With this utility function, the Samuelson ratio is $(G/C)^* \equiv (1-\gamma)/\gamma$.

[17]If g did not crowd out c, then $dc/dg > 0$ and $dy/dg = 1 + dc/dg > 1$, which would violate (5) and can therefore not occur at an optimum.

[18]The linear utility function $\mathscr{U}(c,g) = c + g$ has $\varepsilon = +\infty$.

a long time.[19] The multiplier dY/dG enters through the statistic m defined by (9) in the formula. The statistic m is an increasing function of dY/dG, $m=0$ when $dY/dG=0$, and $m\to+\infty$ when $dY/dG\to Y/G$. Clearly, m and dY/dG have the same sign. The statistic m enters the formula instead of dY/dG because what matters for welfare is dY/dg, not dY/dG, and at x^*, $dY/dg = (1+\tau(x^*))\cdot m > dY/dG$.

If the government-purchases multiplier is zero, government purchases should remain at the level given by the Samuelson formula. If the multiplier is positive, the government purchases-output ratio should be above the Samuelson ratio when unemployment is inefficiently high and below it when unemployment is inefficiently low. If the multiplier is negative, the government purchases-output ratio should be below the Samuelson ratio when unemployment is inefficiently high and above it when unemployment is inefficiently low.

Formula (7) is a first-order approximation of (5) valid up to a remainder that is $O((x-x^*)^2 + (G/C-(G/C)^*)^2)$. In practice, government purchases are never far from the Samuelson level so $(G/C-(G/C)^*)^2$ remains small. However, as we shall see in Section 4, tightness displays large fluctuates. Hence, it could sometimes be far from its efficient level, so $(x-x^*)^2$ could be large and formula (7) could be inaccurate. To alleviate this concern, we show that under some reasonable assumptions, a formula similar to (7) provides a good approximation of (5) even far from the efficient tightness.

LEMMA 2. *Assume that the separation rate s and matching cost ρ are small enough compared to the matching rates $f(x)$ and $q(x)$. Then the following is a good approximation:*

$$\frac{\eta}{1-\eta}\cdot\frac{\tau(x)}{u(x)} \approx \frac{x}{x^*}. \tag{10}$$

Proof. Assume that $s \ll f(x)$ and $s\cdot\rho \ll q(x)$. In that case, $f(x)$ is a good approximation of $s+f(x)$ and $q(x)$ of $q(x)-s\cdot\rho$. Therefore, we can approximate

$$\frac{\tau(x)}{u(x)} = \frac{s\cdot\rho}{q(x)-s\cdot\rho}\cdot\frac{s+f(x)}{s} \approx \frac{s\cdot\rho}{q(x)}\cdot\frac{f(x)}{s} = \rho\cdot x.$$

This approximation implies that $\tau(x^*)/u(x^*) \approx \rho\cdot x^*$. The efficiency condition (4) implies that

[19] For instance, using a heuristic method, Woodford [2011] and Nakamura and Steinsson [2014] show that the multiplier affects the optimal level of government purchases in a new Keynesian model.

$\tau(x^*)/u(x^*) = (1-\eta)/\eta$. Hence, $(1-\eta)/\eta \approx \rho \cdot x^*$. Combining $\tau(x)/u(x) \approx \rho \cdot x$ and $\eta/(1-\eta) \approx 1/(\rho \cdot x^*)$, we find that (10) is a good approximation. □

Combining the results from Lemmas 1 and 2, we obtain the following formula:

PROPOSITION 3. *Assume that the separation rate s and matching cost ρ are small compared to the matching rates $f(x)$ and $q(x)$. Assume also that G/C remains in the neighborhood of $(G/C)^*$. Then a good approximation for optimal government purchases is*

$$\frac{G/C-(G/C)^*}{(G/C)^*} \approx -\varepsilon \cdot \frac{dY/dG}{1-(x/x^*)\cdot(G/Y)\cdot(dY/dG)} \cdot \frac{x-x^*}{x^*}. \tag{11}$$

All the statistics are defined above and listed in Table 2. The statistics ε and dY/dG are evaluated at $[G/C, x]$.

Proof. We start from (6). Using Lemma 2, we approximate $1-[\eta/(1-\eta)]\cdot(\tau/u)$ by $-(x-x^*)/x^*$ and $[\eta/(1-\eta)]\cdot(\tau/u)$ by x/x^*. Combining these approximations yields (11). □

An advantage of formula (11) is that it applies even when business cycles have triggered large departures of the tightness from its efficient level. While the formula relies on assumptions on the size of some parameters and variables (s and ρ small, gap between G/C and $(G/C)^*$ small), we find that these assumptions are satisfied in US data, suggesting that (11) would be accurate in the US. Appendix A finds that in US monthly data $s = 3.3\%$, $s \cdot \rho = 6.5\%$, $f(x^*) = 56\%$, and $q(x^*) = 94\%$, validating the assumptions that $s \ll f(x)$ and $s \cdot \rho \ll q(x)$. Appendix A also finds that in US data for 1951–2014, the approximation of Lemma 2 is extremely accurate. Panel B of Figure 3 shows that in US data the deviation of G/C from $(G/C)^*$ is never more than 8%, validating the assumption that the gap between G/C and $(G/C)^*$ is small.

To ascertain the conditions under which formula (7) holds far from the efficient tightness, it suffices to compare (7) and (11). A first condition is that the elasticity $(G/Y)\cdot(dY/dG)$ is small enough such that the large fluctuations of x/x^* in the denominator of the right-hand side of (11) do not generate large deviations of the right-hand side of (11) from the right-hand side of (7). This condition seems satisfied in US data. Section 4 shows that $G/Y = 0.17$ on average in US data and argues that a reasonable estimate of the multiplier is $dY/dG = 0.6$, suggesting that the elasticity

$(G/Y) \cdot (dY/dG)$ is fairly small. Appendix A also finds that $1-(x/x^*) \cdot (G/Y) \cdot (dY/dG)$ is always quite close to $1-(G/Y) \cdot (dY/dG)$.

A second condition is that elasticity of substitution ε and the multiplier dY/dG are fairly stable when tightness varies a lot. Indeed, ε and dY/dG are evaluated at $[(G/C)^*, x^*]$ in (7) but at $[G/C, x]$ in (11). If these two statistics varied a lot with tightness, the right-hand sides of (7) and (11) would be very different. We know little about ε and its possible variations over the business cycle. On the other hand, there is growing evidence that multipliers are higher when the unemployment rate is higher.[20] Of course, the variations of dY/dG when x varies only have a second-order effect in formula (7); but these second-order effects could be large is tightness drifts far from efficiency.[21] To gauge the quality of formula (7) when dY/dG responds to x, Section 5 develops a specific model in which dY/dG is countercyclical and investigates numerically whether formula (7) provides a good approximation to the exact formula (5).

3.3. An Explicit Formula in Estimable Statistics

While formula (7) is useful for certain applications, we cannot use the formula to answer the following question: if the tightness is 50% above its efficient level and government purchases are at the Samuelson level, what should be the optimal increase in government purchases? This is because the formula describes the optimal policy implicitly.[22] This is a typical limitation of sufficient-statistics optimal policy formulas, and a typical criticism addressed to the sufficient-statistics approach [Chetty, 2009]. Here we develop an explicit sufficient-statistics formula that can be used to address this question.

We assume that the tightness is initially at an inefficient level $x_0 \neq x^*$. As government purchases change, tightness endogenously responds. Once we have described the endogenous response, we obtain the following explicit formula:

Proposition 4. *Assume that the economy is at an equilibrium $[(G/C)^*, x_0]$, where the tightness*

[20] See for instance Auerbach and Gorodnichenko [2012, 2013].

[21] By second-order effect, we mean that accounting for the variation of dY/dG when x deviates from x^* would only add a term that is $O((x-x^*)^2)$ to formula (7).

[22] The ratio G/C is implicitly defined by (7) because the right-hand side of (7) is endogenous to G/C.

$x_0 \neq x^*$ is inefficient. Then optimal government purchases are approximately given by

$$\frac{G/C - (G/C)^*}{(G/C)^*} \approx \frac{-\varepsilon \cdot m}{1 + a \cdot \varepsilon \cdot m^2} \cdot \frac{x_0 - x^*}{x^*}, \tag{12}$$

where

$$a \equiv \frac{(G/Y) \cdot (1 - G/Y)}{(1 - \eta) \cdot u}, \tag{13}$$

$m = (dY/dG)/(1 - (G/Y) \cdot (dY/dG))$ and $(G/C)^*$, ε, x^*, and $1 - \eta$ are defined above and listed in Table 2. The statistics ε, m, and a are evaluated at $[(G/C)^*, x^*]$. Furthermore, the equilibrium level of tightness once optimal government purchases are in place is approximately given by

$$x \approx x^* + \frac{1}{1 + a \cdot \varepsilon \cdot m^2} \cdot (x_0 - x^*). \tag{14}$$

The two approximations are valid up to a remainder that is $O((x_0 - x^*)^2 + (G/C - (G/C)^*)^2)$.

The complete proof is in Appendix C. It is easy to derive the result heuristically starting from the implicit formula (7) and recognizing that $x \approx x_0 + \alpha \cdot (G/C - (G/C)^*)$ where α is the marginal effect of G/C on x. Plugging this expression into (7) and re-arranging, we obtain

$$\frac{G/C - (G/C)^*}{(G/C)^*} \approx \frac{-\varepsilon \cdot m}{1 + \varepsilon \cdot m \cdot \alpha \cdot (G/C)^*/x^*} \cdot \frac{x_0 - x^*}{x^*}.$$

The proof shows that $\alpha \approx m \cdot a \cdot x^*/(G/C)^*$ which immediately yields (12).[23] Taking the ratio of (7) and (12) immediately implies that $x - x^* \approx (x_0 - x^*)/(1 + a \cdot \varepsilon \cdot m)$, which proves (14).

Formula (12) links the deviation of the optimal government purchases-output ratio from the Samuelson ratio to the initial deviation of tightness from its efficient level. The formula can be directly applied by policymakers to determine the optimal response of government purchases to a shock that leads to a departure of tightness from its efficient level. Since formula (12) builds on formula (7), formula (12) requires the same conditions as formula (7) to be accurate when tightness moves far from its efficient level. Section 5 uses numerical simulations to investigate the accuracy of the formula.

[23] As α captures the effect of G/C on x, it is not surprising that it is proportional to the multiplier m.

Formula (12) shows that the optimal percent deviation of the government purchases-output ratio from the Samuelson ratio when tightness is below the efficient tightness by 1 percent is

$$\Delta = \frac{\varepsilon \cdot m}{1 + a \cdot \varepsilon \cdot m^2}, \tag{15}$$

where ε is the elasticity of substitution between government and personal consumption, m is the increasing function of the multiplier dY/dG given by (9), and a is given by (13). Equation (15) implies that when government and personal consumption are more substitutable, the purchases-output ratio should respond more strongly to fluctuations in tightness. Formally, $\partial\Delta/\partial\varepsilon > 0$. Equation (15) also indicates that a higher multiplier does not necessarily imply a stronger response of government purchases to fluctuations in tightness. The following proposition formalizes this statement:

PROPOSITION 5. *The function $m \mapsto \Delta(m)$ is positive on $[0, +\infty)$, with $\Delta(0) = \lim_{m\to+\infty} \Delta(m) = 0$. The function increases on $[0, 1/\sqrt{\varepsilon \cdot a}]$, and decreases on $[1/\sqrt{\varepsilon \cdot a}, +\infty)$. It is maximized at $1/\sqrt{\varepsilon \cdot a}$; the maximum is $\Delta^m = 0.5 \cdot \sqrt{\varepsilon / a}$. The function is odd so $\Delta(-m) = -\Delta(m)$.*

Proof. All the results follow from some routine algebra. □

Since m is an increasing function of the multiplier dY/dG, the proposition implies that the optimal increase in government purchases following a given fall in tightness is not monotonically increasing with the multiplier; instead, the optimal increase in government purchases is a hump-shaped function of the multiplier.

There is a simple intuition behind this apparently surprising result. Consider first a small multiplier: $dY/dG \to 0$. We can neglect the feedback effect of G on x because the multiplier is small so $x \approx x_0$. Hence, the application of formula (7) yields

$$\frac{G/C - (G/C)^*}{(G/C)^*} \approx \varepsilon \cdot m \cdot \frac{x^* - x_0}{x^*}.$$

From this formula, it is clear that when $dY/dG \to 0$, $[G/C - (G/C)^*]/(G/C)^*$ increases with m and thus dY/dG.[24] The intuition is that for small multipliers, the optimal amount of government

[24] The explicit formula (12) indeed simplifies to the same expression when $dY/dG \to 0$ and thus $m \to 0$.

purchases is determined by the crowding out of personal consumption by government consumption; a higher multiplier means less crowding out and thus higher optimal government purchases.

Consider next a very large multiplier, $d\ln(Y)/d\ln(G) \to 1$ and $m \to +\infty$. With such a large multiplier, G/C remains constant as G increases.[25] Since the marginal rate of substitution between government and personal consumption only depends on G/C, increasing G fills the tightness gap without changing the marginal rate of substitution. The optimal policy therefore is to maintain G/C at $(G/C)^*$ while entirely filling the tightness gap $x_0 - x^*$. Equation (A14) indicates that filling the tightness gap necessitates

$$\frac{G/C-(G/C)^*}{(G/C)^*} \approx \frac{1}{a\cdot m}\cdot\frac{x^*-x_0}{x^*}$$

Clearly, when $m \to +\infty$, $[G/C-(G/C)^*]/(G/C)^*$ decreases with m and accordingly with dY/dG.[26] The intuition is that if the multiplier is high, government purchases are a very potent policy that can bring the economy close to the efficient tightness without distorting the allocation of output between personal and government consumption. As the multiplier rises, fewer government purchases are required to bring tightness to its efficient level.

The maximum Δ^m gives the strongest possible response of government purchases to a given fall in tightness, for any possible multiplier. This upper bound is useful given that empirical research has not yet reached a consensus on the precise value of the multiplier. The maximum depends critically on the elasticity of substitution.

4. Policy Applications

In this section, we propose estimates for the sufficient statistics of Section 3 and combine these estimates with formulas (7) and (12) for two policy applications for the United States over the 1951–2014 period. First, we show that US government purchases are not countercyclical enough for conventional estimates of the multiplier and the elasticity of substitution between government and personal consumption. In fact, we find that US government purchases would be optimal only if the multiplier and the elasticity of substitution took minuscule values. Second, we determine the

[25] As $C+G=Y$, we have $dC/dG = dY/dG-1$ and hence $d\ln(G/C)/d\ln(G) = (Y/C)\cdot(1-d\ln(Y)/d\ln(G))$. Therefore, $d\ln(G/C)/d\ln(G) \to 0$ when $d\ln(Y)/d\ln(G) \to 1$.

[26] The explicit formula (12) indeed simplifies to the same expression when $m \to +\infty$.

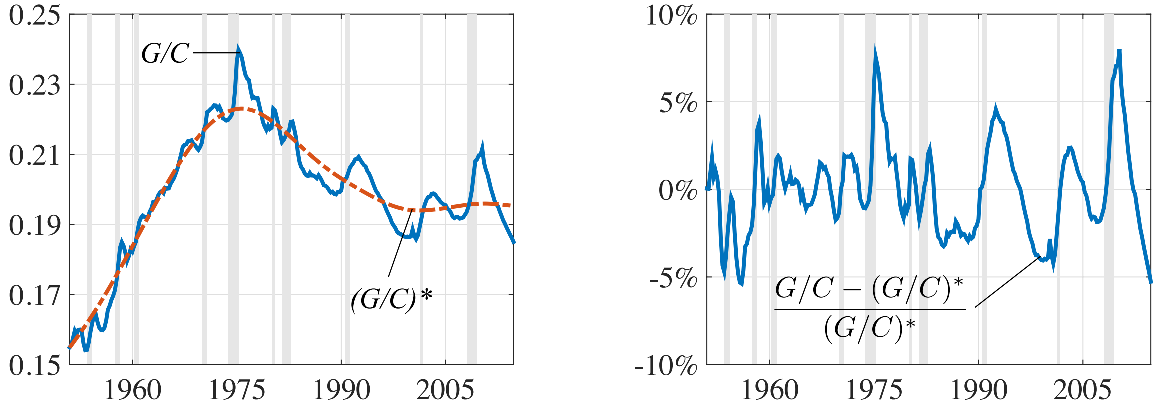


Figure 3: The Ratio of Government Consumption to Personal Consumption in the United States, 1951–2014

Notes: Government consumption G is the quarterly average of the seasonally adjusted monthly number of employees in the government industry constructed by the BLS from the CES survey. Personal consumption C is the quarterly average of the seasonally adjusted monthly number of employees in the private industry constructed by the BLS from the CES survey. The ratio $(G/C)^*$ is the low-frequency trend of G/C produced using a HP filter with smoothing parameter 10^5. The shaded areas represent the recessions identified by the NBER.

optimal response of government purchases when the unemployment rate increases from its average level of 5.9% to a high level of 9%. We highlight how the optimal policy response depends on the value of the multiplier and elasticity of substitution.

4.1. Assessment of US Government Purchases

We assess whether the fluctuations of US government purchases are optimal given the observed fluctuations of the market tightness. For the assessment, we evaluate

$$\frac{G/C-(G/C)^*}{(G/C)^*}+\varepsilon\cdot m\cdot\frac{x-x^*}{x^*}. \tag{16}$$

Formula (7) establishes that if this expression is zero, the level of government purchases is optimal; if this expression is positive, government purchases are too high; and if this expression is negative, government purchases are too low.

To evaluate the expression, we need a measure of government purchases. Using employment data constructed by the BLS from the Current Employment Statistics (CES) survey, we measure G/C as the ratio of employment in the government industry to employment in the private indus-

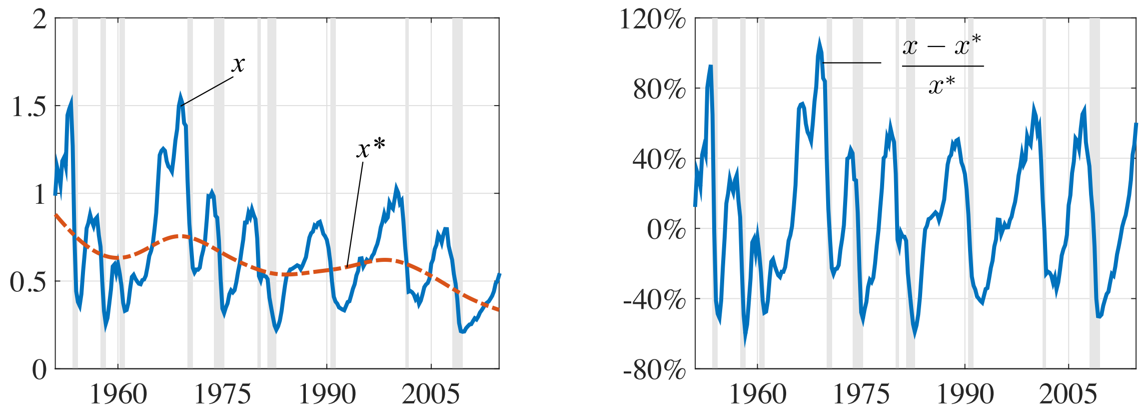


Figure 4: Market Tightness in the United States, 1951–2014

Notes: The market tightness x is the ratio of number of vacancies to unemployment level. The number of vacancies is the quarterly average of the monthly vacancy index constructed by Barnichon [2010], scaled to match the number of vacancies in JOLTS for 2001–2014. The unemployment level is the quarterly average of the seasonally adjusted monthly number of unemployed persons constructed by the BLS from the CPS. The efficient market tightness x^* is the low-frequency trend of x produced using a HP filter with smoothing parameter 10^5. The shaded areas represent the recessions identified by the NBER.

try.[27] As showed in the left-panel of Figure 3, the ratio G/C starts at 15.5% in 1951, peaks at 24.0% in 1975, falls back to 20.0% in 1990, and hovers around this level since. The average of G/C over the 1951–2014 period is 19.9%, and the average of $G/Y = (G/C)/(1+G/C)$ is 16.6%.

We assume that the government determines the trend of government purchases using the well-known Samuelson formula.[28] Thus, the ratio $(G/C)^*$ can be measured as the low-frequency trend of G/C. From the quarterly series for G/C, we produce this trend using a Hodrick-Prescott (HP) filter with smoothing parameter 10^5. The left-panel of Figure 3 also displays $(G/C)^*$ in dashed line. The right-panel of Figure 3 displays the relative deviation $(G/C - (G/C)^*)/(G/C)^*$. As the relative deviation increases in recessions and falls in expansions, US government purchases are mildly countercyclical.

To evaluate (16), we also need a measure of market tightness. Following the standard practice, we measure tightness by the ratio of vacancies to unemployment. We measure unemployment by the number of unemployed persons constructed by the BLS from the CPS. The number of

[27] We measure G/C with employment data to be consistent with our measure of tightness based on labor market data. As a robustness check, Appendix D constructs an alternative measure of G/C using consumption expenditures data constructed by the Bureau of Economic Analysis (BEA) as part of the NIPA. The cyclical behavior of the two series is similar for 1951–2014 and almost undistinguishable after 1980.

[28] If the trend of tightness is efficient, as we assume below, determining the trend of government purchases with the Samuelson formula is optimal.

vacancies is measured by the BLS using data collected in the Job Opening and Labor Turnover Survey (JOLTS) since December 2000. Before December 2000, we construct a proxy for vacancies by rescaling the help-wanted advertising index of Barnichon [2010].[29] We scale up the index such that its average value between December 2000 and December 2014 matches the average number of vacancies posted over the same period.[30] The left-panel of Figure 4 plots the market tightness. Tightness averages 0.65 between 1951 and 2014, and it is very procyclical.

Next, we measure the efficient tightness x^*. We assume that the trend of the economy is efficient. This assumption has a long tradition: macroeconomists have made it at least since Okun [1963] did in his famous paper that proposed Okun's law. This assumption can also be justified on theoretical grounds. Merging the direct-search theory of Moen [1997] and the costly price-adjustment framework of Rotemberg [1982], Michaillat and Saez [2015*b*] propose a price mechanism that adjusts sluggishly to shocks but eventually converges to the efficient price, driven by the market forces described in Moen [1997]. With this mechanism, tightness is temporarily inefficient in response to demand or supply shocks, but it eventually converges to its efficient level such that the trend of tightness is efficient. We therefore measure x^* as the low-frequency trend of the tightness x. From the quarterly series for x, we produce this trend using a HP filter with smoothing parameter 10^5. Figure 4 displays x^* and the relative deviation $(x-x^*)/x^*$. The tightness is systematically inefficiently high in booms and inefficiently low in slumps; these fluctuations are very large. Our methodology would not be altered if a different efficient tightness was selected; only the numbers would change.[31]

Finally, we need estimates of ε and m to evaluate (16). The statistic ε is the elasticity of substitution between government and personal consumption. A Leontief utility function has an elasticity of 0. A Cobb-Douglas utility function has an elasticity of 1. A linear utility function has an elasticity of $+\infty$. We take $\varepsilon = 1$ as a baseline.

[29] This index combines the online and print help-wanted advertising indices constructed by the Conference Board.

[30] The average value of the Barnichon index between December 2000 and December 2014 is 80.59. The average number of vacancies from JOLTS between December 2000 and December 2014 is 3.707 millions. (The JOLTS only started in December 2000.) Hence we multiply the Barnichon index by $3.707 \times 10^6/80.59 = 45,996$ to obtain a proxy for the number of vacancies since 1951.

[31] For example, if prices adjusted very quickly, then tightness would continuously be efficient, and the observed tightness would be the efficient tightness. In this case, the fluctuations in tightness depicted in Figure 4 would be fluctuations in the efficient tightness. In our model, these fluctuations would only occur with large variations in the matching technology, as x^* depends solely on it. See Michaillat and Saez [2015*b*] for a longer discussion of this point.

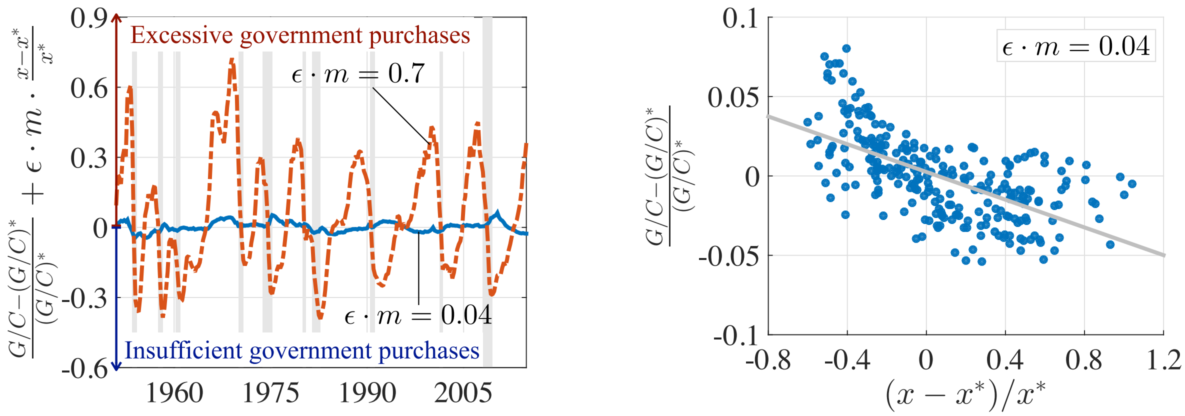
The statistic m, given by (9), is an increasing function of the government-purchases multiplier dY/dG. Ramey [2011a] discusses the estimates of dY/dG in the literature. Table 1 in Ramey [2011a] shows that in aggregate analyzes on postwar US data, dY/dG is between 0.6 and 1.6 when the increase in G is financed by deficit spending. Under the standard assumption that households are Ricardian, lump-sum taxation is equivalent to deficit financing and the range 0.6–1.6 also applies to an increase in G financed by lump-sum taxes. If households are not Ricardian, deficit spending could stimulate output beyond what a balanced-budget increase in G would achieve. In that case, the range 0.6–1.6 overstates the value of dY/dG achieved when the increase in G is financed by lump-sum taxes. Romer and Bernstein [2009] propose to reduce by 1 the value of dY/dG obtained with a deficit-financed increase in G in order to describe the value of dY/dG obtained with a balanced-budget increase in G. Overall, a multiplier of $dY/dG = 0.6$ seems to be a reasonable estimate. We therefore take $m = 0.6/(1-0.166\times 0.6) = 0.7$ as a baseline.[32]

Panel A of figure 5 plots (16) using our estimates of the sufficient statistics. In particular, we set ε and m to their baseline values of $\varepsilon = 1$ and $m = 0.7$ (corresponding to $dY/dG = 0.6$). We find that the expression is systematically positive in booms and negative in slumps (this is clearly visible in the recession years of 1982, 1991, 2001, and 2009). We conclude that for a multiplier of 0.6 and an elasticity of substitution of 1, US government purchases should be higher in slumps and lower in booms. This finding is not surprising because government purchases have not been actively used for stabilization in the United States.[33]

US government purchases are not countercyclical enough if ε and m take baseline values motivated by the estimates in the literature. Exploiting formula (7), we now determine the values of ε and m consistent with the observed fluctuations of government purchases and tightness in the United States. When government purchases are optimal, the observed values of government purchases and tightness satisfy formula (7). Therefore, we can regress $(G/C-(G/C)^*)/(G/C)^*$ on

[32]We compute m using (9), $dY/dG = 0.6$, and $G/Y = (G/Y)^* = 16.6\%$.

[33]During the Great Recession, government expenditure dramatically increased with the American Recovery and Reinvestment Act of 2009. But government purchases of goods and services did not increase; the federal government increased transfers and tax rebates, and federal government purchases increases were offset by reduced state and local government purchases. See the description of US public expenditure during the 2000–2010 period by Taylor [2011].



A. Assessment for various values of the statistics

B. Estimating the statistics consistent with policy

Figure 5: Assessment of US Government Purchases, 1951–2014

Notes: The data used in the figure are quarterly US data covering the 1951–2014 period. The actual and efficient market tightnesses, x and x^*, are described in Figure 4. The ratios of government consumption to personal consumption, G/C and $(G/C)^*$, are described in Figure 3. Panel A evaluates equation (7) for $\varepsilon \cdot m = 0.04$ and $\varepsilon \cdot m = 0.7$. The shaded areas represent the recessions identified by the NBER. Panel B displays a scatter plot of $(G/C - (G/C)^*)/(G/C)^*$ and $(x - x^*)/x^*$. The plot also includes the regression line used to estimate $\varepsilon \cdot m$.

$(x - x^*)/x^*$ to estimate $\varepsilon \cdot m$. We estimate the linear regression

$$\frac{G_t/C_t - (G_t/C_t)^*}{(G_t/C_t)^*} = \hat{\beta} \cdot \frac{x_t - x_t^*}{x_t^*} + \varepsilon_t$$

using ordinary least squares and find $\hat{\beta} = 0.043$ (robust standard error: 0.012). The regression analysis is illustrated in Panel B of Figure 5. The analysis implies that US government purchases are optimal under statistics $\varepsilon \cdot m = 0.04$. For an elasticity of substitution of $\varepsilon = 1$, the result implies that the US policy is optimal under a minuscule multiplier of $dY/dG = 0.04$.[34]

Panel A of Figure 5 plots (16) with $\varepsilon \cdot m = 0.04$. Consistent with the regression result, we find that, with $\varepsilon \cdot m = 0.04$, US government purchases are nearly optimal—formula (7) nearly holds at all time. In sum, either the product of the government-purchases multiplier by the elasticity of substitution is tiny ($\varepsilon \cdot m \approx 0.04$) and US government purchases are optimal, or the product is larger and US government purchases are not countercyclical enough.

[34]With such a small multiplier, $m \approx dY/dG$.

4.2. *Optimal Response of Government Purchases to an Unemployment Increase*

For policy applications we re-express the results of Propositions 4 and 5 in terms of the unemployment rate u and the government purchases-output ratio G/Y. This is simple as the deviation of G/Y from the Samuelson ratio $(G/Y)^*$ is related to the deviation of G/C from $(G/C)^*$ by

$$\frac{G/Y-(G/Y)^*}{(G/Y)^*} \approx [1-(G/Y)^*] \cdot \frac{G/C-(G/C)^*}{(G/C)^*} \tag{17}$$

and the deviation of u from its efficient level u^* is related to the deviation of x from x^* by

$$\frac{u-u^*}{u^*} \approx -(1-u^*) \cdot (1-\eta) \cdot \frac{x-x^*}{x^*}. \tag{18}$$

The approximations are valid up to remainders that are $O((G/C-(G/C)^*)^2)$ and $O((x-x^*)^2)$, respectively.[35]

Assume that the economy is at an equilibrium $[(G/Y)^*, u_0]$, where the unemployment rate $u_0 \neq u^*$ is inefficient. Then Proposition 4, combined with (17) and (18), indicates that the optimal government purchases-output ratio is approximately given by

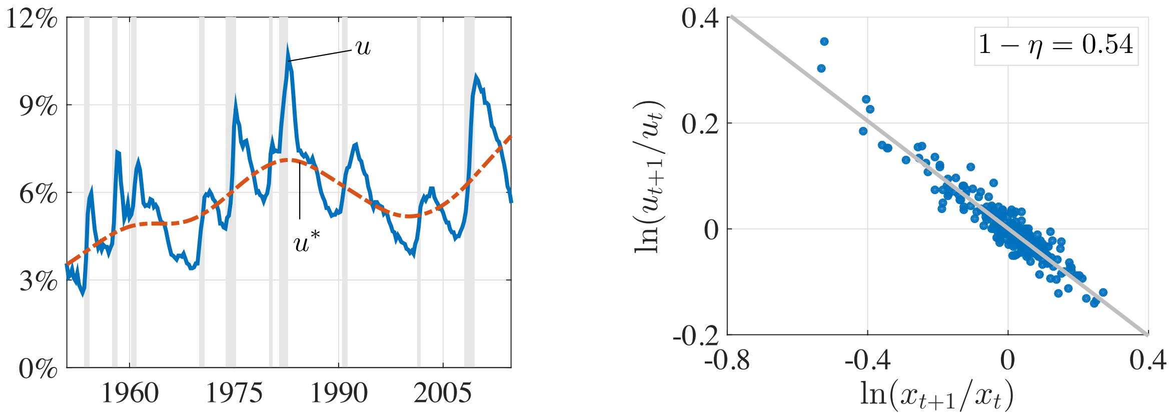
$$G/Y-(G/Y)^* \approx \frac{a}{1-u^*} \cdot \frac{\varepsilon \cdot m}{1+a \cdot \varepsilon \cdot m^2} \cdot (u_0-u^*). \tag{19}$$

All the statistics are defined above and listed in Table 2. We exploit this formula to compute the optimal response of government purchases to a given increase in unemployment. Before proceeding, we need an estimate of the efficient unemployment rate u^* and the elasticity of the selling rate $1-\eta$, which determines the value of the statistic a.

We measure u using the unemployment rate constructed by the BLS from the CPS. We construct the efficient unemployment rate u^* as the low-frequency trend of u. From the quarterly series for u, we produce this trend using a HP filter with smoothing parameter 10^5. Figure 6 displays u and u^*. The average value of u^* for 1951–2014 is 5.9%.

The elasticity of $u(x)$ with respect to x is $-(1-\eta) \cdot (1-u)$. Therefore, using the measures of

[35] Since $G/Y = (G/C)/(1+G/C)$, we have $d\ln(G/Y)/d\ln(G/C) = 1-G/Y$. A first-order Taylor approximation of G/Y around $(G/C)^*$ therefore yields (17). In a feasible allocation, $u = u(x)$, where the elasticity of $u(x)$ is $-(1-\eta) \cdot (1-u)$. A first-order Taylor approximation of $u(x)$ around x^* therefore yields (18).



A. Unemployment rate

B. Estimating $1-\eta$

Figure 6: Unemployment rate in the United States, 1951–2014

Notes: Panel A: The unemployment rate u is the quarterly average of the seasonally adjusted monthly unemployment rate constructed by the BLS from the CPS. The efficient unemployment rate u^* is the low-frequency trend of u produced using a HP filter with smoothing parameter 10^5. The shaded areas represent the recessions identified by the NBER. Panel B: The actual and efficient unemployment rates, u and u^*, are described in Panel A. The actual and efficient tightnesses, x and x^*, are described in Figure 4. The plot also displays the regression line used to estimate $1-\eta$.

x and u described in Figures 4 and 6, we regress $\ln(u_{t+1}) - \ln(u_t)$ on $\ln(x_{t+1}) - \ln(x_t)$ to estimate $1-\eta$. We estimate the linear regression

$$\ln\left(\frac{u_{t+1}}{u_t}\right) = \hat{\beta}\cdot\ln\left(\frac{x_{t+1}}{x_t}\right) + \varepsilon_t \tag{20}$$

using ordinary least squares and find $\hat{\beta} = -0.51$ (standard error: 0.011). The regression analysis is illustrated in Panel B of Figure 6. Since $1-u$ is broadly constant over time with an average value $1-0.059 = 0.941$, we infer that $1-\eta = 0.51/0.941 = 0.54$ and $\eta = 0.46$. This estimate is consistent with those found in the literature.[36]

As an illustration, we consider that the unemployment rate reaches a high level of $u_0 = 9\%$ while government purchases follow the Samuelson formula. We set the efficient unemployment rate to 5.9%, the average value of u^* between 1951 and 2014. The Samuelson level of government purchases is optimal at the efficient unemployment rate $u^* = 5.9\%$ but not at $u_0 = 9\%$. Panel A of Figure 7 displays the optimal increase of the government purchases-output ratio in that situation. Since we do not have precise estimates of the elasticity of substitution between government and personal consumption and the government-purchases multiplier, we consider several values of the

[36] Petrongolo and Pissarides [2001] find that most estimates of η in the literature are between 0.4 and 0.7.

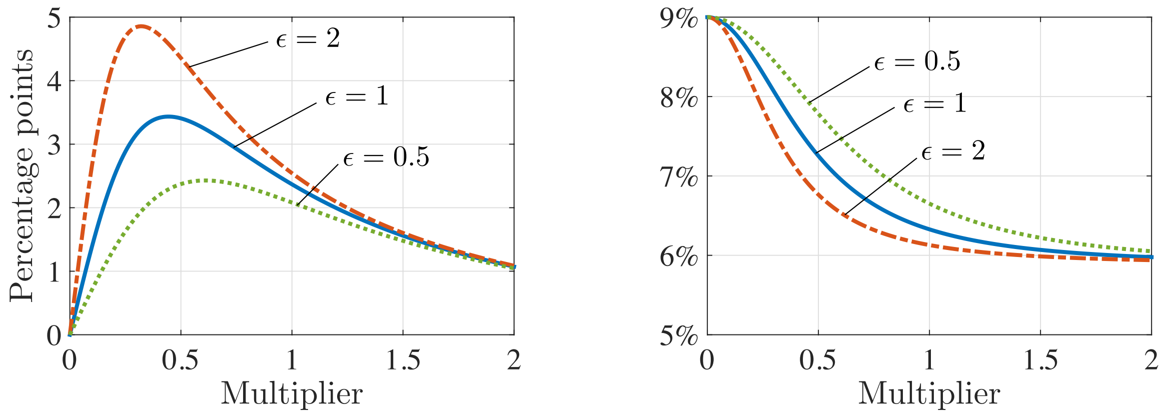
elasticity of substitution and multiplier.

The first observation is that the government purchases-output ratio should remain at the Samuelson level if the multiplier is 0. For positive multipliers, the ratio should rise above the Samuelson level. The same pattern would appear for negative multipliers, except that in that case the government purchases-output ratio should fall below the Samuelson level.

The second observation is that even with a small multiplier of 0.2, government purchases should increase significantly above the Samuelson level when the unemployment rate reaches 9%. With an elasticity of substitution of 0.5, the government purchases-output ratio should increase by 1.4 percentage points from 16.6% to 18.0%; for an elasticity of substitution of 1, it should increase by 2.5 points from 16.6% to 19.1%; and for an elasticity of substitution of 2, it should increase by 4.3 points from 16.6% to 20.9%. Thus, our theory suggests that government purchases should be markedly countercyclical even for small positive multipliers.

The third observation is that the optimal increase in government purchases rises with the elasticity of substitution between government and personal consumption. For instance, with a multiplier of 0.6, the government purchases-output ratio should increase by 2.4 percentage points with a low elasticity of 0.5, by 3.3 points with an elasticity of 1, and by 3.9 points with a high elasticity of 2. Hence, the elasticity of substitution significantly influences the optimal response of government purchases to unemployment fluctuations.

The fourth observation is that the optimal increase in government purchases does not rise monotonically with the multiplier. Instead, as we formally saw in Proposition 5, it is a hump-shaped function of the multiplier. It is true that the optimal increase in government purchases rises with the multiplier for low values of the multiplier. For instance, with an elasticity of substitution of 1, the government purchases-output ratio should increase by 2.5 percentage points with a low multiplier of 0.2, but it should increase by 3.4 points with a multiplier of 0.5. However, the optimal increase in the government purchases-output ratio diminishes for higher values of the multiplier. For instance, with the same elasticity of substitution of 1, the government purchases-output ratio should only increase by 2.4 percentage points with a multiplier of 1, by 1.6 points with a multiplier of 1.5, and by 1.1 points with a multiplier of 2. As discussed above, with large multipliers the optimal policy is to fill the unemployment gap $u_0 - u^*$, and the larger the multiplier the smaller the amount of additional government purchases required to fill the gap. Since the optimal policy sim-



A. Optimal deviation $G/Y - (G/Y)^*$

B. Unemployment rate with the optimal G/Y

Figure 7: Optimal Deviation of the Government Purchases-Output Ratio from the Samuelson Ratio When the Unemployment Rate Reaches 9%

Notes: Initially, the government purchases-output ratio is the Samuelson ratio $(G/Y)^* = 16.6\%$ and the unemployment rate is $u_0 = 9\%$. The Samuelson ratio is optimal at the efficient unemployment rate $u^* = 5.9\%$ but not at $u_0 = 9\%$. Panel A displays $G/Y - (G/Y)^*$ where G/Y is the optimal government purchases-output ratio. The values of $G/Y - (G/Y)^*$ are computed using formula (19), dY/dG between 0 and 2, ε equal to 0.5, 1, and 2, and $\eta = 0.46$. Panel B displays the equilibrium unemployment rate once the government purchases-output ratio has adjusted from $(G/Y)^*$ to its optimal level G/Y.

ply is to fill the unemployment gap for large multipliers, the value of the elasticity of substitution becomes less important when the multiplier is large.

Finally, we compute the unemployment rate that prevails in equilibrium once government purchases have been adjusted to their optimal level. Proposition 4, combined with (17) and (18), indicates that once government purchases are optimal, the equilibrium unemployment rate is approximately given by

$$u \approx u^* + \frac{1}{1 + a \cdot \varepsilon \cdot m^2} \cdot (u_0 - u^*). \tag{21}$$

Panel B of Figure 7 displays the equilibrium unemployment rate after optimal government purchases have taken place for various elasticities of substitution and multipliers.

The first observation is that for small values of the multiplier the unemployment rate barely falls below 9%, even though government purchases increase significantly. With a low multiplier of 0.2, the unemployment rate only falls to 8.7% with an elasticity of substitution of 0.5, to 8.5% with an elasticity of substitution of 1, and to 8.2% with an elasticity of substitution of 2.

The second observation is that despite the hump-shaped pattern of the increase in government purchases, the equilibrium unemployment rate decreases with the multiplier.

The third observation is that the equilibrium unemployment rate decreases with the elasticity of substitution. The reason is that the increase in government purchases rises with the elasticity of substitution. The effect of the elasticity of substitution is substantial: with a multiplier of 0.5, the unemployment rate falls to 7.8% with an elasticity of substitution of 0.5, to 7.3% with an elasticity of substitution of 1, and to 6.8% with an elasticity of substitution of 2.

The last observation is that, with a multiplier above 1.5, the stabilization of the unemployment rate is almost perfect. Irrespective of the elasticity of substitution, the equilibrium unemployment rate is very close to the efficient unemployment rate of 5.9%. Of course, we know from (7) that the stabilization cannot be perfect: since $G/Y > (G/Y)^*$, the optimality condition (7) imposes that $u > u^*$. But with large multipliers, government purchases are so potent that the gap between u and u^* is negligible: with a multiplier of 2, the unemployment rate falls to 6.05%, 5.98%, and 5.94% with elasticities of substitution of 0.5, 1, and 2.

5. Numerical Simulations with a Countercyclical Multiplier

In this section, we describe, calibrate, and simulate a specific model of unemployment and government purchases. We use the model of Michaillat and Saez [2015*b*] that generates an aggregate demand by having real wealth enter the utility function. This model naturally generates a positive government multiplier. The size of the multiplier depends on how flexible the interest rate is to economic conditions so that it can easily be calibrated to match a desired magnitude. The specificity of this model is that it generates a government-purchases multiplier that is sharply countercyclical.[37] We use the numerical simulations to investigate the accuracy of the approximate sufficient-statistics formula derived in Section 3. In the calibrated model, we find that our formulas are very accurate, even for large fluctuations in tightness.

[37] The countercyclicality of the multiplier arises naturally in a matching market, following the mechanism described by Michaillat [2014].

5.1. A Specific Model

Overview. The specific model adds an aggregate demand and an interest-rate schedule to the generic model of Section 2. This aggregate demand and interest-rate schedule determine a specific equilibrium function $x(g)$. Following Michaillat and Saez [2015*b*], real wealth enters households' utility. The households' trade-off between consumption of services and holding of wealth generates an aggregate demand. Aggregate demand shocks are parameterized by the marginal utility of wealth; with higher marginal utility of wealth, households desire to save more and consume less, which depresses aggregate demand. The amplitude of the fluctuations in tightness and unemployment are governed by the rigidity of the interest rate. The rigidity of the interest rate also governs the size of the multiplier.

Households. The representative household spends part of its labor income on services and saves part of it as bonds. Government purchases are financed by a lump-sum tax. The law of motion of the household's assets is

$$\dot{b}(t) = Y(x(t)) - (1+\tau(x(t)))\cdot c(t) + r(t)\cdot b(t) - T(t). \tag{22}$$

Here, $b(t)$ are real bond holdings, $c(t)$ is net personal consumption, $(1+\tau(x(t)))\cdot c(t)$ is gross personal consumption, $Y(x(t))$ is labor income, $r(t)$ is the real interest rate, and $T(t)$ is the lump-sum tax paid to the government.

The representative household derives utility from consuming the $c(t)$ services that it purchases, consuming the $g(t)$ services purchased by the government, and holding $b(t)$ units of real wealth. Its instantaneous utility function is separable: $\mathscr{U}(c(t),g(t))+\mathscr{V}(b(t))$. The function $\mathscr{V}$ is strictly increasing and concave. Utility for wealth is a simple way to introduce an aggregate demand in a real economy without money (See Michaillat and Saez [2015*b*] for more details).[38]

The utility function of a household at time 0 is the discounted sum of instantaneous utilities $\int_0^{+\infty} e^{-\delta\cdot t}\cdot[\mathscr{U}(c(t),g(t))+\mathscr{V}(b(t))]\,dt$, where $\delta>0$ is the subjective discount rate. Given initial real wealth $b(0)=0$ and the paths for market tightness, government purchases, real interest rate,

[38] Another possibility would be to introduce utility for money as in Michaillat and Saez [2015*a*], or utility for a nonproduced good as in Hart [1982]. The formulation with wealth is the simplest: it does not require the introduction of an additional good, and it captures the idea that shifts in thriftiness create aggregate demand shocks.

and tax $[x(t), g(t), r(t), T(t)]_{t=0}^{+\infty}$, the household chooses paths for consumption and real wealth $[c(t), b(t)]_{t=0}^{+\infty}$ to maximize its utility function subject to (22).

Market for Bonds. Households can issue or buy riskless real bonds. Household hold bonds to smooth future consumption and because they derive utility from real wealth, which can only be stored in bonds. At time t, a household holds $b(t)$ bonds, and the rate of return on bonds is the real interest rate $r(t)$.

Bonds are traded on a perfectly competitive market. Bonds are in zero net supply because the government funds government purchases using lump-sum taxes. In equilibrium, the bond market clears and $b(t) = 0$. Accordingly, the aggregate real wealth in the economy is zero. This means that wealth is irrelevant for social welfare maximization; therefore, our formulas for optimal government purchases apply to this model.

In the economy there are two goods—labor services and bonds—and hence one relative price. The price of bonds relative to services is determined by the real interest rate. On a Walrasian market, the real interest rate would be determined such that supply equals demand on the market for labor services. On a matching market, things are different: we specify a price mechanism for the real interest rate and the market tightness adjusts such that supply equals demand on the market for labor services.[39] Here we specify a general price mechanism: $r = r(x, g)$.

Equilibrium. The equilibrium is a dynamical system that we describe in Appendix E. The system is a source, with no state variables, so that it converges immediately to its steady state. We therefore only describe the steady-state equilibrium. The equilibrium is composed of five variables: g, c, y, r, and x. Net government consumption g is chosen by the government. The real interest rate is $r = r(x, g)$. Net output is $y = y(x)$, where $y(x)$ is given by (3). Solving the household's problem, we find that consumption satisfies

$$\frac{\partial \mathcal{U}}{\partial c}(c, g) = \frac{(1 + \tau(x)) \cdot \mathcal{V}'(0)}{\delta - r}. \tag{23}$$

[39] Michaillat and Saez [2015*a*] explain the similarities and differences between a Walrasian and a matching market.

This equation is the standard Euler equation modified by the utility of wealth and evaluated in steady state.[40] Let $c(x,g,r)$ be the amount of net personal consumption implicitly defined by (23). The constraint that $c+g=y$ can be written as $c(x,g,r(x,g))+g=y(x)$, which determines equilibrium tightness $x(g)$.

The joint behavior of inflation and monetary policy is taken as given in our analysis. This joint behavior is not explicitly described in the model, but it is parsimoniously summarized by the interest-rate schedule $r(g)$. If $r(x,g)$ ensures that $x(g)=x^*$ for all g, the interest rate is always efficient. This happens when monetary policy perfectly stabilizes the economy. If $r(x,g)=r_0$ for all g, the interest rate is totally rigid. This happens when the central bank is unable to affect the real interest rate, for instance if inflation is fixed and the nominal interest rate at the zero lower bound. More generally, the interest rate follows a Taylor rule $r=r(\pi)$, inflation is given by a Phillips curve $\pi=\pi(x)$, and shocks lead to fluctuations in both tightness and inflation.[41] Nevertheless, it is still the case that in general equilibrium tightness is a function $x(g)$ and the real interest rate is a function $r(x,g)$. Hence, our formulas carry over to the general case.

Since we take the real interest rate and therefore monetary policy as given when we determine the optimal level of government purchases, the relevant multiplier in our formula should not control for monetary policy. The multiplier should take into account the mechanical response of monetary policy to a change in government purchases.

5.2. *Calibration*

We calibrate the model to US data for 1951–2014. The calibration ensures that the two sufficient statistics at the heart of our formulas—the elasticity of substitution between government and personal consumption, ε, and the government-purchases multiplier, dY/dG— match the empirical evidence. As discussed in Section 4, reasonable estimates of the elasticity of substitution and multiplier are $\varepsilon=1$ and $dY/dG=0.6$.

We specify the utility function as follows:

$$\mathscr{U}(c,g)=\left[(1-\gamma)\cdot c^{\frac{\varepsilon-1}{\varepsilon}}+\gamma\cdot g^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{24}$$

[40] We can accommodate any interest schedule $r(x,g)$ such that $r(x,g)<\delta$, including a negative interest rate.
[41] See Michaillat and Saez [2015*b*] for details.

where $\gamma \in (0,1)$ indicates the value of government consumption relative to personal consumption, and $\varepsilon > 0$ is the elasticity of substitution. We set $\varepsilon = 1$, obtaining a Cobb-Douglas utility function $\mathscr{U}(c,g) = c^{\gamma} \cdot g^{1-\gamma}$.

We need to specify an interest rate schedule $r(x,g)$. A common assumption in the matching literature is that prices are efficient—they maintain the economy at the efficient unemployment rate u^*.[42] The efficient real interest rate is the interest rate r^* that ensures that tightness x is efficient at x^*. Hence, using equation (23), we have:

$$r^* = \delta - (1+\tau(x^*)) \cdot \frac{\mathscr{V}'(0)}{\frac{\partial \mathscr{U}}{\partial c}(y^* - g, g)}. \tag{25}$$

If the interest rate is continuously at r^*, the tightness is always at x^*, net output is always at its efficient level y^* and government purchases have no effect on tightness or net output, and the Samuelson formula holds.

In practice, however, the economy experiences business cycles with fluctuations in tightness. To describe these variations, we assume that the interest rate is not as flexible as the efficient interest rate. We consider an interest-rate schedule of the form

$$r(g) = \delta - \mu \cdot \frac{\mathscr{V}'(0)^{1-\alpha}}{\frac{\partial \mathscr{U}}{\partial c}(y^* - g, g)^{1-\beta}}. \tag{26}$$

The parameter $\mu > 0$ governs the level of the real interest rate. The parameter $\alpha \in [0,1]$ measures the rigidity of the real interest rate with respect to aggregate demand shocks: if $\alpha = 1$, the real interest rate does not respond at all to aggregate demand shocks; if $\alpha = 0$, the real interest rate responds as much to aggregate demand shocks as the efficient real interest rate. When $\alpha = 0$ aggregate demand shocks are absorbed by the real interest rate, but when $\alpha > 0$ aggregate demand shocks generate fluctuations in tightness. The parameter $\beta \in [0,1]$ measures the rigidity of the real interest rate with respect to changes in the marginal utility of personal consumption: if $\beta = 1$, the real interest rate does not respond at all to shocks to the marginal utility of personal consumption; if $\beta = 0$, the real interest rate responds as much to shocks to the marginal utility of personal consumption as the efficient real interest rate.

[42] Another typical assumption is that prices are determined by bargaining. Bargained prices usually have similar properties to efficient prices [Michaillat and Saez, 2015*a*].

Appendix F shows that when the unemployment rate is efficient and government purchases are optimal, the multiplier simplifies to

$$dY/dG = \frac{\beta}{(G/Y)^* + \varepsilon \cdot [1 - (G/Y)^*]},$$

where β is the coefficient of the interest-rate schedule (26). When $\beta = 0$, government purchases shocks are absorbed by the real interest rate, the aggregate demand does not depend on government purchases, and the multiplier is zero. When $\beta > 0$, however, the multiplier is positive. With $\varepsilon = 1$, we set $\beta = 0.6$ to match $dY/dG = 0.6$ in the average state.

The rest of the calibration is standard and relegated to Appendix G. Since there is considerable uncertainty about the values of the elasticity of substitution and multiplier, Appendix H presents additional simulations targeting $\varepsilon = 0.5$, $\varepsilon = 2$, $dY/dG = 0.2$, and $dY/dG = 1$.

5.3. Simulations

We simulate our model under aggregate demand shocks. Appendix E shows that the economy jumps from one steady-state equilibrium to another in response to unexpected permanent shocks. Hence, we represent the different stages of the business cycle as a succession of steady states. We simulate business cycles generated by aggregate demand shocks by computing a collection of steady states parameterized by different values for the marginal utility of wealth, $\mathscr{V}'(0)$. In each case, we perform two simulations: one in which the government purchases-output ratio G/Y remains constant at 16.6%, its average value for 1951–2014, and one in which G/Y is at its optimal level, given by (6).

Figure 8 displays the results of the simulations. Each steady state is indexed by a marginal utility of wealth $\mathscr{V}'(0) \in [0.97, 1.03]$. On the one hand, the steady states with low $\mathscr{V}'(0)$ represent booms: they have a relatively low interest rate and low unemployment. On the other hand, the steady states with high $\mathscr{V}'(0)$ represent slumps: they have a relatively high interest rate and high unemployment. Unemployment rises from 4.0% to 9.9%, and output falls accordingly, when $\mathscr{V}'(0)$ increases from 0.97 to 1.03 and G/Y remains constant.

On average the multiplier is 0.6, matching empirical evidence. The multiplier is sharply countercyclical, increasing from 0.3 to 1.3 when the unemployment rate increases from 4.0% to 9.9%.

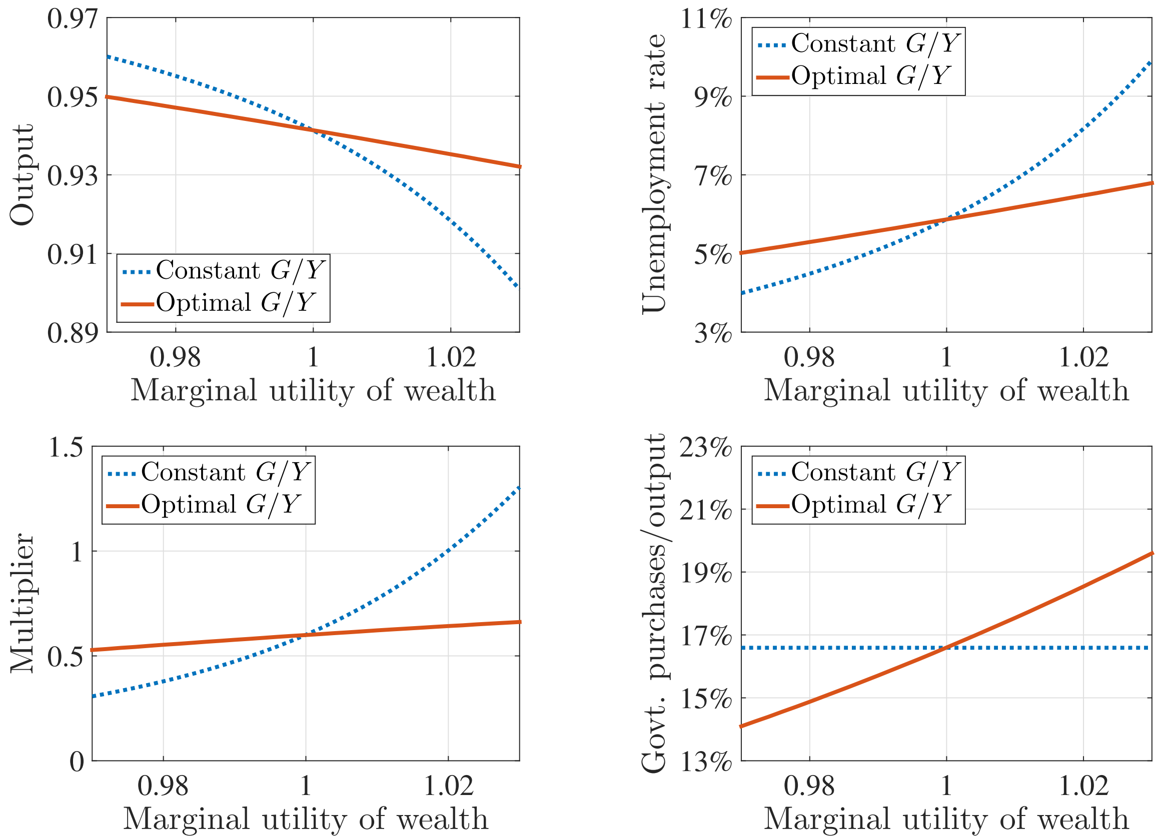


Figure 8: Simulations of Specific Model Under Aggregate Demand Shocks

This sharp increase of the multiplier when the unemployment rate is high and output is low is consistent with the empirical evidence provided by Auerbach and Gorodnichenko [2012, 2013]. The mechanism behind the countercyclicality of the multiplier is described by Michaillat [2014]. The size of the multiplier depends on the extent of crowding-out of personal consumption by government consumption; the crowding-out is determined by the amplitude of the increase in tightness. When unemployment is high, the government needs to advertise few vacancies to purchase additional services because the matching process is congested by sellers of services. Moreover, the idle capacity is so large that the vacancies posted and services purchased by the government have little influence on tightness. Consequently, when unemployment is high, the increase in tightness is small and crowding-out is weak after an increase in government purchases.

The model is calibrated so that the unemployment rate is efficient when $\mathscr{V}'(0) = 1$. Hence, the unemployment rate is inefficiently high when $\mathscr{V}'(0) > 1$ and inefficiently low when $\mathscr{V}'(0) < 1$. Since the multiplier is positive, G/Y should be more generous than the Samuelson ratio when

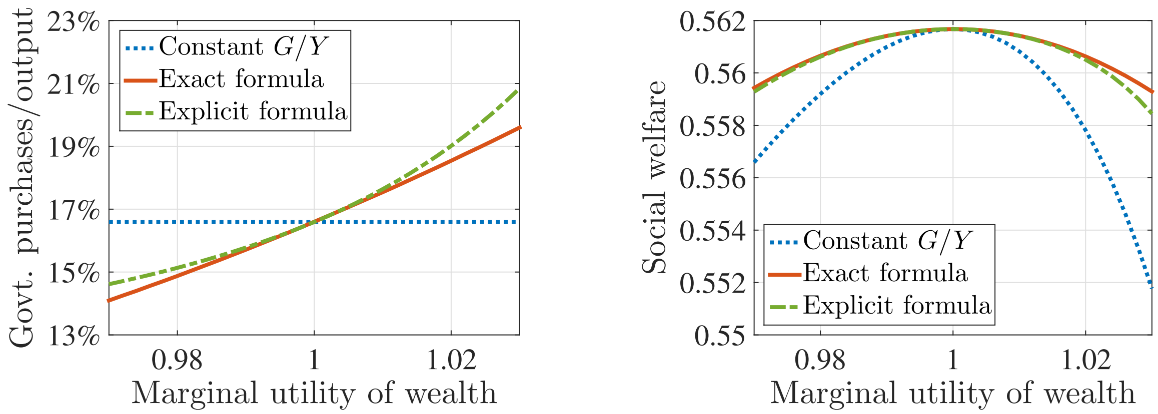


Figure 9: Accuracy of the Explicit Formula

$\mathscr{V}'(0) > 1$ and less generous when $\mathscr{V}'(0) < 1$. Indeed, the optimal G/Y is markedly countercyclical, increasing from 14.1% when $\mathscr{V}'(0) = 0.97$ to 19.6% when $\mathscr{V}'(0) = 1.03$.

Of course, the unemployment rate responds to the adjustment of G/Y from its original level of 16.6% to its optimal level. When $\mathscr{V}'(0) > 1$, the optimal G/Y is higher than 16.6% so the unemployment rate is below its original level: at $\mathscr{V}'(0) = 1.03$ the unemployment rate falls by 3.1 percentage points from 9.9% to 6.8%. When $\mathscr{V}'(0) < 1$, the optimal G/Y is below 16.6% so the unemployment rate is above its original level: at $\mathscr{V}'(0) = 0.97$ the unemployment rate increases by 1 percentage point from 4.0% to 5.0%.

Although the multiplier varies, which could be a source of inaccuracy, the approximate explicit formula (12) is quite accurate. Figure 9 compares the ratios G/Y obtained with the exact formula (6) and with the explicit formula (12). As expected, at $\mathscr{V}'(0) = 1$, the two formulas gives the same G/Y. The approximation is less precise when the initial unemployment rate is further away from its efficient level, but it remains satisfactory: at $\mathscr{V}'(0) = 0.97$, the exact formula gives $G/Y = 14.1\%$ while the explicit formula gives $G/Y = 14.6\%$; at $\mathscr{V}'(0) = 1.03$, the exact formula gives $G/Y = 19.6\%$ while the approximate explicit formula gives $G/Y = 20.8\%$. Despite these discrepancies, the social welfare values resulting from the two formulas are nearly identical.

6. Heterogeneity and Endogenous Labor Supply

The formulas of Section 3 are obtained with a representative agent supplying an exogenous productive capacity of 1. This section shows that the formulas remain valid with heterogeneous households and endogenous labor supply decisions. This finding mirrors the result that the Samuelson formula is valid with heterogeneity and endogenous labor supply under standard separability assumptions [Kaplow, 1996; Samuelson, 1954].[43]

To study endogenous labor supply, we assume that households supply a productive capacity k. The model of Section 2 is a special case where k is exogenous and equal to 1. The matching market works as with $k = 1$ except that traded quantities are scaled by k. Hence, unemployment rate $u(x)$ and matching wedge $\tau(x)$ are exactly the same, but tightness is given by $x = v/(k - Y)$, gross output is $Y = (1 - u(x)) \cdot k$, and net output is given by $y = \phi(x) \cdot k$ where

$$\phi(x) \equiv \frac{1 - u(x)}{1 + \tau(x)}.$$

Following standard practice in public economics, we apply the benefit principle whereby a change in government purchases is financed by a change in individual taxes designed to leave all individual utilities unchanged. A Pareto improvement is possible if the reform generates a government budget surplus or deficit. Hence, the formula for optimal government purchases obtains when the effect of the reform leaves the government budget balanced.

6.1. Heterogeneity

Households have heterogeneous preferences $\mathscr{U}_i(c_i, g)$ and heterogeneous productive capacity k_i indexed by i.[44] The marginal rate of substitution between government and personal consumption for household i is $MRS_i \equiv (\partial \mathscr{U}_i / \partial g) / (\partial \mathscr{U}_i / \partial c)$. The productive capacity k_i is exogenous. Household i is subject to a lump-sum tax T_i. Paralleling the analysis of Samuelson [1954], we find that (5) remains valid with heterogeneity once we replace the appropriate statistics by their averages:

[43] Kreiner and Verdelin [2012] explain the connection between the analysis of Kaplow [1996] and the earlier literature on public-good provision in the presence of distortionary taxation.

[44] Formally, i is distributed over a space $\mathscr{I}$ on which ν is a measure. There is a measure 1 of households so $\int_{i \in \mathscr{I}} d\nu(i) = 1$. To economize on notation, if z_i is the value of some variable z for household i, we denote the average $\int_{i \in \mathscr{I}} z_i d\nu(i)$ by $\int_i z_i$.

Proposition 6. *With heterogeneous households, optimal government purchases satisfy*

$$1 = \int_i MRS_i + \phi'(x) \cdot x'(g) \cdot \int_i k_i. \tag{27}$$

Proof. Starting from an allocation $[\{c_i\}, g, y, x]$, we implement a small change dg. We follow the benefit principle: the change dg is funded by a change each individual tax by dT_i to keep each household indifferent. As a result of these changes, personal consumption levels change by dc_i. Since household i is indifferent, we have $\mathscr{U}_i(c_i + dc_i, g + dg) = \mathscr{U}_i(c_i, g)$, which implies that $(\partial \mathscr{U}_i / \partial c) \cdot dc_i + (\partial \mathscr{U}_i / \partial g) \cdot dg = 0$ and hence $dc_i = -MRS_i \cdot dg$.

The budget constraint of household i is $c_i = y_i - T_i$, where $y_i = \phi(x) \cdot k_i$ is the net output sold by household i. By integrating over household budgets, we find that the effect of the reform on tax revenue is $\int_i dT_i = \int_i dy_i - \int_i dc_i$. Since $y_i = \phi(x) \cdot k_i$ and $x = x(g)$, the effect of the reform on net output is $dy_i = \phi'(x) \cdot x'(g) \cdot dg \cdot k_i$. Accordingly, the effect of the reform on the government budget balance R (defined as tax revenue minus government spending) is $dR = \int_i dT_i - dg = \int_i dy_i - \int_i dc_i - dg = [\phi'(x) \cdot x'(g) \cdot \int_i k_i + \int_i MRS_i - 1] \cdot dg$. Hence, if (27) does not hold, the reform creates a first-order government budget surplus or deficit. A surplus could be redistributed back to households, thus creating a Pareto improvement. With a deficit, the opposite of the proposed reform would create a surplus and hence make a Pareto improvement possible. To conclude, if (27) does not hold, government purchases are not at their optimal level. □

Despite the heterogeneity, it is possible to to re-express formula (27) in terms of estimable sufficient statistics. The correction term $\phi'(x) \cdot x'(g) \cdot \int_i k_i$ only involves macro variables so it can be expressed in terms of estimable statistics exactly as in Section 3. The Samuelson term $1 - \int_i MRS_i$ can be expressed as a function of an elasticity ε that is the properly weighted harmonic mean of the households' elasticities of substitution ε_i.

6.2. *Endogenous Labor Supply*

Representative Household. The representative household supplies a productive capacity k at some utility cost. The household's utility function becomes $\mathscr{U}(c, g, k)$; the function $\mathscr{U}$ decreases with k.

The government imposes a tax $T(k)$ on productive capacity.[45] The household's budget constraint imposes that $(1+\tau(x))\cdot c = (1-u(x))\cdot(k-T(k))$, implying that $c = \phi(x)\cdot(k-T(k))$; therefore, the household chooses k to maximize $\mathscr{U}(\phi(x)\cdot(k-T(k)),g,k)$. In this case, we find that formula (5) remains valid:

Proposition 7. *With endogenous labor supply and an arbitrary tax $T(k)$, optimal government purchases satisfy*

$$1 = MRS_{gc} + \phi'(x)\cdot x'(g)\cdot k. \tag{28}$$

Proof. Starting from an allocation $[c,g,y,x]$, we implement a small change dg. We follow the benefit principle: the change dg is funded by a change in tax $dT(k)$ designed to keep the household's utility constant for any choice of k. For all k, $dT(k)$ satisfies

$$\mathscr{U}(\phi(x)\cdot(k-T(k)),g,k) = \mathscr{U}(\phi(x+dx)\cdot(k-T(k)-dT(k)),g+dg,k).$$

Because of the change $dT(k)$, the household does not change his choice of k after the change dg so that $dk=0$. Taking a first-order expansion around the initial allocation, we obtain

$$\frac{\partial \mathscr{U}}{\partial c}\cdot\left[(k-T(k))\cdot\phi'(x)\cdot dx - \phi(x)\cdot dT(k)\right] + \frac{\partial \mathscr{U}}{\partial g}\cdot dg = 0.$$

Dividing by $\partial\mathscr{U}/\partial c$ and re-arranging yields

$$T(k)\cdot\phi'(x)\cdot dx + \phi(x)\cdot dT(k) = MRS_{gc}\cdot dg + k\cdot\phi'(x)\cdot dx.$$

We use this equation to obtain the effect of the reform on the government budget balance $R = \phi(x)\cdot T(k) - g$:

$$dR = T(k)\cdot\phi'(x)\cdot dx + \phi(x)\cdot dT(k) - dg = (MRS_{gc}-1)\cdot dg + k\cdot\phi'(x)\cdot dx.$$

(We used again $dk=0$.) As explained in the proof of Proposition 6, $dR=0$ at the optimum; thus,

[45] It would be equivalent to base the income tax on output Y instead of capacity k. Basing the tax on capacity simplifies notations and derivations.

equation (28) holds at the optimum. □

Proposition 7 establishes that the generalized Samuelson formula (5) also applies when labor supply is endogenous. The supply responses are orthogonal to the proper level of government purchases in Samuelson's theory as in our generalization. We could express the formula in terms of estimable statistics as we did in Section 3. The relevant multiplier dY/dG would measure the response of Y to a change in G associated with a change in the tax system that leaves the household's utility and labor supply choices constant. This multiplier might not be the same as a multiplier estimated when a change in G is not associated with compensating tax adjustments.

The modern public economics literature, following Kaplow [1996], has showed that optimal public-good spending is conceptually orthogonal to labor supply responses. This result seems to contradict the received wisdom that if taxes are distortionary, raising revenue for public spending creates substantial efficiency losses that make public spending less desirable. How can we reconcile this contradiction? First, with a representative agent, it is naturally preferable to raise revenue for public spending with a lump-sum tax T independent of k to avoid efficiency costs.[46] Second, if for some unspecified reason, the government uses a distortionary tax $T(k)$, then expanding public good spending by increasing distortionary taxation involves deadweight loss proportional to the size of the labor supply responses. However, this mixes an increase in public spending with an inefficient increase in distortionary taxes; as a result, increasing the public good seems unappealing because it is bundled with an inefficient tax increase. Conversely, if the public good expansion were bundled with a tax increase that reduced tax distortion (an increase in the lump-sum tax with a reduction of the marginal tax rate), then increasing the public good beyond the generalized Samuelson formula would be desirable. If the expansion in public good is financed by increasing taxes without creating extra distortions (as in our proof above), then supply side responses are irrelevant. This benefits principle, proposed by Kaplow [1996], follows the spirit of the original derivation by Samuelson [1954].

Macroeconomists often use simple linear taxes with a lump-sum rebate so that $T(k) = \tau \cdot k - E$. In this case, the generalized Samuelson formula carries over if the tax reform required to offset utility changes due to dg leaves the tax system linear with an adjustment $d\tau, dE$. This happens

[46] With a lump-sum tax, the household utility is $\mathscr{U}(\phi(x) \cdot k - g, g, k)$. Because k is chosen to maximize household utility, by the envelope theorem, the generalized Samuelson formula (5) can be obtained immediately.

when the utility function for (c,g) is Cobb-Douglas so that $\mathscr{U}(c,g,k) = \mathscr{W}(c^{\alpha} \cdot g^{1-\alpha}, k)$.[47]

Heterogeneous Households. The best theoretical justification for using a distortionary tax system is when there is heterogeneity across households. Following traditional optimal income tax models, if the government values redistribution, then a uniform lump-sum tax is not an attractive option for raising revenue for public spending.[48] We therefore assume that households are heterogeneous with household i having utility $\mathscr{U}_i(c,g,k) = \mathscr{W}_i(w(c,g),k)$. The assumption that the subutility of consumption $w(c,g)$ is separable from labor supply k and homogeneous across all households is important; this is the classical Atkinson and Stiglitz [1976] separability assumption needed to obtain robustness of the Samuelson formula [Boadway and Keen, 1993; Gauthier and Laroque, 2009; Kaplow, 1996; Kreiner and Verdelin, 2012].[49] A tax system $k \mapsto T(k)$ and a level of public spending g will map into an allocation where household i chooses k_i to maximize $k \mapsto \mathscr{W}_i(w(\phi(x) \cdot (k - T(k)), g), k)$, taking x and g as given.

PROPOSITION 8. *With endogenous labor supply, heterogeneous households with utilities $\mathscr{U}_i(c,g,k) = \mathscr{W}_i(w(c,g),k)$, and an arbitrary tax $T(k)$, optimal government purchases g satisfy*

$$1 = \int_i MRS_i + \phi'(x) \cdot x'(g) \cdot \int_i k_i. \tag{29}$$

Proof. We implement a small change dg and follow the benefit principle: the change dg is funded by a tax change $dT(k)$ that leaves household's subutility $w(c,g)$ unchanged for any choice of k. For all k, $dT(k)$ satisfies

$$w(\phi(x) \cdot (k - T(k)), g) = w(\phi(x+dx) \cdot (k - T(k) - dT(k)), g + dg). \tag{30}$$

The reform $dT(k)$ does not need to be tailored to each household i because we assume that $w(c,g)$

[47]Leaving utility and labor supply constant requires a reform $d\tau, dE$ such that $\alpha \cdot dc/c + (1-\alpha) \cdot dg/g = 0$ for all k. As $c = \phi(x) \cdot [k \cdot (1-\tau) + E]$ and k stays constant, we have $dc/c = \phi'(x) \cdot dx/\phi(x) + [-k \cdot d\tau + dE] / [k \cdot (1-\tau) + E]$. Hence $\alpha \cdot dc/c + (1-\alpha) \cdot dg/g = \alpha \cdot [-k \cdot d\tau + dE] / [k \cdot (1-\tau) + E] + \alpha \cdot \phi'(x) \cdot dx/\phi(x) + (1-\alpha) \cdot dg/g$. By choosing $dE, d\tau$ so that $dE/E = -d\tau/(1-\tau) = -\phi'(x) \cdot dx/\phi(x) - [(1-\alpha)/\alpha] \cdot dg/g$, we ensure that $\alpha \cdot dc/c + (1-\alpha) \cdot dg/g = 0$ for all k so that the utility and choice of k remain unchanged.

[48]Even if the government does not value redistribution, if incomes are heterogeneous, lower incomes might not be able to pay the uniform lump-sum tax. In that case, taxes have to depend on earnings.

[49]The result carries over if utilities take the slightly more general form $\mathscr{U}_i(c,g,k) = \mathscr{W}_i(w(c,g,k),k)$ with $w(c,g,k)$ homogeneous across households [Kreiner and Verdelin, 2012]. In words, any heterogeneity in earnings capacity must be separable from the (c,g) choice.

is homogeneous across households. As a result, for each household, the functions $k \mapsto \mathscr{W}_i(w(\phi(x) \cdot (k - T(k)), g), k)$ and $k \mapsto \mathscr{W}_i(w(\phi(x+dx) \cdot (k - T(k) - dT(k)), g + dg), k)$ are identical. This implies that the choices k_i and resulting utilities $\mathscr{W}_i$ are not affected by the reform.

Taking a first-order expansion of equation (30) around the initial allocation (c_i, k_i, g) for household i and using $dk_i = 0$, we obtain

$$\frac{\partial w}{\partial c}(c_i, g) \cdot [(k_i - T(k_i)) \cdot \phi'(x) \cdot dx - \phi(x) \cdot dT(k_i)] + \frac{\partial w}{\partial g}(c_i, g) \cdot dg = 0.$$

Dividing by $\partial w / \partial c$ and re-arranging yields

$$T(k_i) \cdot \phi'(x) \cdot dx + \phi(x) \cdot dT(k_i) = MRS_i \cdot dg + k_i \cdot \phi'(x) \cdot dx,$$

where $MRS_i = (\partial \mathscr{U}_i / \partial g) / (\partial \mathscr{U}_i / \partial c) = (\partial w / \partial g) / (\partial w / \partial c)$ is the marginal rate of substitution between g and c evaluated at (c_i, g).[50] We use this equation to obtain the effect of the reform on the government budget balance $R = \phi(x) \cdot \int_i T(k_i) - g$:

$$dR = \int_i [T(k_i) \cdot \phi'(x) \cdot dx + \phi(x) \cdot dT(k_i)] - dg = \left(\int_i MRS_i - 1 \right) \cdot dg + \phi'(x) \cdot dx \cdot \int_i k_i.$$

(We used again $dk_i = 0$ for all i.) At the optimum, $dR = 0$ and thus (29) holds. □

Proposition 8 shows that the generalized Samuelson formula (5) remains valid with both heterogeneity and labor supply responses. If the separability and homogeneity assumption does not hold, then the Samuelson formula needs to be modified. For instance, Kreiner and Verdelin [2012] provide formulas in the case of a unidimensional "ability" heterogeneity across households. Even in that case, it is not the strength of labor supply responses that governs the departure from the Samuelson formula, but the extent to which taste for the public good correlates with ability conditional on earnings. We conjecture that the Kreiner and Verdelin [2012] method and formula would apply to our setting.

[50] Given the separability and homogeneity assumptions, all households have the same function $(c, g) \mapsto MRS_{gc}$. Because the c_i differ across households, however, MRS_i differs across households: MRS_i is higher for households with high earnings capacity because they have a higher c_i.

7. Conclusion

This paper provides several general insights on optimal government purchases to stabilize business cycles. Some of these insights are unsurprising. First, even in a macroeconomic model with unemployment, the Samuelson [1954] formula holds as long as the unemployment rate is efficient. Second, the government-purchases multiplier, one of the most commonly estimated statistic in macroeconomics, does matter for the optimal level of government purchases.

Other insights are more unexpected. First, the cutoff value of the multiplier that justifies an increase in government purchases in slumps is 0, and not 1 as in macroeconomic models in which government purchases are wasteful. With any positive multiplier, it is optimal to increase government purchases above the Samuelson level when the unemployment rate is inefficiently high, even though government purchases crowd out personal consumption. With statistics calibrated to the US economy, even for small multipliers, optimal government purchases deviate by significant amounts from the Samuelson level when unemployment deviates from its efficient level.

Second, for positive multipliers, the relation between the size of the multiplier and the deviation of the optimal government purchases-output ratio from the Samuelson ratio is not increasing but hump-shaped, with a peak for a multiplier of about 0.5. The optimal ratio increases less for multipliers above 0.5 because when multipliers are large, a higher multiplier means that fewer government purchases are required to fill the unemployment gap. The optimal ratio increases less for multipliers below 0.5 because when multipliers are small, a smaller multiplier means that government purchases crowd out personal consumption more and are therefore less desirable.

Third, there is another statistic that has been neglected but is as important as the multiplier to determine the optimal level of government purchases: the elasticity of substitution between government and personal consumption. On the one hand, if the elasticity of substitution is zero, government purchases should remain at the Samuelson level. On the other hand, if the elasticity of substitution is infinite, government purchases should perfectly stabilize unemployment. For positive, finite elasticities, the deviation of the optimal government purchases-output ratio from the Samuelson ratio when unemployment deviates from its efficient level is larger for larger elasticities.

Fourth, a negative multiplier does not mean that government purchases should not respond to unemployment fluctuations; it means that government purchases should be below the Samuelson level when unemployment is inefficiently high and above the Samuelson level when unemployment is inefficiently low. It is only for a multiplier of 0 that government purchases should follow the Samuelson formula.

Our analysis suggests that as soon as the government-purchases multiplier and the elasticity of substitution between government and personal consumption are positive, balanced-budget government purchases are a key tool for macroeconomic stabilization. Whenever the unemployment rate is inefficient, government purchases should be adjusted, sometimes by a sizable amount. Government purchases are therefore particularly useful for macroeconomic stabilization whenever monetary policy is not available. This situation could arise because of the zero lower bound on nominal interest rates. It could also arise for member countries of a monetary union, which face a fixed monetary policy but can tailor government purchases and taxes to local conditions. For instance, a country in the eurozone or a state in the United States could respond to local shocks with government purchases using our formulas. Budget balanced government purchases are particularly relevant for the US states which cannot run deficits or European Union countries that face severe limits on their budget deficits.

In practice, adjusting government purchases may take time. To reduce the time lags between decision and implementation of government purchases, the government should automatize adjustments of government purchases, much in the same way as extensions of unemployment insurance are automatic in the United States. A possibility would be to keep a long list of useful government purchases (either services or investment projects valued by society) and go up or down the list as the amount of government purchases is adjusted over the business cycle.

The methodology developed in this paper could help bridge the gap between the analysis of optimal taxation, transfers, social insurance, and public-good provision in public economics and the analysis of stabilization policies in macroeconomics. This agenda is related to the new dynamic public finance literature, which analyzes optimal policy in macroeconomic models [Golosov, Tsyvinski and Werning, 2006; Golosov and Tsyvinski, 2015; Golosov, Troshkin and Tsyvinski, 2011; Kocherlakota, 2010], and to the work of Farhi and Werning [2013], who propose a macroeconomic framework to study optimal macroprudential policies in financial markets.

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Appendix A: Validation of the Approximations

In this appendix, we use US data for 1951–2014 to validate the three approximations made in the analysis.

Absence of Transitional Dynamics

In the analysis, we abstract from transitional dynamics for the unemployment rate and output. We argue here that these transitional dynamics are quantitatively negligible. Given that output and unemployment rate are linearly related in the model ($Y = 1 - u$), it suffices to make the case for one of the two variables. Using the same labor market data as in Section 4, we focus on the unemployment rate.

We begin by constructing a time series for the selling rate f_t. We measure one unit of service by one job. The selling rate therefore is a selling rate. We assume that unemployed workers find a job according to a Poisson process with arrival rate f_t. Under this assumption, the monthly selling rate satisfies $f_t = -\ln(1 - F_t)$, where F_t is the monthly job-finding probability. We construct a time series for F_t following the method developed by Shimer [2012]. We use the relationship

$$F_t = 1 - \frac{u_{t+1} - u^s_{t+1}}{u_t},$$

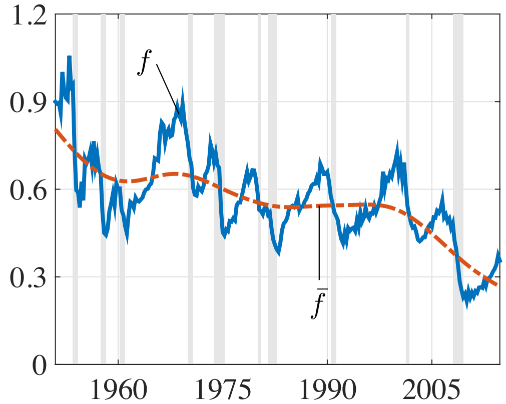
where u_t is the number of unemployed persons at time t and u^s_t is the number of short-term unemployed persons at time t. We measure u_t and u^s_t in the data constructed by the BLS from the CPS. The number of short-term unemployed persons is the number of unemployed persons with zero to four weeks duration, adjusted as in Shimer [2012] for the 1994–2014 period. Panel A of Figure A1 displays the monthly selling rate and its trend. The selling rate averages 56% between 1951 and 2014.

Next, we construct the separation rate following the method developed by Shimer [2012]. The separation rate s_t is implicitly defined by

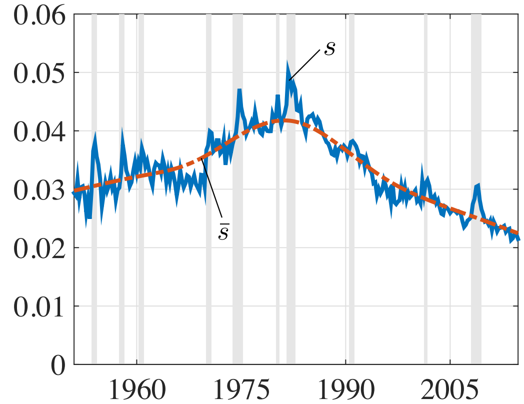
$$u_{t+1} = \left(1 - e^{-f_t - s_t}\right) \cdot \frac{s_t}{f_t + s_t} \cdot h_t + e^{-f_t - s_t} \cdot u_t,$$

where h_t is the number of persons in the labor force at time t, u_t is the number of unemployed persons at time t, and f_t is the monthly selling rate. We measure u_t and h_t in the data constructed by the BLS from the CPS, and we use the series that we have just constructed for f_t. Panel B of Figure A1 displays the monthly separation rate and its trend. The separation rate averages 3.3% between 1951 and 2014.[51]

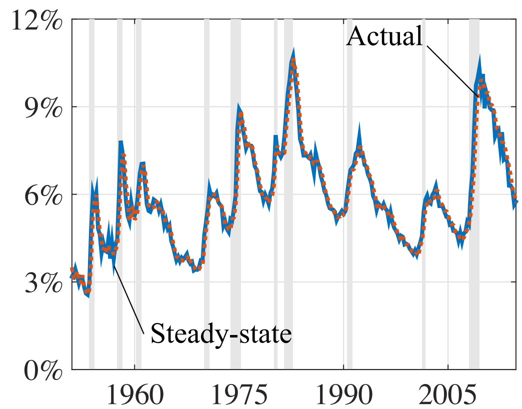
[51] One concern is that increases in government purchases cannot be undone because long-term relationships created by the government, especially employment relationships, are effectively permanent. It is true that government relationships separate more slowly than private relationships: using data constructed by the BLS from the JOLTS for 2000–2014, we find that the average monthly separation rate is 3.9% for jobs in the private sector and 1.4% for jobs in the government sector. Nevertheless, the separation rate for government relationships remains sizable. If no new relationships were created by the government, the level of government purchases would rapidly decrease: with a hiring freeze, US government employment would fall by $1 - \exp(-0.014 \cdot 12) = 15\%$ in one year.



A. Selling rate (per month)



B. Separation rate (per month)



C. Actual and steady-state unemployment rates

Figure A1: The Irrelevance of Transitional Dynamics For the Unemployment Rate

Notes: Panel A: The selling rate f is constructed from CPS data following the methodology of Shimer [2012]. The series $\bar{f}$ is the low-frequency trend of f produced using a HP filter with smoothing parameter 10^5. Panel B: The separation rate s is constructed from CPS data following the methodology of Shimer [2012]. The series $\bar{s}$ is the low-frequency trend of s produced using a HP filter with smoothing parameter 10^5. Panel C: The actual unemployment rate is the quarterly average of the seasonally adjusted monthly unemployment rate constructed by the BLS from the CPS. The steady-state unemployment rate is computed using (A1): $u_t = s_t/(f_t + s_t)$; this rate abstracts from transitional dynamics. The two series are almost identical showing that transitional dynamics are quantitatively unimportant. The shaded areas represent the recessions identified by the NBER.

Finally, we compare the actual unemployment rate and the steady-state unemployment rate

$$u_t = \frac{s_t}{f_t + s_t}. \tag{A1}$$

The two series, displayed in Panel C of Figure A1, are almost identical. Since the actual unemployment rate barely departs from its steady-state level, the transitional dynamics of the unemployment rate are unimportant.[52]

Approximation of $[\eta/(1-\eta)] \cdot \tau/u$ by x/x^*

Lemma 2 gives conditions under which x/x^* is a good approximation for $[\eta/(1-\eta)] \cdot \tau/u$. We show here that this approximation is very accurate in US data.

We begin by constructing the matching wedge

$$\tau = \frac{s \cdot \rho}{q - s \cdot \rho}. \tag{A2}$$

We use the separation rate s described in Panel B of Figure A1. We construct the vacancy-filling rate $q = f/x$ using the selling rate f described in Panel A of Figure A1 and the tightness x displayed in Figure 4. Panel A of Figure A2 displays the monthly vacancy-filling rate and its trend. The vacancy-filling rate averages 94% between 1951 and 2014. Last, we construct the matching cost ρ as a slow-moving variable such that the market is efficient on average. If the market is efficient on average, then (4) implies that

$$\overline{\tau} = \frac{1-\eta}{\eta} \cdot \overline{u}, \tag{A3}$$

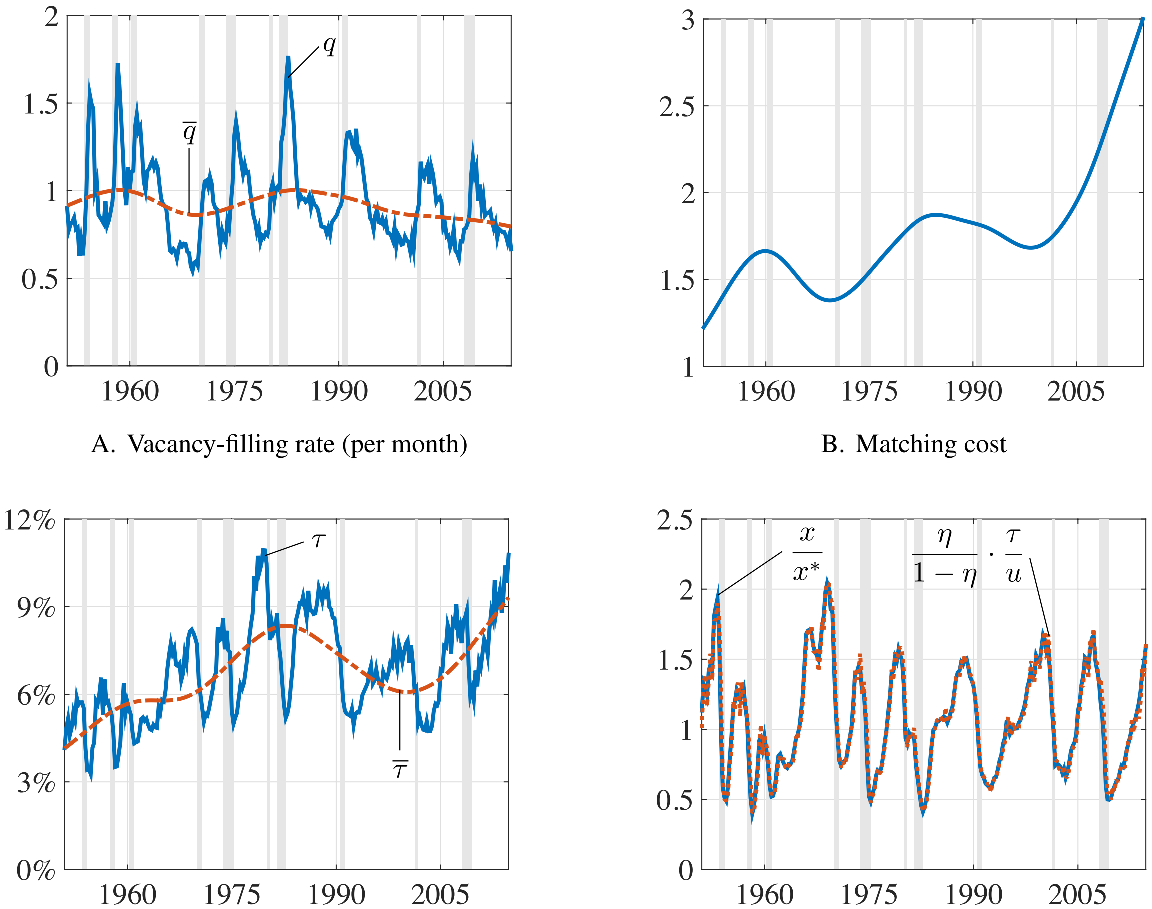
where $\overline{\tau}$ is the average value of the matching wedge and $\overline{u}$ the average value of the unemployment rate. We set $\eta = 0.46$ as estimated in the main text, produce $\overline{u}$ using a HP filter with smoothing parameter 10^5, and obtain $\overline{\tau}$. We produce the average values of the vacancy-filling and separation rates, $\overline{q}$ and $\overline{s}$, using a HP filter with smoothing parameter 10^5. Then, we construct

$$\rho = \frac{\overline{q}}{\overline{s}} \cdot \frac{\overline{\tau}}{1 + \overline{\tau}}. \tag{A4}$$

This equation is (A2) evaluated in the average state. Panel B of Figure A2 displays the resulting matching cost. The matching cost ρ averages 1.8 between 1951 and 2014. Using the series for s, q, and ρ , we construct τ from (A2). Panel C of Figure A2 displays the matching wedge and its trend. The matching wedge averages 6.8% between 1951 and 2014.

Finally, we construct $[\eta/(1-\eta)] \cdot \tau/u$ using our series for τ, $\eta = 0.46$, and the unemployment rate u described in Figure 6. Panel D of Figure A2 displays $[\eta/(1-\eta)] \cdot \tau/u$. Panel D also displays x/x^*, constructed using the tightnesses x and x^* described in Figure 4. These series of $[\eta/(1-\eta)] \cdot \tau/u$ and x/x^* are nearly indistinguishable, which validates the approximation.

[52] Panel C of Figure A1 is similar to Figure 1 in Hall [2005]. Even though we use different measures of the job-finding and separation rates and a longer time period, Hall's conclusion that transitional dynamics are irrelevant remains valid.



A. Vacancy-filling rate (per month)

B. Matching cost

C. Matching wedge

D. Assessing the approximation

Figure A2: The Similarity of $\frac{\eta}{1-\eta} \cdot \frac{\tau}{u}$ and $\frac{x}{x^*}$ in the United States, 1951–2014

Notes: Panel A: The vacancy-filling rate q is computed using $q = f/x$, where the selling rate f is described in Panel A of Figure A1 and the tightness x is described in Figure 4. The series $\overline{q}$ is the low-frequency trend of q produced using a HP filter with smoothing parameter 10^5. Panel B: The matching cost is computed using (A4). Panel C: The matching wedge τ is computed using (A2). The series $\overline{\tau}$ is computed using (A3). Panel D: The term $1\left[\eta/(1-\eta)\right] \cdot \tau/u$ is computed with $\eta = 0.46$, the unemployment rate u described in Figure 6, and the matching wedge τ described in Panel C. The term x/x^* is computed with the tightnesses x and x^* described in Figure 4. These series of $[\eta/(1-\eta)] \cdot \tau/u$ and x/x^* are nearly indistinguishable, which validates the approximation. The shaded areas represent the recessions identified by the NBER.

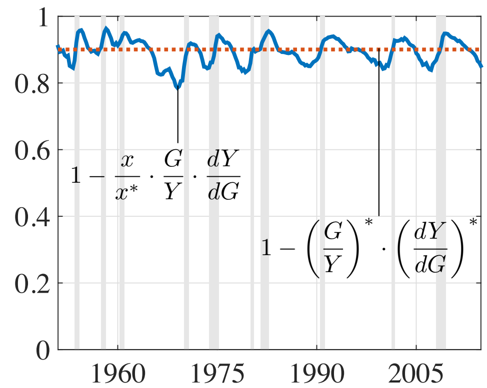


Figure A3: The Proximity of $1-(x/x^*)\cdot(G/Y)\cdot(dY/dG)$ with $1-(G/Y)^*\cdot(dY/dG)$ in the United States, 1951–2014

Notes: The term x/x^* is computed using the tightnesses x and x^* described in Figure 4. The term $G/Y=(G/C)/(1+G/C)$ is computed using the ratio G/C from Figure 3. We set $(G/Y)^*=16.6\%$. We set dY/dG and $(dY/dG)^*$ at a constant value of 0.6. The shaded areas represent the recessions identified by the NBER.

Approximation of $1-(x/x^*)\cdot(G/Y)\cdot(dY/dG)$ *by* $1-(G/Y)^*\cdot(dY/dG)^*$

In Section 3, we explain that $1-(x/x^*)\cdot(G/Y)\cdot(dY/dG)$ and $1-(G/Y)^*\cdot(dY/dG)^*$ are close when $(G/Y)\cdot(dY/dG)$ is small, and we argue that this is the case in US data. Here we assess the proximity of the two terms. We set $dY/dG=(dY/dG)^*=0.6$ and $\eta=0.46$, as discussed in Section 4. Using the ratio G/C described in Figure 3, we construct $G/Y=(G/C)/(1+G/C)$. We use the tightnesses x and x^* described in Figure 4. Figure A3 shows that the resulting time series for $1-(x/x^*)\cdot(G/Y)\cdot(dY/dG)$ and $1-(G/Y)^*\cdot(dY/dG)^*$ are close between 1951 and 2014.

Appendix B: Okun's Law

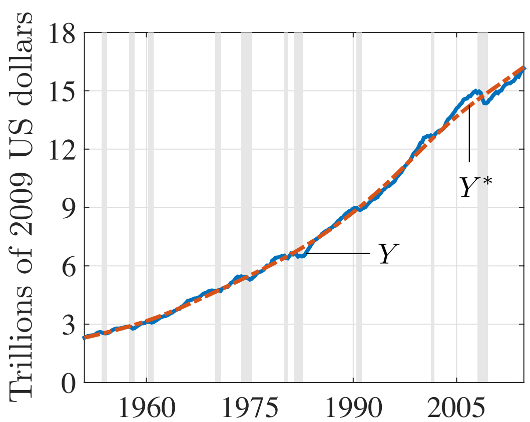
In this appendix, we revisit Okun's law using US data for 1951–2014. Okun's law is the statistical relationship between deviations of output from trend and deviations of unemployment from trend. It was first proposed by Okun [1963], who found that in US data for 1954–1962, output was 3 percent below trend when the unemployment rate was 1 percentage point above trend. We estimate Okun's law for the entire 1951–2014 period and for the recent 1994–2014 period. The relationship between output gap and unemployment gap has evolved over time.

We measure output Y the real gross domestic product constructed by the BEA as part of the NIPA. We produce the trend Y^* of output using a HP filter with smoothing parameter 10^5. Panel A of Figure A4 displays Y and Y^*. We use the unemployment rate u and the unemployment trend u^* described in Figure 6. For reference, Panel B of Figure A4 displays u and u^*.

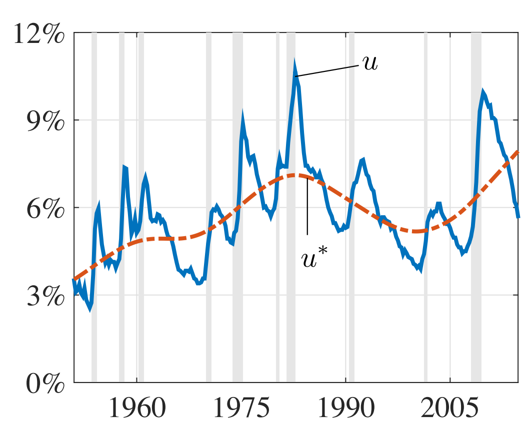
Okun's law is the following linear relationship:

$$\frac{Y_t - Y_t^*}{Y_t^*} = -\chi \cdot (u_t - u_t^*) \tag{A5}$$

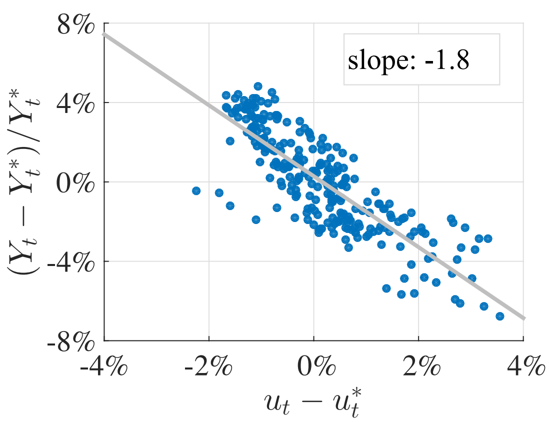
The coefficient χ was estimated around 3 by Okun [1963] in US data for 1954–1962. Regressing $(Y_t-Y_t^*)/Y_t^*$ on $(u_t-u_t^*)$ with ordinary least squares, we estimate a coefficient of 1.8 on the 1951–



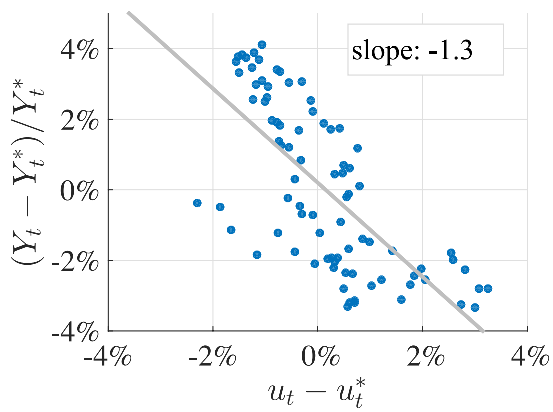
A. Output



B. Unemployment rate



C. Okun's law for 1951–2014



D. Okun's law for 1994–2014

Figure A4: Okun's Laws in the United States, 1951–2014

Notes: Panel A: Output Y is seasonally adjusted quarterly real gross domestic product in chained 2009 dollars constructed by the BEA as part of the NIPA. The series Y^* is the low-frequency trend of Y produced using a HP filter with smoothing parameter 10^5. Panel B: The series u and u^* are the series described in Panel A of Figure 6. The shaded areas in Panels A and B represent the recessions identified by the NBER. Panel C: The series Y, Y^*, u, and u^* used to construct the scatter plot are from Panels A and B. These series cover the 1951–2014 period. The plot also displays the regression line used to estimate the coefficient χ in (A5). Panel D: This plot is obtained from the scatter plot in Panel C by restricting the data to the 1994–2014 period.

2014 period and a coefficient of 1.3 on the 1994–2014 period. Panels C and D illustrate Okun's law for the two periods. We conclude that when the unemployment rate is 1 percentage point above trend, output is 1.8 percent below trend in the 1951–2014 period and 1.3 percent below trend in the 1994–2014 period. In our model, we have $Y(t) = 1 - u(t)$ and hence $dY/Y = -du/(1-u)$ which leads to an Okun's coefficient of $1/(1-u) \approx 1.06$ as the average unemployment rate is $u = 5.9\%$, which is slightly below the empirical estimate of 1.3 for the recent 1994–2014 period.

Appendix C: Proof of Propositions 2 and 4

Proof of Proposition 2

In a feasible allocation, all the variables are functions of g and x. Equivalently, they can be defined as functions of G/C and x. Focusing on feasible allocations, we can therefore write formula (5) as $\Omega(G/C,x) = 0$ where $\Omega(G/C,x) = 1 - MRS_{gc} - y'(x) \cdot x'(g)$. By definition, $MRS_{gc}((G/C)^*) = 1$ and $y'(x^*) = 0$ so $\Omega((G/C)^*, x^*) = 0$; thus, the first-order Taylor expansion of Ω around $[(G/C)^*, x^*]$ is

$$\Omega(G/C,x) = \frac{\partial \Omega}{\partial(G/C)} \cdot [G/C - (G/C)^*] + \frac{\partial \Omega}{\partial x} \cdot (x - x^*) + O(\|\mathbf{w}\|^2) \tag{A6}$$

where the derivatives are evaluated at $[(G/C)^*, x^*]$ and $\mathbf{w} \equiv [G/C - (G/C)^*, x - x^*] \in \mathbb{R}^2$ and $\|\cdot\|$ is any norm on $\mathbb{R}^2$.

The first step to computing the partial derivatives of $\Omega = 1 - MRS_{gc} - y'(x) \cdot x'(g)$ is to compute the partial derivatives of $1 - MRS_{gc}$ at $[(G/C)^*, x^*]$. With homothetic preferences, MRS_{gc} is a function of G/C only and $\partial MRS_{gc}/\partial x = 0$. Furthermore, by definition of ε,

$$-\frac{\partial MRS_{gc}}{\partial(G/C)} = \frac{1}{\varepsilon} \cdot \frac{MRS_{gc}((G/C)^*)}{(G/C)^*} = \frac{1}{\varepsilon} \cdot \frac{1}{(G/C)^*}.$$

The second step to computing the partial derivatives of $\Omega = 1 - MRS_{gc} - y'(x) \cdot x'(g)$ is to compute the partial derivatives of $y'(x) \cdot x'(g)$ at $[(G/C)^*, x^*]$. Net output $y(x)$ is a function of x only, so $y'(x)$ is a function of x only. We therefore have

$$\begin{aligned}
\frac{\partial(y'(x) \cdot x'(g))}{\partial(G/C)} &= y'(x^*) \cdot \frac{\partial x'(g)}{\partial(G/C)} = 0 \\
\frac{\partial(y'(x) \cdot x'(g))}{\partial x} &= y''(x^*) \cdot x'(g^*) + y'(x^*) \cdot \frac{\partial x'(g)}{\partial x} = y''(x^*) \cdot x'(g^*).
\end{aligned} \tag{A7}$$

From this we infer that

$$\frac{\partial \Omega}{\partial(G/C)} = \frac{1}{\varepsilon} \cdot \frac{1}{(G/C)^*}. \tag{A8}$$

It only remains to compute $y''(x^*)$ and $x'(g^*)$.

The elasticity of $u(x)$ is $-(1-\eta) \cdot (1 - u(x))$ and the elasticity of $\tau(x)$ is $\eta \cdot (1 + \tau(x))$ so the elasticity of $y(x) = (1-u(x))/(1+\tau(x))$ is $(1-\eta)\cdot u(x) - \eta\cdot\tau(x)$ and hence

$$y'(x) = \frac{y(x)}{x}\cdot(1-\eta)\cdot u(x)\cdot\underbrace{\left(1-\frac{\eta}{1-\eta}\cdot\frac{\tau(x)}{u(x)}\right)}_{\equiv z(x)}. \tag{A9}$$

Since $z(x^*) = 0$, we have

$$y''(x^*) = \frac{y(x^*)}{x^*}\cdot(1-\eta)\cdot u(x^*)\cdot z'(x^*).$$

Using the elasticity of τ and u, we infer that the elasticity of τ/u is $\eta\cdot(1+\tau(x)) + (1-\eta)\cdot(1-u(x)) = 1+\eta\cdot\tau(x)-(1-\eta)\cdot u(x)$. At $x = x^*$, $\eta\cdot\tau(x^*)-(1-\eta)\cdot u(x^*) = 0$ so the elasticity of τ/u is 1 and

$$z'(x^*) = -\frac{\eta}{1-\eta}\cdot\frac{\tau(x^*)/u(x^*)}{x^*} = -\frac{1}{x^*}.$$

We conclude that

$$y''(x^*) = -\frac{y(x^*)}{(x^*)^2}\cdot(1-\eta)\cdot u(x^*). \tag{A10}$$

In equilibrium, $Y = Y(x)$, $x = x(g)$, and $G = (1+\tau(x(g)))\cdot g$. We can therefore differentiate Y in two different ways:

$$\frac{d\ln(Y)}{d\ln(g)} = \frac{d\ln(Y)}{d\ln(G)}\cdot\frac{d\ln(G)}{d\ln(g)} = \frac{d\ln(Y)}{d\ln(x)}\cdot\frac{d\ln(x)}{d\ln(g)}.$$

As the elasticity of $1+\tau(x)$ is $\eta\cdot\tau$ and $G = (1+\tau(x))\cdot g$, we find that

$$\frac{d\ln(G)}{d\ln(g)} = 1+\eta\cdot\tau\cdot\frac{d\ln(x)}{d\ln(g)}.$$

Since the elasticity of $Y(x)$ is $(1-\eta)\cdot u$, we conclude that

$$\frac{d\ln(Y)}{d\ln(G)}\cdot\left(1+\eta\cdot\tau(x)\cdot\frac{d\ln(x)}{d\ln(g)}\right) = (1-\eta)\cdot u(x)\cdot\frac{d\ln(x)}{d\ln(g)}.$$

Some algebra yields

$$x'(g) = \frac{x}{g}\cdot\frac{1}{(1-\eta)\cdot u}\cdot\frac{(G/Y)\cdot(dY/dG)}{1-\frac{\eta}{1-\eta}\cdot\frac{\tau}{u}\cdot(G/Y)\cdot(dY/dG)}. \tag{A11}$$

Using the results that $(1-\eta)\cdot u(x^*) = \eta\cdot\tau(x^*)$ and $G/Y = g/y$, we find that at $[(G/C)^*, x^*]$

$$x'(g^*) = \frac{x^*}{y(x^*)}\cdot\frac{1}{(1-\eta)\cdot u^*}\cdot m,$$

where m is defined by (9). Combining this equation with (A10) as showed by (A7), we conclude that

$$\frac{\partial \Omega}{\partial x} = \frac{m}{x^*}. \tag{A12}$$

The combination of (A6), (A8), and (A12) yields (7).

Proof of Proposition 4

In a feasible allocation, all the variables can be expressed as a function of G/C and x; in particular, $g = y \cdot (g/y) = y \cdot (G/Y) = y(x) \cdot (G/C)/(1+G/C)$. Among all the feasible allocations, the equilibrium allocations satisfy the constraint $x = x(g)$, or equivalently $g = x^{-1}(x)$. An equilibrium allocation must therefore satisfy

$$\Lambda(G/C, x) \equiv x^{-1}(x) - y(x) \cdot \frac{G/C}{1+G/C} = 0.$$

By definition, $[(G/C)^*, x_0]$ is an equilibrium so $\Lambda((G/C)^*, x_0) = 0$. Accordingly, the first-order Taylor expansion of Λ around $[(G/C)^*, x_0]$ is

$$\Lambda(G/C, x) = \frac{\partial \Lambda}{\partial (G/C)} \cdot [G/C - (G/C)^*] + \frac{\partial \Lambda}{\partial x} \cdot (x - x_0) + O(\|\mathbf{w}\|^2), \tag{A13}$$

where the derivatives are evaluated at $[(G/C)^*, x_0]$ and $\mathbf{w} \equiv [G/C - (G/C)^*, x - x_0] \in \mathbb{R}^2$.

Using (A11), we infer that at $[(G/C)^*, x_0]$,

$$\frac{dx^{-1}}{dx} = \frac{y(x_0)}{x_0} \cdot (1-\eta) \cdot u(x_0) \cdot \frac{1 - \frac{\eta}{1-\eta} \cdot \frac{\tau(x_0)}{u(x_0)} \cdot (G/Y)^* \cdot (dY/dG)_0}{dY/dG}.$$

Here the multiplier $(dY/dG)_0$ is evaluated at $[(G/C)^*, x_0]$. Equation (A9) gives the expression for $y'(x_0)$. Last, simple algebra indicates yields

$$\frac{\partial\, [G/C/(1+G/C)]}{\partial (G/C)} = \frac{1}{(1+G/C)^2} = (C/Y)^2 = \frac{C}{Y} \cdot \frac{C}{G} \cdot \frac{G}{Y}.$$

Combining these results, we find that at $[(G/C)^*, x_0]$

$$\frac{\partial \Lambda}{\partial (G/C)} = y(x_0) \cdot (C/Y)^* \cdot \frac{(G/Y)^*}{(G/C)^*}, \qquad \frac{\partial \Lambda}{\partial x} = \frac{y(x_0)}{x_0} \cdot \frac{(1-\eta) \cdot u(x_0)}{m_0},$$

where m_0 is defined by (9) with G/Y and dY/dG evaluated at $[(G/C)^*, x_0]$.

In an equilibrium allocation, $\Lambda(G/C, x) = 0$. Using (A13), we infer that tightness in an equilibrium allocation is related to government purchases by

$$x - x_0 \approx \frac{-\partial \Lambda / \partial (G/C)}{\partial \Lambda / \partial x} \cdot (G/C - (G/C)^*),$$

where the approximation is valid up to a remainder that is $O((x-x_0)^2+(G/C-(G/C)^*)^2)$. Combining this approximation with the expressions for the partial derivatives of Λ, we obtain

$$\frac{x-x_0}{x_0} \approx -a_0 \cdot m_0 \cdot \frac{G/C-(G/C)^*}{(G/C)^*}. \tag{A14}$$

where a_0 is defined by (13) with G/Y, C/Y and u evaluated at $[(G/C)^*, x_0]$.

Formula (7) describes how government purchases are related to tightness in a feasible allocation in which government purchases are optimal. In an equilibrium, starting from an inefficient allocation $[(G/C)^*, x_0]$, tightness responds endogenously to government purchases as described by (A14). In an equilibrium in which government purchases are optimal, both (7) and (A14) are satisfied simultaneously. Substituting x in (7) using (A14) and doing a bit of algebra, we find that the optimal level of government purchases approximately satisfy

$$\frac{G/C-(G/C)^*}{(G/C)^*} \approx \underbrace{\frac{-\varepsilon^* \cdot m^*}{1+\varepsilon^* \cdot m^* \cdot a_0 \cdot m_0 \cdot \frac{x_0}{x^*}}}_{\equiv \Gamma(x_0, x^*, (G/C)^*)} \cdot \frac{x_0-x^*}{x^*},$$

where m^* is defined by (9) with G/Y and dY/dG evaluated at $[(G/C)^*, x^*]$ and ε^* is defined by (8) with G/C evaluated at $(G/C)^*$. This approximation is valid up to a remainder that is $O((x_0-x^*)^2 + (G/C-(G/C)^*)^2)$. With a first-order Taylor approximation around $[x^*, x^*, (G/C)^*]$ we can write $\Gamma(x_0, x^*, (G/C)^*) = \Gamma(x^*, x^*, (G/C)^*) + (\partial\Gamma/\partial x_0) \cdot (x_0-x^*) + O((x_0-x^*)^2)$. Hence we have

$$\frac{G/C-(G/C)^*}{(G/C)^*} \approx \Gamma(x^*, x^*, (G/C)^*) \cdot \frac{x_0-x^*}{x^*}$$

up to a remainder that is $O((x_0-x^*)^2 + (G/C-(G/C)^*)^2)$, which establishes (12).

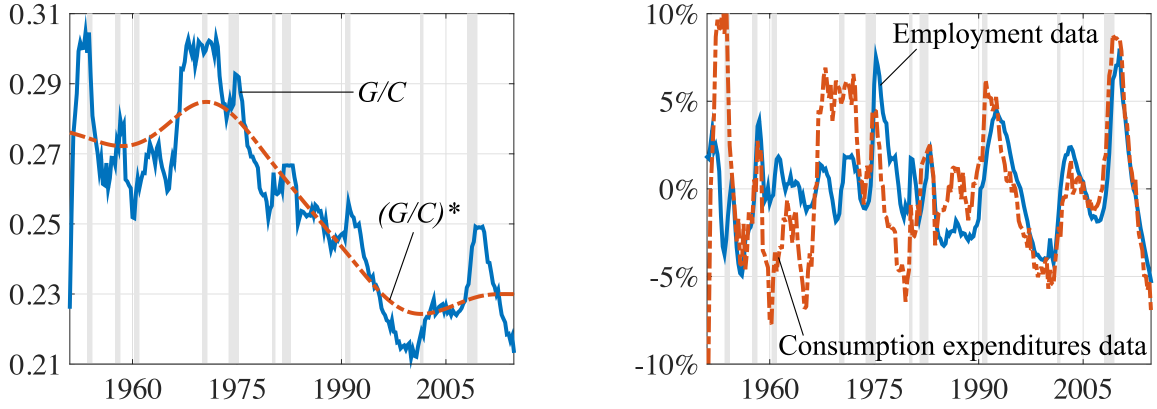
Equation (14) is obtained by identifying (12) and (7), both of which hold in the equilibrium with optimal government purchases.

Appendix D: Government Consumption Expenditures

In this appendix, we construct an alternative measure of the government consumption-personal consumption ratio G/C for the United States. We measure G by the government consumption expenditures constructed by the BEA as part of the NIPA. We measure C by the personal consumption expenditures constructed by the BEA as part of the NIPA. Figure A5 displays the resulting series for G/C between 1951 and 2014.

The measure of G/C based on employment data and plotted in Figure 3 and the measure of G/C based on consumption expenditures data and plotted in Figure A5 have fairly different levels. The ratio G/C based on consumption expenditures data is always higher. The gap between the two measures is large at the beginning of the period, shrinks until 1990, and is roughly constant after 1990. Indeed, since 1990, the measure based on consumption expenditures data has been hovering between 0.21 and 0.25 and the measure based on employment data between 0.19 and 0.21.

Despite this difference in levels, the ratios $(G/C-(G/C)^*)/(G/C)^*$ obtained from government expenditures data and employment data nearly perfectly overlap since 1980. The ratio $(G/C-$



A. Consumption expenditures data

B. Two measures of $\frac{G/C-(G/C)^*}{(G/C)^*}$

Figure A5: Government Consumption Expenditures and Government Employment in the United States, 1951–2014

Notes: Panel A: Government consumption G is seasonally adjusted quarterly government consumption expenditures in dollars constructed by the BEA as part of the NIPA. Personal consumption C is seasonally adjusted quarterly personal consumption expenditures in dollars constructed by the BEA as part of the NIPA. The ratio $(G/C)^*$ is the low-frequency trend of G/C produced using a HP filter with smoothing parameter 10^5. Panel B: The ratio $[G/C-(G/C)^*]/(G/C)^*$ measured from consumption expenditures data uses the ratios G/C and $(G/C)^*$ described in Panel A. The ratio $[G/C-(G/C)^*]/(G/C)^*$ measured from employment data uses the ratios G/C and $(G/C)^*$ described in Figure 3. The shaded areas represent the recessions identified by the NBER.

$(G/C)^*)/(G/C)^*$ determines the value of $1-MRS_{gc}$ so it enters into most of our formulas. Since deviations of government expenditures from trend almost exactly track deviations of government employment from trend, the two measures of government purchases have the same implications in our analysis.

Before 1980, the two ratios do not overlap as well because of the Korean and Vietnam wars. The ratio based on consumption expenditures data is especially high in 1951–1953 during the Korean war and in 1967–1972 during the Vietnam war. Since military personnel does not count as government employees in BLS data, and since wars trigger important purchases of military equipment, government expenditures during wars rise whereas government employment does not change much. Accordingly, wars create a discrepancy between our two measures of $(G/C-(G/C)^*)/(G/C)^*$.

Appendix E: The Equilibrium of the Model of Section 5

In this appendix we derive and analyze the dynamical system describing the equilibrium of the model of Section 5. An equilibrium consists of paths for market tightness, net personal consumption, net government consumption, net output, real wealth, and real interest rate, $[x(t), c(t), g(t), y(t), b(t), r(t)]_{t=0}^{+\infty}$. The equilibrium consists of 6 variables, so it requires 6 conditions.

The first condition is that the government chooses a fixed amount of government consumption: $g(t)=g$. The second condition is that a price mechanism determines the real interest rate: $r(t)$ is given by (26). The third condition is that the bond market is in equilibrium: $b(t)=0$. The fourth condition is that the market for services is in equilibrium: $c(t) + g(t) = y(t)$. The fifth condition is that the matching process determines net output: $y(t) = y(x(t))$.

The sixth condition is that the representative household chooses net personal consumption to maximize utility subject to its budget constraint. To solve the household's problem, we set up the current-value Hamiltonian:

$$\mathscr{H}(t, c(t), b(t)) = \mathscr{U}(c(t), g(t)) + \mathscr{V}(b(t)) + \lambda(t) \cdot [Y(x(t)) - (1 + \tau(x(t))) \cdot c(t) + r(t) \cdot b(t) - T(t)]$$

with control variable $c(t)$, state variable $b(t)$, and current-value costate variable $\lambda(t)$. The necessary conditions for an interior solution to this maximization problem are $\partial\mathscr{H}/\partial c = 0$, $\partial\mathscr{H}/\partial b = \delta \cdot \lambda(t) - \dot{\lambda}(t)$, and the transversality condition $\lim_{t \to +\infty} e^{-\delta \cdot t} \cdot \lambda(t) \cdot b(t) = 0$. Given that $\mathscr{U}$ and $\mathscr{V}$ are concave and that $\mathscr{H}$ is the sum of $\mathscr{U}$, $\mathscr{V}$, and a linear function of (c, b), $\mathscr{H}$ is concave in (c, b) and these conditions are also sufficient. These two first-order conditions imply that

$$\frac{\partial\mathscr{U}}{\partial c}(c(t), g(t)) = \lambda(t) \cdot (1 + \tau(x(t))) \tag{A15}$$

$$\mathscr{V}'(b(t)) = (\delta - r(t)) \cdot \lambda(t) - \dot{\lambda}(t). \tag{A16}$$

Recombining these equations, we obtain the consumption Euler equation

$$(1 + \tau(x(t))) \cdot \frac{\mathscr{V}'(b(t))}{\frac{\partial\mathscr{U}}{\partial c}(c(t), g(t))} + (r(t) - \delta) = -\frac{\dot{\lambda}(t)}{\lambda(t)},$$

where $\dot{\lambda}(t)/\lambda(t)$ can be expressed as a function of $c(t)$, $g(t)$, and $x(t)$, and their time derivatives using (A15). The Euler equation represents a demand for saving in part from intertemporal consumption-smoothing considerations and in part from the utility provided by wealth. The equation implies that at the margin, the household is indifferent between spending income on consumption and holding real wealth. The equation determines the level of aggregate demand.

We have obtained the six equations that define the dynamical system representing the equilibrium. We now describe the transitional dynamics toward the steady state. The dynamic system is simple to study because it can be described by one single endogenous variable: the costate variable $\lambda(t)$. All the variables can be recovered from $\lambda(t)$. The law of motion for $\lambda(t)$ is given by (A16):

$$\dot{\lambda}(t) = (\delta - r) \cdot \lambda(t) - \mathscr{V}'(0).$$

Note that $r(t) = r$ is constant over time (see (26) and note that $g(t)$ is constant over time). The steady-state value of the costate variable is $\lambda = \mathscr{V}'(0)/(\delta - r) > 0$. Since $\delta - r > 0$, we infer that the dynamical system is a source. As there is no state variable, our source system jumps from one steady state to the other in response to unexpected permanent shocks. Therefore, the steady-state analysis that we carried out to derive optimal government purchase formulas in Section 3 apply to this dynamic model.

Appendix F: The Multiplier in the Model of Section 5

In this appendix we derive an expression for the government-purchases multiplier in the model of Section 5. We use this expression for the calibration for he model. The expression also shows that the multiplier is higher when the unemployment rate is higher.

PROPOSITION A1. *In the model of Section 5, the multiplier satisfies*

$$\frac{d\ln(Y)}{d\ln(G)} = \left[1-(1-\beta)\cdot\frac{1-G/Y}{1-G/Y^*}\cdot\frac{z(G/(Y-G))}{z(G/(Y^*-G))}\right]\cdot\left[1+\varepsilon\cdot\frac{\eta}{1-\eta}\cdot\frac{\tau}{u}\cdot\frac{C}{Y}\cdot z\left(\frac{G}{C}\right)\right]^{-1} \quad \text{(A17)}$$

where the auxiliary function z is defined by

$$z(\theta) = 1+\frac{1-\gamma}{\gamma}\cdot\theta^{\frac{1-\varepsilon}{\varepsilon}}. \quad \text{(A18)}$$

When the unemployment rate is efficient and government purchases are given by the Samuelson formula, the multiplier simplifies to

$$dY/dG = \frac{\beta}{(G/Y)^*+\varepsilon\cdot[1-(G/Y)^*]}.$$

In addition, if the elasticity of substitution between government and personal consumption is $\varepsilon = 1$, then the multiplier simplifies to $dY/dG = \beta$.

Proof. The proof of the proposition proceeds in five steps.

Step 1. Using equation (24) and simple algebra, we write the marginal utility of consumption as

$$\frac{\partial\mathcal{U}}{\partial c}(c,g) = (1-\gamma)\cdot d\left(\frac{G}{C}\right)^{\frac{1}{\varepsilon}}$$

where $d(\theta)\equiv\mathcal{U}(1,\theta)$. Furthermore, the elasticity of $d(\theta)$ is $1/z(\theta)$, where $z(\theta)$ is given by (A18).

Step 2. Using the results from step 1 and equation (26), we rewrite the interest-rate schedule as

$$\delta - r = \frac{\mu}{(1-\gamma)^{1-\beta}}\cdot\mathcal{V}'(0)^{1-\alpha}\cdot d\left(\frac{G}{Y^*-G}\right)^{-\frac{1-\beta}{\varepsilon}}.$$

The results from step 1 and more algebra imply that the elasticity of the interest-rate schedule is

$$\frac{d\ln(\delta-r)}{d\ln(G)} = -\frac{1-\beta}{\varepsilon}\cdot\frac{1}{z(G/(Y^*-G))}\cdot\frac{Y^*}{Y^*-G}.$$

Note that $\delta - r$ depends only on G and not on x.

Step 3. We implicitly define $C(G,x)$ as the solution of

$$d\left(\frac{G}{C}\right)^{-\frac{1}{\varepsilon}} = \frac{1-\gamma}{\mathscr{V}'(0)} \cdot \frac{\delta - r(G)}{1+\tau(x)}.$$

The function $C(G,x)$ is the gross personal consumption that satisfies the Euler equation (23) for a gross government consumption G and a tightness x. The results from steps 1 and 2 and simple algebra imply that

$$\frac{\partial \ln(C)}{\partial \ln(x)} = -\varepsilon \cdot \eta \cdot \tau \cdot z\left(\frac{G}{C}\right)$$
$$\frac{\partial \ln(C)}{\partial \ln(G)} = 1-(1-\beta)\cdot \frac{Y^*}{Y^*-G} \cdot \frac{z(G/C)}{z\left(G/\left(Y^*-G\right)\right)}.$$

Step 4. The equilibrium condition determining market tightness is

$$Y = C(G,x) + G.$$

We differentiate this equilibrium condition with respect to G:

$$\frac{d\ln(Y)}{d\ln(G)} = \frac{C}{Y}\cdot\left(\frac{\partial \ln(C)}{\partial \ln(G)} + \frac{\partial \ln(C)}{\partial \ln(x)} \cdot \frac{d\ln(x)}{d\ln(G)}\right) + \frac{G}{Y}$$

Equation (1) implies that

$$\frac{d\ln(x)}{d\ln(G)} = \frac{1}{(1-\eta)\cdot u} \cdot \frac{d\ln(Y)}{d\ln(G)}.$$

Using these equations and the elasticities from step 3, we obtain

$$\left[1+\frac{C}{Y}\cdot\varepsilon\cdot\frac{\eta}{1-\eta}\cdot\frac{\tau}{u}\cdot z\left(\frac{G}{C}\right)\right]\cdot\frac{d\ln(Y)}{d\ln(G)} = 1-(1-\beta)\cdot\frac{Y-G}{Y}\cdot\frac{Y^*}{Y^*-G}\cdot\frac{z(G/(Y-G))}{z\left(G/\left(Y^*-G\right)\right)}$$
$$\frac{d\ln(Y)}{d\ln(G)} = \frac{1-(1-\beta)\cdot\frac{1-G/Y}{1-G/Y^*}\cdot\frac{z(G/(Y-G))}{z(G/(Y^*-G))}}{1+\frac{C}{Y}\cdot\varepsilon\cdot\frac{\eta}{1-\eta}\cdot\frac{\tau}{u}\cdot z\left(\frac{G}{C}\right)}.$$

Step 5. When the unemployment rate is efficient, $Y = Y^*$ and $1 = [\eta/(1-\eta)]\cdot\tau/u$. Hence, the expression for the multiplier simplifies to

$$\frac{d\ln(Y)}{d\ln(G)} = \frac{\beta}{1+\varepsilon\cdot\frac{C}{Y}\cdot z\left(\frac{G}{C}\right)}.$$

Table A1: Parameter Values Used in the Simulations of Section 5

	Description	Value	Source
	Panel A. Average values targeted in calibration		
$\overline{u}$	unemployment rate	5.9%	CPS, 1951–2014
$\overline{x}$	market tightness	0.65	Barnichon [2010], JOLTS, CPS, 1951–2014
$\overline{G/Y}$	government purchases-output ratio	16.6%	CES, 1951–2014
$\overline{\tau}$	matching wedge	6.8%	efficiency on average (see Appendix A)
$\overline{dY/dG}$	government-purchases multiplier	0.6	literature (see Section 4)
$\overline{\mathscr{V}'(0)}$	marginal utility of wealth	1	normalization
	Panel B. Calibrated parameters		
$1-\eta$	elasticity of the selling rate $f(x)$	0.54	Barnichon [2010], JOLTS, CPS, 1951–2014
s	separation rate (monthly)	3.3%	CPS, 1951–2014
ω	matching efficacy	0.67	matches average values
ρ	matching cost	1.6	matches average values
ε	elasticity of substitution	1	Section 4
γ	parameter of utility function	0.17	matches $\overline{G/Y} = 16.6\%$
α	parameter of interest-rate schedule	1	normalization
β	parameter of interest-rate schedule	0.6	matches $\overline{dY/dG} = 0.6$
μ	parameter of interest-rate schedule	1.4	matches average values

Furthermore, when government purchases satisfy the Samuelson formula, $G/C = (G/C)^* = [\gamma/(1-\gamma)]^{\varepsilon}$ so $z(G/C) = (Y/G)^*$ and the multiplier simplifies to

$$\frac{dY}{dG} = \frac{\beta}{(G/Y)^* + \varepsilon \cdot [1-(G/Y)^*]}.$$

Finally, if $\varepsilon = 1$, then $dY/dG = \beta$. □

Appendix G: Calibration of the Model of Section 5

In this appendix we calibrate the model of Section 5 to US data for 1951–2014. The calibration is summarized in Table A1.

We calibrate several parameters such that variables in the average state are equal to their average value measured in the data. We target an average unemployment rate $\overline{u} = 5.9\%$, an average market tightness $\overline{x} = 0.65$, an average government purchases-output ratio $\overline{G/Y} = 16.6\%$, and an average matching wedge $\overline{\tau} = 6.8\%$. These average values come from the times series constructed in Section 4 and Appendix A. We also normalize the values of marginal utility of wealth in the average state to $\overline{\mathscr{V}'(0)} = 1$.

We begin by calibrating the three parameters determining the sufficient statistics at the heart of our formulas. Based on the discussion in Section 4, we calibrate the model to obtain an elasticity of substitution between government and personal consumption of 1, a elasticity of the selling rate of 0.54, and a multiplier in the average state of 0.6; hence, we set $\varepsilon = 1$, $\eta = 0.46$, and $\beta = 0.6$.[53]

Next, we calibrate parameters related to matching. We set the separation rate to its average value for 1951–2014: $s = 3.3\%$ (see Appendix A). To calibrate the matching efficacy, we exploit the relationship $\overline{u} \cdot f(\overline{x}) = s \cdot (1-\overline{u})$, which implies $\omega = s \cdot (\overline{x})^{\eta-1} \cdot (1-\overline{u})/\overline{u} = 0.67$. To calibrate the vacancy-filling cost, we exploit the relationship $\overline{\tau} = \rho \cdot s / [\omega \cdot (\overline{x})^{-\eta} - \rho \cdot s]$, which implies $\rho = \omega \cdot (\overline{x})^{-\eta} \cdot \overline{\tau} / [s \cdot (1+\overline{\tau})] = 1.6$.

Then, we calibrate the parameters of the utility function. We find that $MRS_{gc} = [\gamma/(1-\gamma)] \cdot (G/C)^{-1/\varepsilon}$. Given that $MRS_{gc}((G/C)^*) = 1$, we infer that $\gamma/(1-\gamma) = ((G/C)^*)^{1/\varepsilon}$. We assume that the average of the ratio G/C is the Samuelson ratio so $(G/C)^* = 19.9\%$. With $\varepsilon = 1$ and $(G/C)^* = 19.9\%$, we set $\gamma = 0.17$.

Last, we calibrate the parameters of the interest-rate schedule. For aggregate demand shocks to generate fluctuations, we need $\alpha > 0$. The value of α determines the elasticity of output to the marginal utility of wealth, $\mathscr{V}'(0)$. Since we do not know the amplitude of the fluctuations of $\mathscr{V}'(0)$, the exact value of α is irrelevant; we arbitrarily set $\alpha = 1$. Last, using (26) in the average state and the expression (24) for $\mathscr{U}$, we find that $\mu = (\delta - r^*) \cdot \left[(1-\gamma) \cdot \mathscr{U}(1, (G/C)^*)^{1/\varepsilon}\right]^{1-\beta}$. The expression (25) implies that in the average state $\delta - r^* = (1+\tau(x^*)) \Big/ \left[(1-\gamma) \cdot \mathscr{U}(1, (G/C)^*)^{1/\varepsilon}\right]$. Combining these expressions, we infer that $\mu = (1+\overline{\tau}) / \left[(1-\gamma) \cdot \mathscr{U}(1, \overline{G/C})^{1/\varepsilon}\right]^{\beta}$. Using the calibrated values of all the parameters, we obtain $\mu = 1.4$.

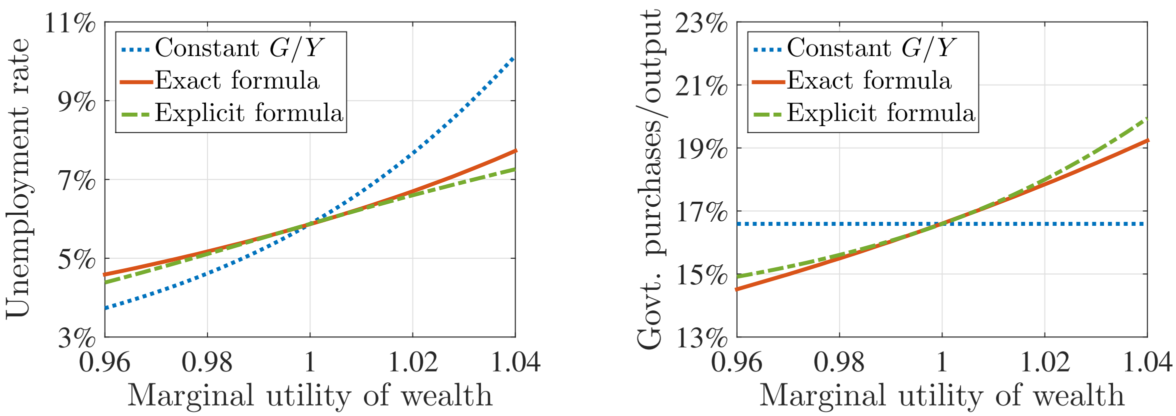
Appendix H: Robustness of the Simulation Results of Section 5

The simulation results in Section 5 are obtained for an elasticity of substitution between government and personal consumption of $\varepsilon = 1$ and an average government-purchases multiplier of $dY/dG = 0.6$. In this appendix we repeat the simulations for alternative values of ε and dY/dG.

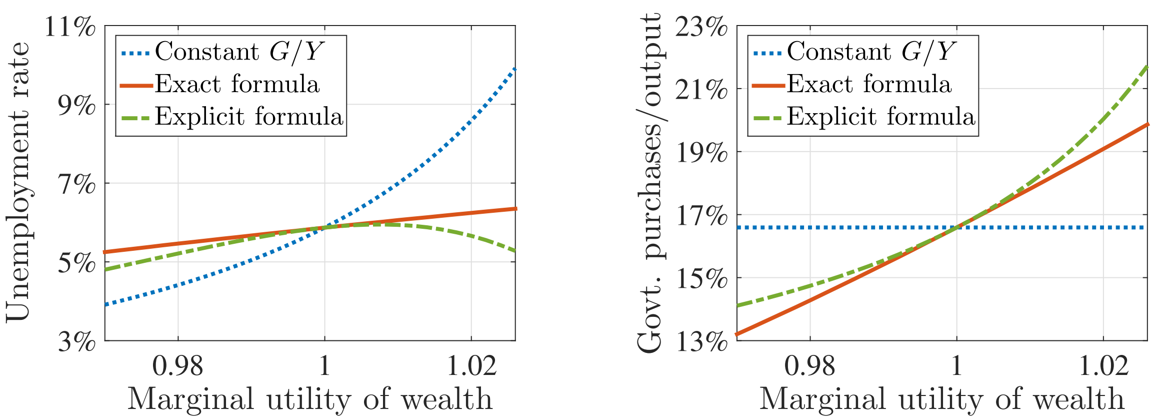
Figure A6 displays simulations for $\varepsilon = 0.5$ and $\varepsilon = 2$. The figure shows that when ε is lower, the optimal government purchases-output ratio responds less to fluctuations in unemployment, and consequently, fluctuations in unemployment are less attenuated. When $\varepsilon = 0.5$ and the unemployment rate reaches 9.9%, optimal government purchases increase to $G/Y = 19.1\%$, which reduces the unemployment rate to 7.6%. But when $\varepsilon = 2$ and the unemployment rate reaches 9.9%, optimal government purchases increase to $G/Y = 19.9\%$, which reduces the unemployment rate to 6.3%. The figure also shows that the explicit formula (12) is more accurate for lower values of ε.

Figure A7 displays simulations for $dY/dG = 0.2$ and $dY/dG = 1$. The figure shows that a higher value of dY/dG does not imply that optimal government purchases respond more strongly to a rise in unemployment; it does imply, however, that fluctuations in unemployment are more attenuated. When $dY/dG = 0.2$ and the unemployment rate reaches 9.9%, optimal government purchases increase to $G/Y = 19.7\%$, which only reduces the unemployment rate to 8.8%. But when $dY/dG = 1$ and the unemployment rate reaches 9.9%, optimal government purchases increase to $G/Y = 18.5\%$, which reduces the unemployment rate to 6.2%. The figure also shows that the explicit formula (12) is more accurate for lower values of dY/dG.

[53] Appendix F establishes the link between β and the multiplier.

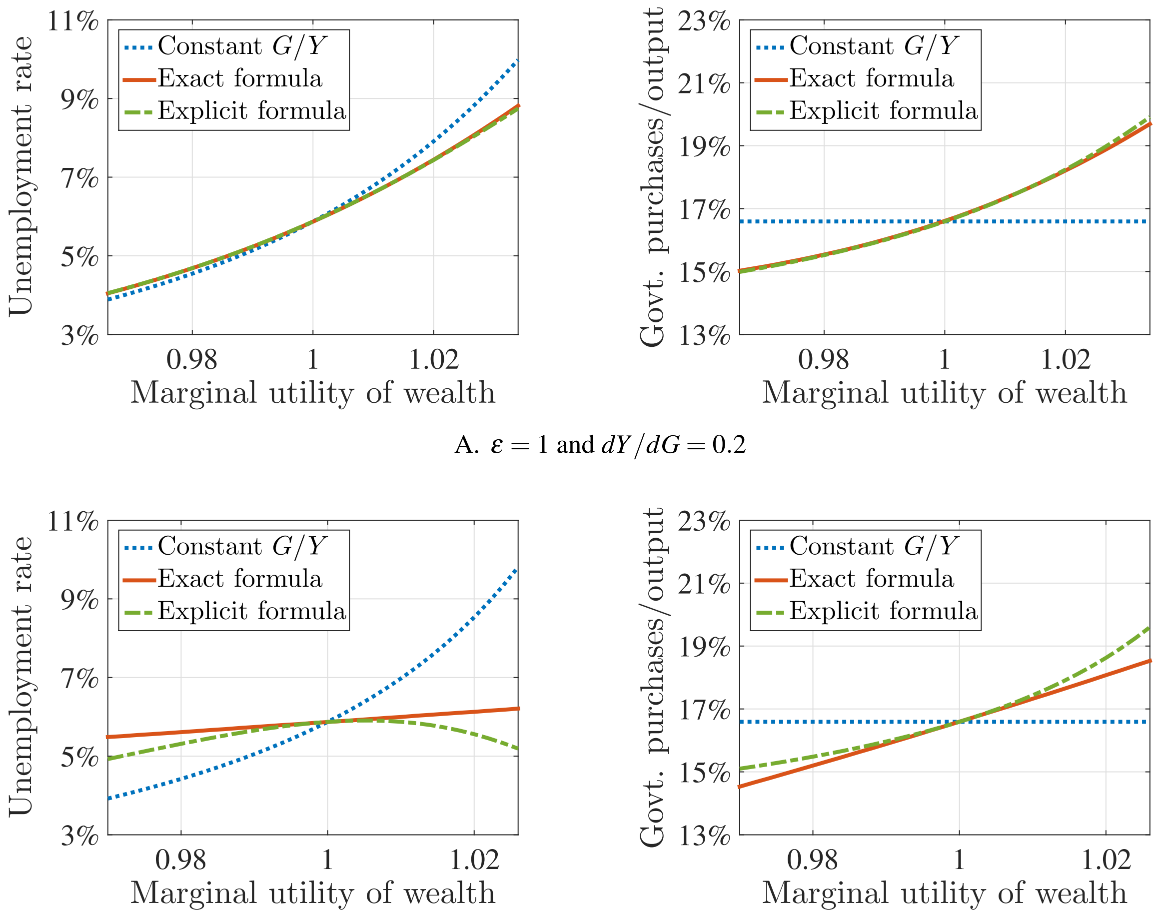


A. $\varepsilon = 0.5$ and $dY/dG = 0.6$



B. $\varepsilon = 2$ and $dY/dG = 0.6$

Figure A6: Simulations for Various Elasticities of Substitution



A. $\varepsilon = 1$ and $dY/dG = 0.2$

B. $\varepsilon = 1$ and $dY/dG = 1$

Figure A7: Simulations for Various Multipliers