ON THE EXISTENCE OF OPTIMAL DEVELOPMENT PLANS

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I. Introduction

A plan for economic development is a description of the production activities required of each firm and the commodity vectors assigned to each supplier of resources and consumer unit, over the lifetime of an economy. The objective of development planning is to choose from the set of feasible plans one that is "best" in terms of the planner's imputation of the society's welfare. In practical applications, development plans are usually to maximize an objective function over a finite horizon, subject to terminal conditions. However, the terminal conditions are derivable in principle from an optimization beyond the finite horizon of the plan, and at the most fundamental level, optimization must be considered over an infinite horizon.

In the infinite-horizon case, the set of feasible plans and the planner's preference relation generally lack the compactness and continuity properties which would automatically ensure the existence of "best" plans. Consequently, existence is conditioned on the specific structure of the economy.

The following example illustrates this observation, and suggests a relationship between existence and boundedness that is the principal result of this paper. Let b_t and c_t be real numbers denoting, respectively, aggregate output and consumption in period t, and assume that a feasible consumption program satisfies the inequalities

$$\begin{cases} b_{t+1} \leqq 4(b_t - c_t) & t = 0, 1, \ldots \\ b_t \geqq 0, \quad c_t \geqq 0 & t = 0, 1, \ldots \\ b_o \leqq 1. \end{cases}$$

These inequalities state that the output of the commodity in period t can be either diverted to consumption, or reinvested and quadrupled in the following period, and that initially one unit of the commodity is available. We may rewrite this system as a discrete-time control

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The question of existence of optimal growth programs has been treated previously by Koopmans [7] and Weizsacker [13] in the case of a one-sector model in which non-produced resources limit production; by Gale [3] in a multi-sector analogue of this model; by Gale and Sutherland [4] in the case of a one-sector model with unbounded production, but decreasing returns; and by the author [9] in the case of a one-sector model with only produced commodities and constant returns.

problem:

$$\Delta b_t = f(b_t, u_t) = (3 - 4u_t)b_t ,$$

$$c_t = u_t b_t ,$$

$$u_t \in [0, 1] .$$

Hence, the existence questions we consider are closely related to the existence of optima in variable-end-point discrete control problems.

Suppose that the economy has an objective function $\omega(c_t)$ measuring its performance in period t. A program (c_t') is said to strictly surpass a program (c_t'') if $\liminf_{H \to \infty} \sum_{t=0}^{H} [\omega(c_t') - \omega(c_t'')] > 0$; this relation is taken to define a partial ordering over feasible programs. We wish to find a feasible program which is maximal; i.e., strictly surpassed by no other feasible program.[1]

Consider the objective function

$$\omega(c_t) = \frac{1}{1 - \alpha} c_t^{1-\alpha}$$

for $\alpha > 0$, $\alpha \neq 1$. An elementary computation establishes that feasible programs satisfy $\sum_{t=0}^{\infty} 4^{-t} c_t \leqq 1$, $c_t \geqq 0$, and that a maximal program must be of the form $c_t' = K \cdot 4^{t/\alpha}$ for some scalar $K > 0$. But (c_t') of this form can satisfy $\sum_{t=0}^{\infty} 4^{-t} c_t' \leqq 1$ if and only if $\alpha > 1$. But this is just the case in which $\omega(c_t)$ bounded above, and we conclude that $\omega(c_t)$ bounded above is *necessary and sufficient* for the existence of a maximal program. This result, extended to a many-sector, many-consumer economy with a strictly productive von Neumann technology, is established in the remainder of this paper.

II. An Economic Model

Consider time as an infinite sequence of short periods, numbered $t = 0, 1, \ldots$. *In period* t, assume there are a finite number of commodities N_t, which may be indexed by $(1t), (2t), \ldots, (N_t t)$. Let

[1]This ordering relation is a natural way of making meaningful the maximization of a sum $\sum_{t=0}^{\infty} \omega(c_t)$ which may be divergent.

$$N = \{(nt) \mid n = 1, \ldots, N_t;\ t = 0, 1, \ldots\}$$

denote the set of all possible commodity indices.

A commodity vector is a real-valued function on N (hence, an infinite sequence of real numbers), and will be denoted by one of the lower case letters a, b, c, w, x, y, z. The notation c_t will be employed for the subvector of c containing the commodities of period t (i.e., the components of c with second index t), and the notation c_{nt} will be used for the (nt)-component of c.

Let $\mathfrak{Z}$ denote the real linear vector space of all commodity vectors, and let Ω denote its nonnegative orthant (defined as the cone of vectors for which every component is nonnegative).

For any finite horizon H, define a linear operator $M_H: \mathfrak{Z} \to \mathfrak{Z}$ by the condition

$$c' = M_H c \text{ if } c_{nt}' = \begin{cases} c_{nt} & \text{for } t < H \\ 0 & \text{for } t \geqq H . \end{cases}$$

Then, M_H is simply the projection operator from $\mathfrak{Z}$ into the subspace of commodities antedating the horizon H. Let $M_H': \mathfrak{Z} \to \mathfrak{Z}$ denote the projection operator into the complementary subspace; i.e.,

$$c'' = M_H' c \text{ if } c_{nt}'' = \begin{cases} 0 & \text{for } t < H \\ c_{nt} & \text{for } t \geqq H . \end{cases}$$

An aggregate production program for the economy is given by a vector $y \in \mathfrak{Z}$ of net outputs of the production sector; i.e., y_{nt} is negative if there is a net input of commodity nt to this sector, and is nonnegative otherwise. The aggregate production possibilities of the economy are defined by a subset Y of $\mathfrak{Z}$ containing all net output vectors y which are consistent with the production technology. The following assumption is made on the set Y.[2]

Assumption 1. The production possibility set Y is a subset of $\mathfrak{Z}$ with the properties:

(i) Non-increasing returns to factors and constant returns to scale: Y is a convex cone;

(ii) Outputs require chronologically prior inputs: $y \in Y$ and $M_H y = 0$ imply $y_H \leqq 0$, for H = 0, 1, 2, ...;

(iii) Free disposal: $y \in Y$ and $y' \leq y$ imply $y' \in Y$.

(iv) The set Y is closed under coördinate-wise convergence: $M_H Y$ is a closed subset of the finite-dimensional subspace $M_H \mathfrak{Z}$.[3]

[2] Brief economic justifications for these conditions can be found in Debreu [1] and Koopmans [7].

[3] Hence, if a sequence $\{y^\nu \mid \nu = 1, 2, \ldots\} \subsetneq Y$ has $y^\nu_{nt} \to y^o_{nt}$ for each

The economy has a given commodity vector $\bar{w} \in \Omega$ of initial resources. Addition of a net output program $y \in Y$ and the initial resource vector $\bar{w}$ yields an aggregate net supply program $c = y + \bar{w}$ to the consumption sector. Let $F = \{c \in \mathcal{J} \mid c = y + \bar{w},\ y \in Y\}$ denote the set of possible net supply programs.

A possible net supply program $c \in F$ is feasible if it is non-negative (i.e., requires no inputs of resources from outside the economy, other than those possibilities for "aid" and "trade" already included in the specification of $\bar{w}$ and Y). A consistent development plan requires a net supply program chosen from the set $F_o = \{c \in F \mid c \geqq 0\}$ of feasible commodity vectors.

If a positive amount of a commodity (nt) is unobtainable in any commodity vector in the feasible set F_o, it is irrelevant to choices among feasible alternatives, and can be deleted from the set of commodities N. Hence, we can assume without loss of generality that the projection $M_H F_o$ spans the space $M_H \mathcal{J}$ for any finite H.

Let $\mathcal{X}$ denote the linear space spanned by F_o, and define its nonnegative orthant $\Gamma = \Omega \cap \mathcal{X}$ and the set $F_1 = F \cap \mathcal{X}$ of possible net supply programs it contains. Define the set $F_2 = \{\theta c' - (1-\theta)c'' \mid 0 \leqq \theta \leqq 1;\ c',\ c'' \in F_o\}$. When F_o is convex, any point $c \in \mathcal{X}$ has a representation $c = \alpha c' - \beta c''$ with $\alpha,\ \beta \geqq 0$ and $c',\ c'' \in F_o$. Then $\frac{1}{\mu} c \in F_2$ for $\mu \geqq \alpha + \beta$, and the support function $\phi(c) = \inf\{\mu \mid \mu > 0,\ \frac{1}{\mu} c \in F_2\}$ exists and is nonnegative, positively linear homogeneous, and convex on $\mathcal{X}$ (cf. Dunford [2], p. 411). For any scalar α, $\phi(\alpha c) = |\alpha|\, \phi(c)$, since $F_2 = -F_2$. Hence $\phi(c)$ is an pseudo-norm on $\mathcal{X}$. (Cf. Kelley [6], p. 15.) This conclusion is strengthened by the following lemma when Assumption 1 holds.

Lemma 1. If Assumption 1 holds, then the set F_o is convex, is sequentially compact in the product topology of $\mathcal{X}$, and has the free disposal property that $c \in F_o$ and $0 \leqq c' \leqq c$ imply $c' \in F_o$. The support function $\phi(c)$ of F_2 is a norm on the space $\mathcal{X}$, and $\mathcal{X}$ is a complete space in this norm (i.e., a Banach space).

Corollary. The norm ϕ satisfies the bound $|c_{nt}| \leqq \phi(c)\sigma_{nt}$ for $c \in \mathcal{X}$, where $\sigma_{nt} = \sup\{c_{nt} \mid c \in F_o\} > 0$.

Proof: It follows trivially from Assumption 1 that F_o is closed in the product topology of $\mathcal{X}$, is convex, and has the free disposal property. A standard argument by contradiction will establish that F_o is bounded. Suppose $M_H F_o$ is a bounded subset of $M_H \mathcal{X}$, but

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(nt) $\in$ N, then $y^o \in Y$. Equivalently, Y is a closed subset of $\mathcal{J}$ in the product topology of $\mathcal{J}$.

that $M_{H+1} F_o$ contains a sequence $\{c^\nu\}$ with $\{c^\nu_H\}$ unbounded. Then, $y^\nu = c^\nu - M_{H+1}\dot{w}$ has $\{M_H y^\nu\}$ bounded and $\{y^\nu_H\}$ unbounded. Define $\hat{y}^\nu = y^\nu / |y^\nu_H|$. Then $\{\hat{y}^\nu\}$ is contained in $M_{H+1} Y$ by Assumption 1(i), and is bounded. Hence, there exists a subsequence of the $\{\hat{y}^\nu\}$ converging to a point $\hat{y}^o \epsilon M_{H+1}\mathfrak{X}$ satisfying $\hat{y}_t = 0$ for $t \neq H$ and $\hat{y}^o_H \geqq 0$, $|\hat{y}^o_H| = 1$. By Assumption 1(iv), $y^o \epsilon M_{H+1} Y$, contradicting Assumption 1(ii). Hence $M_{H+1} F_o$ is bounded for all H. Since F_o is coordinate-wise closed and bounded, it follows that out of every sequence in F_o can be extracted a coordinate-wise convergent subsequence. Hence, F_o is sequentially compact in the product topology.

The support function $\phi(c)$ of F_2 has been shown to be a pseudo-norm. Now consider $c \epsilon \mathfrak{X}$, $c \neq 0$. Then $c_{nt} \neq 0$ for some $nt \epsilon N$. By the boundedness of F_o, $\sigma_{nt} = \sup\{c_{nt} | c \epsilon F_o\}$ is finite. Hence, there exists a positive scalar ε such that $|c_{nt}| > \varepsilon \sigma_{nt}$, implying $\frac{1}{\varepsilon} c \notin F_2$ and $\phi(c) \geqq \varepsilon > 0$. It follows that $\phi(c)\sigma_{nt} \geqq |c_{nt}|$ for all $nt \epsilon N$, and ϕ is a norm on $\mathfrak{X}$.

A sequence $\{c^\nu\} \subseteq \mathfrak{X}$ is a Cauchy sequence in the ϕ-norm topology of $\mathfrak{X}$ (or more briefly, a ϕ-Cauchy sequence) if, for each $\varepsilon > 0$, there exists ν_o such that $\phi(c^{\nu'} - c^{\nu''}) < \varepsilon$ for $\nu', \nu'' \geqq \nu_o$. By the bound $\phi(c)\sigma_{nt} \geqq |c_{nt}|$ derived above, every ϕ-Cauchy sequence $\{c^\nu\} \subseteq \mathfrak{X}$ converges coordinate-wise to some point $c^o \epsilon \mathfrak{Z}$ (cf. Kelley [6], p. 57). Let $\tilde{\mathfrak{X}}$ denote the linear space of all such coordinate-wise limits of ϕ-Cauchy sequences in $\mathfrak{X}$. We now will show that $\tilde{\mathfrak{X}} = \mathfrak{X}$.

Suppose $c^o \epsilon \tilde{\mathfrak{X}}$. There exists a ϕ-Cauchy sequence $\{c^\nu\} \subseteq \mathfrak{X}$ converging coordinate-wise to c^o. Since $\{\phi(c^\nu)\}$ is bounded, we can assume (by rescaling) that $\phi(c^\nu) < 1$. Then, by construction there exists a scalar θ_ν, $0 \leqq \theta_\nu \leqq 1$, and commodity vectors a^ν, $b^\nu \epsilon F_o$ such that $c^\nu = \theta_\nu a^\nu - (1 - \theta_\nu)b^\nu$. By the sequential compactness of F_o, there exists a subsequence $\{a^\nu, b^\nu, \theta_\nu\}$ (retain notation) such that a^ν and b^ν converge coordinate-wise to respective limits a^o and b^o in F_o and θ_ν converges to a scalar θ_o. But then $c^o = \theta_o a^o - (1 - \theta_o)b^o$, implying $c^o \epsilon \mathfrak{X}$. Q. E. D.

The analysis of this paper will be restricted to economies in which (1) all commodities can be produced, (2) initial resources are eventually negligible relative to potential production, and (3) the technology exhibits constant returns and sufficient flexibility to allow substitution of "free" inputs for "non-free" ones. Such economies will generally have the property of <u>reachability</u> defined below.

Definition. A commodity vector $c \in F_o$ is reachable if for any positive scalar μ, there exists a horizon H such that $\mu M_H' c \in F_o$.

Reachability requires that starting from an arbitrarily small proportion of the initial resource vector, one can reach, by pure accumulation for a sufficiently long time, the levels of a previously feasible commodity vector. The economy will be called reachable if every $c \in F_o$ is reachable. The following lemma characterizes the space of continuous linear functionals on $\mathfrak{X}$ (in its ϕ-norm topology) for reachable economies.

Lemma 2. If Assumption 1 holds and the economy is reachable, then the space $\mathfrak{X}^*$ of all continuous linear functionals on $\mathfrak{X}$ (in its ϕ-norm topology) is a sequence space; i.e., every functional $p \in \mathfrak{X}^*$ can be represented as a vector $p \in \mathfrak{Z}$, with the value of the linear functional given by the infinite inner product $p(c) = \sum_{nt \in N} p_{nt} c_{nt} = p \cdot c$ for $c \in \mathfrak{X}$. Further, $p = (p_{nt}) \in \mathfrak{X}^*$ satisfies the bound $\sigma_{nt}^* |p_{nt}| \leqq \phi^*(p)$ for $nt \in N$, where ϕ^* is the dual norm on $\mathfrak{X}^*$

Proof: Consider any $p \in \mathfrak{X}^*$ Over the subspace $\mathfrak{X}_o$ of vectors in $\mathfrak{X}$ with a finite number of non-zero components, p can be represented by an infinite sequence $p' = (p_{nt}) \in \mathfrak{Z}$.* Hence, $p(M_H c) = p' \cdot M_H c$ for all finite H and $c \in \mathfrak{X}$. But c can be written $c = \alpha c' - \beta c''$ with α, β non-negative scalars and $c', c'' \in F_o$. Given $\varepsilon > 0$, reachability of the economy implies the existence of a horizon H_o such that $\phi(\frac{\alpha}{\varepsilon} M_H' c') < \frac{1}{2}$ and $\phi(\frac{\beta}{\varepsilon} M_H' c'') < \frac{1}{2}$ for $H \geqq H_o$. Hence,

$$\phi(c - M_H c) = \phi(M_H' c) \leqq \varepsilon[\phi(\frac{\alpha}{\varepsilon} M_H' c') + \phi(\frac{\beta}{\varepsilon} M_H' c'')] < \varepsilon ,$$

and the sequence $\{M_H c\}$ converges to c in the ϕ-norm. Hence, the continuous linear functional p satisfies $p(c) = \lim_{H \to \infty} p' \cdot M_H c = \sum_{nt \in N} p_{nt} \cdot c_{nt}$.

The bound on p_{nt} follows from the definition $\phi^*(p) = \sup\{p \cdot c \mid \phi(c) \leq 1\}$ of the dual norm. Q. E. D.

A finite number of consumer units K_t are assumed to come into existence in the economy in each time period t, and may be indexed by (1t), ..., $(K_t t)$. Let $K = \{(kt) \mid k = 1, \ldots, K_t; t = 0, 1, \ldots\}$ denote the set of all consumer units.

*The linear functional p is continuous on any finite-dimensional subspace of $\mathfrak{X}$, and hence has a unique representation as a vector on this subspace. Induction over successively large finite-dimensional subspaces establishes the conclusion.

Each consumer unit is assumed to have a finite lifetime. Let $\mathfrak{X}_{kt}$ denote the finite-dimensional linear subspace of $\mathfrak{X}$ containing all commodity vectors which have zero components outside the lifetime of the consumer unit kt. A desired set D_{kt} contained in the non-negative orthant $\Gamma_{kt} = \Gamma \cap \mathfrak{X}_{kt}$ of $\mathfrak{X}_{kt}$ specifies the commodity vectors on which this consumer unit can subsist.

A consumption plan for the economy is a list of the commodity vectors going to each of the consumer units; i.e., a function $s: K \to \mathfrak{X}$, representable as a vector $s = (c^{kt})$ with vector components $c^{kt} \in \mathfrak{X}_{kt}$ for each $kt \in K$. A consumption plan $s = (c^{kt})$ is desired if every consumer unit receives a commodity vector in its desired set and is feasible if the aggregate commodity vector is a feasible net supply vector. Let S denote the set of desired feasible consumption plans.

The problem of optimal economic planning can now be stated formally. The economic planner has a preference ordering over the set of desired feasible consumption plans which reflects his evaluation of social welfare, and he wishes to select a "best" plan. At the most general level, each consumer unit can be assumed to have a preference ordering over its desired set, and the planner can be taken to have a partial or complete ordering determined as a function of his information (real or fancied, ordinal or cardinal) on an individual's preferences.[4] In discussing the existence of "best" programs, we shall restrict our attention to a special case in which the planner assigns to each consumer unit a concave "unit welfare" function over its desired set, and assumes that "social welfare" for any finite group of consumer units is given by the sum of their unit welfare functions.[5]

Let $u(c, kt): D_{kt} \to R$ denote the unit welfare function assigned by the planner to the unit $kt \in K$. For a shift from the first to the second of any two desired consumption plans $s = (c^{kt})$ and $\hat{s} = (\hat{c}^{kt})$, the gain in social welfare for the consumer units coming into existence before time H is measured by the planner as

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[4]A particular example would be the case when the planner has a partial ordering consistent with the Pareto ordering of plans by the consumer units. The possibility of competitive decentralization in this model has been explored by the author in [10].

[5]We avoid the question of whether the unit welfare functions are consistent with individual preferences. If they are, our decentralization results can be taken to apply to maximization of "true" preferences; if not, units must be viewed as carrying out a "Lange-Lerner" type maximization by command of the planner. The assumption that social welfare is a separable function of unit welfare levels is one of the analytically most tractable of a variety of possible forms for a social welfare function, but there seems to be no deeper justification for its privileged position in the economics literature.

$$V_H(\hat{s}, s) = \sum_{kt \in K_H} [u(\hat{c}^{kt}, kt) - u(c^{kt}, kt)],$$

where $K_H = \{kt \in K | t < H\}$. The plan $\hat{s}$ catches up to the plan s if $\liminf_{H \to \infty} V_H(\hat{s}, s) \geqq 0$, and $\hat{s}$ strictly surpasses s if $\liminf_{H \to \infty} V_H(\hat{s}, s) > 0$. We assume that the planner partially orders consumption plans by the catches up to relation.[6] A desired feasible consumption plan $\bar{s}$ is maximal in S if no other plan in S strictly surpasses it, and is optimal in S if it catches up to all other programs in S.

A desired, feasible consumption plan $\bar{s} = (\bar{c}^{kt})$ and a non-zero "price" vector $p = (p_{nt}) \in \mathfrak{Z}$ define a valuation competitive equilibrium if

(1) p is a continuous linear functional on $\mathfrak{X}$ in the ϕ-norm topology of $\mathfrak{X}$,

(2) $u(c, kt) - p \cdot c \leqq u(\bar{c}^{kt}, kt) - p \cdot \bar{c}^{kt}$ for all $c \in D_{kt}$, and

(3) $p \cdot c \leqq p \cdot \bar{c}$ for all $c \in F_1$, where $\bar{c} = \sum_{kt \in K} \bar{c}^{kt}$.

The first of these conditions requires that the present value of any feasible net supply program be well defined. The second implies maximization of unit welfare subject to a budget constraint, and the third requires maximization of the present value of output among feasible net supply programs.

The desired sets and unit welfare functions are required to meet the following regularity conditions.

Assumption 2.[7] For each consumer unit $kt \in K$, the desired set D_{kt} is a convex subset of Γ_{kt} with the properties

(i) $c \in \Gamma_{kt}$ and $c \geqq c' \in D_{kt}$ imply $c \in D_{kt}$;

(ii) $\Gamma_{kt} \setminus D_{kt}$ is a closed subset of $\mathfrak{X}_{kt}$.

The unit welfare function u(c, kt) defined on D_{kt} has the properties

(iii) u(c, kt) is concave, nondecreasing, and continuous on D_{kt}, and no finite $c \in D_{kt}$ maximizes u (i. e., nonsatiation

(iv) if a sequence $\{c^{(\nu)}\} \subseteq D_{kt}$ converges to a vector $c^{(0)} \notin D_{kt}$, then $u(c^{(\nu)}, kt)$ converges to $-\infty$.

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[6] This partial ordering has come to be known as the Ramsey-Weizsäcker "overtaking" criterion, cf. [9], p. 27.

[7] Brief discussions of the economic rationale for these conditions are given in Koopmans [7, 8] and in [9], pp. 28, 36.

In reachable economies satisfying Assumptions 1 and 2, the author has established the equivalence of maximal, competitive, and optimal programs ([9], pp. 32 - 35), as summarized in the following theorem.

Theorem 1. Suppose Assumptions 1 and 2 hold and the economy is reachable. If $\bar{s}$ is a maximal desired feasible consumption plan, then there exists a non-zero price vector $p = (p_{nt}) \in \mathfrak{Z}$ such that $(\bar{s}, p)$ is a valuation competitive equilibrium. If $(\bar{s}, p)$ is a valuation competitive equilibrium, then $\bar{s}$ is optimal.

As a consequence of this theorem, the question of existence of "best" plans under the overtaking criterion can be confined to tests for the existence of plans $\bar{s} \in S$ satisfying $\liminf_{H \to \infty} V_H(\bar{s}, s) \geqq 0$ for all $s \in S$.

III. On the Existence of Optimal Development Plans

Investigation of the existence of optimal consumption plans is conveniently divided into the case in which conventional continuity and compactness arguments can be applied and the case in which existence depends on the precise structural characteristics of the economy.

For the first case, suppose the "total social welfare" of the economy can be represented by a real-valued function $v: S \to [-\infty, +\infty)$ over desired feasible consumption plans.[8] Suppose $v(\hat{s}) > -\infty$ for some $\hat{s} \in S$, and suppose that S can be topologized in such a way that $S_1 = \{ s \in S \mid v(s) \geqq v(\hat{s}) \}$ is compact and v is continuous on S_1. One can then conclude immediately that a plan $\bar{s} \in S$ exists maximizing v. An application of this argument, employing the product topology, gives an easy existence theorem for the common economic case of "discounting of the future" (cf. [9], p. 38).

Theorem 2. Suppose Assumptions 1 and 2 hold.[9] Suppose there exist (1) a desired feasible plan $\hat{s} = (\hat{c}^{kt})$, (2) a factorization of each unit welfare function $u(c, kt) = \gamma_{kt} \tilde{u}(c, kt)$ (where the γ_{kt} are nonnegative "discount factors" satisfying $\sum_{kt \in K} \gamma_{kt} \leqq 1$), and (3) a large positive scalar M, such that $\tilde{u}(\hat{c}^{kt}, kt) \geqq -M$ and $\sup\{ \tilde{u}(c, kt) \mid c \in F_o \cap D_{kt} \} \leqq M$. Then, an optimal consumption plan exists.

[8] In our previous terminology, the overtaking criterion for this case examines $V_\infty(s, \hat{s}) = v(s) - v(\hat{s})$ for $s, \hat{s} \in S$ with $v(\hat{s}) > -\infty$.

[9] The conclusion continues to hold if one imposes only conditions (ii) and (iv) of Assumption 1; conditions (i), (ii), and (iv) of Assumption 2; the requirement that unit welfare functions be continuous on their desired sets; and the requirement that output in a period be bounded if all chronologically prior inputs are bounded.

We shall now derive a number of conditions for existence in reachable economies where the hypotheses of Theorem 2 may not be met. We first define the concepts of a "good" plan and an "optimizing sequence" of plans.

Definition. A desired (but not necessarily feasible) consumption plan $\hat{s}$ is good if there is an upper bound on the increase in social welfare which can be achieved, by the group of consumer units existing prior to any finite horizon H, by shifting to any desired feasible plan; i.e., there exists $M \geqq 0$ such that $V_H(s, \hat{s}) \leqq M$ for all $s \in S$ and $H = 1, 2, \ldots$.

Definition. A sequence $\{s^\nu\}$ of desired feasible consumption plans is an optimizing sequence if each s^ν is comparable to all $s \in S$ (i.e., the limit $V_\infty(s, s^\nu) = \lim_{H \to \infty} V_H(s, s^\nu)$ exists and is a value in $[-\infty, +\infty)$ for each $s \in S$) and $\lim_{\nu \to \infty} V_\infty(s, s^\nu) \leqq 0$ for all $s \in S$.

Associated with a consumption plan $s = (c^{kt})$ is an aggregate net supply vector which we will denote by $c(s) = \sum_{kt \in K} c^{kt}$. Let $\mathscr{S}$ denote the linear space of plans s with $c(s) \in \mathfrak{X}$. Recalling that F_1 is the set of possible net supply vectors in $\mathfrak{X}$ and $F_2 \subseteq F_1$, define the support function of F_1, $\psi(c) = \inf\{\mu > 0 \mid \frac{1}{\mu} c \in F_1\}$ for $c \in \mathfrak{X}$. Then, $\psi(c)$ is convex, $0 \leqq \psi(c) \leqq \phi(c)$ for $c \in \mathfrak{X}$, and $\psi(c) = \phi(c)$ for $c \in \Gamma$. For reachable economies, the following relationship can be established between the existence of good programs and optimizing sequences.

Theorem 3. Suppose a reachable economy satisfies Assumptions 1 and 2, and has a good feasible consumption plan $\hat{s}$. Then, the following conclusions hold:

(a) All good plans in S are comparable (i.e., $V_\infty(s, s') = \lim_{H\to\infty} V_H(s, s')$ exists for any good $s, s' \in S$), and any good plan $s \in S$ strictly surpasses any plan $s' \in S$ which is not good (precisely, $V_\infty(s', s) = -\infty$).

(b) There exists an optimizing sequence $\{s^\nu \mid \nu = 1, 2, \ldots\}$ $\subseteq S$ (i.e., $\lim_{\nu\to\infty} V_\infty(s^\nu, \hat{s}) = \bar{v} \geqq V_\infty(s, \hat{s})$ for all $s \in S$), which can be chosen to satisfy $\psi(c(s^\nu)) = 1$.

(c) There exists a non-zero price vector $p = (p_{nt})$ in the space $\mathfrak{X}^*$ of continuous linear functionals on $\mathfrak{X}$ such that $\lim_{\nu\to\infty} p \cdot c(s^\nu) = \lambda > 0$ exists, $p \cdot c \leqq \lambda\psi(c)$ for all $c \in \mathfrak{X}$, and $V_\infty(s, \hat{s}) - \bar{v} \leqq p \cdot c(s) - p \cdot c(s^\nu)$ for all good $s \in \mathscr{S}$ and any $\nu = 1, 2, \ldots$.

(d) The sequence $\{s^\nu\}$ has a subsequence converging coordinate-wise to a desired feasible plan $\bar{s} = (\bar{c}^{kt})$ such that $u(c, kt) - u(\bar{c}^{kt}, kt) \leqq p\cdot(c - \bar{c}^{kt})$ for all $c \in D_{kt}$ and $kt \in K$. The plan $\bar{s}$ is good. If $p\cdot c(\bar{s}) = \lambda$, then $\bar{s}$ is optimal and $(\bar{s}, p)$ defines a valuation competitive equilibrium. If $\lambda > \sup\{p\cdot c(s) \mid s$ is a coordinate-wise cluster point of some optimizing sequence$\}$, then no optimum exists.

(e) If $\lim_{\nu\to\infty} \phi(c(s^\nu) - c(\bar{s})) = 0$, then $\bar{s}$ is optimal.

(f) If $\lim_{H\to\infty} [\sup\{p\cdot M_H' c \mid c \in F_o\}] = 0$, then $\bar{s}$ is optimal.

Proof: The first portion of the proof will establish the existence of a positive scalar λ and a non-zero vector p such that the inequality

$$(1) \qquad \liminf_{H\to\infty} V_H(s, \hat{s}) - \bar{v} + \lambda(1 - \psi(c')) - p\cdot c(s) + p\cdot c' \leqq 0$$

holds for all good $s \in \mathscr{S}$ and all $c' \in \mathscr{X}$, where

$$\bar{v} = \sup\left\{\liminf_{H\to\infty} V_H(s, \hat{s}) \mid \text{good } s \in S\right\}$$

This inequality is used to establish first the conclusions (a) and (b), and then the properties of the price system in (c).

Note that $\hat{s}$ good implies $\bar{v}$ finite. Define a linear space $\mathscr{W} = R^2 \times \mathscr{X}$ with points $w = (\alpha, \beta, x) \in \mathscr{W}$ assigned the norm $\|w\| = |\alpha| + |\beta| + \phi(x)$. Define the set

$$W = \{(\alpha, \beta, x) \in \mathscr{W} \mid \alpha \leqq \liminf_{H\to\infty} V_H(s, \hat{s}),\ \beta \leqq 1 - \psi(c'),$$
$$x \geqq c(s) - c' \text{ for some good } s \in \mathscr{S},\ c' \in \mathscr{X}\}\ .$$

The set of good $s \in \mathscr{S}$ is convex by Assumption 2, $-\psi$ is concave on $\mathscr{X}$, and $\liminf_{H\to\infty} V_H(s, \hat{s})$ is concave on the set of good $s \in \mathscr{S}$. Hence, W is a convex set.

The neighborhood $\|w - \hat{w}\| < \frac{1}{2}$ of $\hat{w} = (-1, -1, 0)$ will be shown to be contained in W. For any (α, β, x) in this neighborhood, take $s = \hat{s}$ and $c' = c(\hat{s}) - x$. Then, $\psi(c') \leqq \phi(c') \leqq \phi(c(\hat{s})) + \phi(x) < \frac{3}{2}$, and the point $(0, -\frac{1}{2}, x)$ is in W. But $\alpha, \beta < -\frac{1}{2}$ then implies $(\alpha, \beta, x) \in W$.

The point $(\bar{v}, 0, 0)$ is in the boundary of W, since $\psi(c') \leq 1$ and $c(s) \geqq c'$ imply the good s is feasible, implying $\bar{v} \geqq \liminf_{H\to\infty} V_H(s, \hat{s})$. Since the interior of W is non-empty there exists a non-zero continuous linear functional $(\gamma, \lambda, -p) \in R^2 \times \mathscr{X}^*$ such that $\gamma\alpha + \lambda\beta - p\cdot x \leq \gamma\bar{v}$ for $(\alpha, \beta, x) \in W$. The construction of W implies that γ, λ, and p are non-negative. If $\lambda = 0$, taking $s = \hat{s}$ and $c' = c(\hat{s}) - x$ yields the inequality

$p \cdot x \leqq \alpha \bar{v}$, implying $p = 0$, $\alpha > 0$, and $\liminf_{H \to \infty} V_H(s, \hat{s}) \leqq \bar{v}$ for all good $s \in \mathscr{S}$. But Assumption 2(iii) implies the existence of $\tilde{c}^{10} \in D_{10}$ such that $u(\tilde{c}^{10}, 10) \geqq \max\{u(c, 10) | c \in F_o \cap D_{10}\} + \varepsilon$ for some $\varepsilon > 0$, and the definition of $\bar{v}$ implies the existence of a good $\check{s} \in S$ such that $\liminf_{H \to \infty} V_H(\check{s}, \hat{s}) \geqq \bar{v} - \varepsilon/2$. Then $s \in \mathscr{S}$, defined by $c^{10} = \check{c}^{10} + \tilde{c}^{10}$ and $c^{kt} = \check{c}^{kt}$ for kt otherwise, has $\liminf_{H \to \infty} V_H(s, \hat{s}) \geqq \bar{v} + \varepsilon/2$, contradicting the supposition that $\lambda = 0$.

Since $c(\hat{s})$ is reachable, there exists H such that $\phi(M_H' c(\hat{s})) < \frac{1}{2}$. Define $\check{c}^{kt} = (1 - \theta)M_H\hat{c}^{kt} + M_H'\hat{c}^{kt}$. Then $\check{s} = (\check{c}^{kt}) \in \mathscr{S}$, and by Assumption 2(ii) and (iv), $\check{s}$ is good, with $\liminf_{H \to \infty} V_H(\check{s}, \hat{s}) < \bar{v}$ for some small positive θ. But $\psi(c(\check{s})) \leqq \phi([(1 - \theta)M_H + M_H']c(\hat{s})) = \phi((1 - \theta)c(\hat{s}) + \theta M_H' c(\hat{s})) \leqq (1 - \theta)\phi(c(\hat{s})) + \frac{\theta}{2} \leqq 1 - \frac{\theta}{2}$. Taking $s = \check{s}$ and $c' = c(\check{s})$ yields the inequality $\gamma \liminf_{H \to \infty} V_H(\check{s}, \hat{s}) - \bar{v} + \lambda(1 - \psi(c(\check{s})) \leqq 0$, implying $\gamma > 0$. Normalize $\gamma = 1$. Then, (1) holds.

Suppose $\tilde{s} = (\tilde{c}^{kt}) \in S$ is a good plan with $V_H(\hat{s}, \tilde{s}) < \tilde{M}$ $(\tilde{M} \geqq 0)$ Let $K^+(\tilde{s}) = \{(kt) \in K | u(\tilde{c}^{kt}, kt) \geqq u(\hat{c}^{kt}, kt)\}$. Then

$$\sum_{kt \in K^+(\tilde{s})} [u(\tilde{c}^{kt}, kt) - u(\hat{c}^{kt}, kt)] \leqq \limsup_{H \to \infty} V_H(\tilde{s} + \hat{s}, \hat{s}) \leqq \bar{v} + \lambda\psi(c(\tilde{s})),$$

implying $\sum_{kt \in K} |u(\tilde{c}^{kt}, kt) - u(\hat{c}^{kt}, kt)| \leqq \tilde{M} + 2\bar{v} + 2\lambda\psi(c(\tilde{s}))$. Hence, $V_\infty(s, \hat{s}) = \lim_{H \to \infty} V_H(s, \hat{s})$ exists for all good $s \in \mathscr{S}$. Alternately, suppose $\tilde{s} \in S$ is not a good plan, and define $s^H = (c^{kt}(H)) \in \mathscr{S}$ by $c^{kt}(H) = \tilde{c}^{kt}$ for $t < H$ and $c^{kt}(H) = \hat{c}^{kt}$ for $t \geqq H$. Then $\psi(c(s^H)) \leqq 2$, implying

$$\sum_{kt \in K^+(s^H)} [u(c^{kt}(H), kt) - u(\hat{c}^{kt}, kt)] \leqq \bar{v} + 2\lambda .$$

Hence, letting $H \to \infty$, $\sum_{kt \in K^+(\tilde{s})} [u(\tilde{c}^{kt}, kt) - u(\hat{c}^{kt}, kt)] \leqq \bar{v} + 2\lambda$. Since $\tilde{s}$ is not good, one then has $\liminf_{H \to \infty} V_H(\hat{s}, \tilde{s}) = +\infty$, and $\hat{s}$ strictly surpasses $\tilde{s}$. We conclude that $V_\infty(s, \hat{s}) = \lim_{H \to \infty} V_H(s, \hat{s})$ exists, and is a point in $[-\infty, \bar{v}]$, for all desired $s \in S$. Hence, a sequence of good $\{s^\nu\} \subseteq S$ with $\lim_{\nu \to \infty} V_\infty(s^\nu, \hat{s}) = \bar{v}$ satisfies the definition of an optimizing sequence. By Assumption 2(iii), $\{s^\nu\}$ can be taken to satisfy $\psi(c(s^\nu)) = 1$. Hence, (a) and (b) hold.

The inequality (1) with $c' = c(s^\nu)$ yields the inequality $V_\infty(s, \hat{s}) - \bar{v} \leqq p \cdot c(s) - p \cdot c(s^\nu)$ for good $s \in \mathscr{S}$. Then $\limsup_{\nu \to \infty} p \cdot c(s^\nu) \leqq p \cdot c(s) + \bar{v} - V_\infty(s, \hat{s})$. Substituting s^ν for s and noting that $\lim_{\nu \to \infty} V_\infty(s^\nu, \hat{s}) = \bar{v}$ we have $\limsup_{\nu \to \infty} p \cdot c(s^\nu) \leqq \liminf_{\nu \to \infty} p \cdot c(s^\nu)$, implying that $\lim p \cdot c(s^\nu)$ exists. The inequality (1) with $s = s^\nu$ yields the inequality (as $\nu \to \infty$) $\lambda[\psi(c') - 1] \geqq p \cdot c' - \lim_{\nu \to \infty} p \cdot c(s^\nu)$. Taking $c' = 0$ and $c' = 2c(s^\nu)$ in this inequality yields (taking $\nu \to \infty$) the equality $\lambda = \lim_{\nu \to \infty} p \cdot c(s^\nu)$, giving the inequality $\lambda\psi(c) \geqq p \cdot c$ for all $c \in \mathscr{X}$. Hence, (c) holds.

Since the optimizing sequence $\{s^\nu\}$ has $c^{kt}(\nu) \in F_o$ and $c(s^\nu) \in F_o$, Lemma 1 implies the existence of a subsequence (retain notation) converging coordinate-wise to a feasible plan $\bar{s} \in \mathscr{S}$. By Assumption 2(ii) and (iv), $\bar{c}^{kt} \in D_{kt}$ for all $kt \in K$. In the inequality (1), take $\tilde{s} = (\tilde{c}^{kt})$ with $\tilde{c}^{kt} = c^{kt}(\nu)$ for $kt \in K$, $kt \neq 10$, and take $c' = c(s^\nu)$, yielding $V_\infty(s^\nu, \hat{s}) - \bar{v} + [u(\tilde{c}^{10}, 10) - u(c^{10}(\nu), 10)] - p \cdot \tilde{c}^{10} + p \cdot c^{10}(\nu) \leqq 0$. The limit of this inequality as $\nu \to \infty$ is then

$$u(\tilde{c}^{10}, 10) - u(\bar{c}^{10}, 10) - p \cdot \tilde{c}^{10} + p \cdot \bar{c}^{10} \leqq 0 .$$

Since 10 can be replaced by any $kt \in K$, $(\bar{s}, p)$ satisfies the first two conditions of a valuation competitive equilibrium. Summing the last inequality over $kt \in K$, one has $V_H(\tilde{s}, \bar{s}) \leqq p \cdot M_H(c(\tilde{s}) - c(\bar{s})) \leqq p \cdot c(\tilde{s}) \leqq \lambda$ for all $\tilde{s} \in S$ and $H = 1, 2, \ldots$, and $\bar{s}$ is good. Now suppose that $\lambda = \lim_{\nu \to \infty} p \cdot c(s^\nu) = p \cdot c(\bar{s})$. Then, the inequality $p \cdot c \leqq \lambda\psi(c)$ for all $c \in \mathscr{X}$ implies $p \cdot c \leqq p \cdot c(\bar{s})$ for $c \in F_1$, and $(\bar{s}, p)$ is a valuation competitive equilibrium. By Theorem 1, $\bar{s}$ is then optimal. If an optimal plan s^* exists, then $\bar{v} = V_\infty(s^*, \hat{s})$, and setting $c' = c(s^*)$ in the inequality (1) yields the inequality $V_\infty(s, \hat{s}) - V_\infty(s^*, \hat{s}) - p \cdot c(s) + p \cdot c(s^*) \leqq 0$, or $V_\infty(s, s^*) \leqq p \cdot c(s) - p \cdot c(s^*)$. Taking $s = s^*$ in (1) yields the inequality $\lambda[1 - \psi(c')] - p \cdot c(s^*) + p \cdot c' \leqq 0$, and $c' = 0$ yields $\lambda \leqq p \cdot c(s^*)$. Then $s^\nu \equiv s^*$ is an optimizing sequence with $\lim_{\nu \to \infty} p \cdot s^\nu = \lambda$. Hence, (d) holds.

If $\lim_{\nu \to \infty} \phi(c(s^\nu) - c(\bar{s})) = 0$, then the continuous linear functional p satisfies $\lim_{\nu \to \infty} p \cdot c(s^\nu) = p \cdot c(\bar{s})$, and conclusion (d) implies (e).

Suppose $\lim_{H\to\infty} [\sup\{p \cdot M_H^{\prime} c \mid c \in F_o\}] = 0$. Then, $\lim_{\nu\to\infty} p \cdot c(s^\nu) = p \cdot M_H c(\bar{s}) + \lim_{\nu\to\infty} p \cdot M_H^{\prime} c(s^\nu)$. Given $\varepsilon > 0$, there exists H_o such that $p \cdot M_H^{\prime} c(s^\nu) < \varepsilon$ for all $\nu = 1, 2, \ldots$ and $H \geqq H_o$. Then, $\lim_{\nu\to\infty} p \cdot c(s^\nu) = \lim_{H\to\infty} p \cdot M_H c(\bar{s}) = p \cdot c(\bar{s})$, and conclusion (d) implies (f). Q. E. D.

A series of examples will illustrate the sensitivity of the existence of good and optimal programs to the precise structure of the economy. Consider an economy with one commodity per period, an initial resource vector $\bar{w}$ with $\bar{w}_o = 0$ for $t > 0$, and a single production activity allowing storage of the commodity without depreciation. Then,

$$F_o = \left\{ c \geqq 0 \,\middle|\, \sum_{t=0}^{\infty} c_t \leqq 1 \right\} .$$

Suppose further that there is one consumer per period, with a "lifetime" of one period, and suppose that the unit welfare function can be written $u(c^{1t}, 1t) = \omega(c_t)$; i. e., all consumers have the same unit welfare function.

If $\omega(c_t) = -c_t^{-1}$, no good plan exists in this economy. If

$$\omega(c_t) = \frac{c_t}{1 + c_t} ,$$

all feasible plans are good, but no optimal plan exists. If $\omega(c_t) = c_t$, an optimal plan exists. We will now verify these conclusions. Suppose in the case $\omega(c_t) = -c_t^{-1}$, a good plan $\tilde{c}$ did exist. Define a feasible desired plan $\hat{c}$ by $\hat{c}_o = \hat{c}_1 = \frac{1}{2}\tilde{c}_o$ and $\hat{c}_t = \tilde{c}_{t-1}$, for $t \geqq 2$. Then,

$$\sum_{t=0}^{H} [\omega(\hat{c}_t) - \omega(\tilde{c}_t)] = -\frac{3}{\tilde{c}_o} + \frac{1}{\tilde{c}_H} .$$

But $\lim_{H\to\infty} \tilde{c}_H = 0$ implies that this sum is unbounded, contradicting the supposition that $\tilde{c}$ is good. In the case

$$\omega(c_t) = \frac{c_t}{1 + c_t} ,$$

one has

$$0 \leqq \sum_{t=0}^{H} \frac{c_t}{1 + c_t} \leqq \sum_{t=0}^{H} c_t \leqq 1$$

for any $c \in F_o$, so that any plan in F_o is good. However, an arbitrage argument using the strict concavity of ω implies that an optimal program would have to have equal consumption levels in any two periods, a condition which can be met for $c \in F_o$ only at the least desirable program $c = 0$. Finally, in the case $\omega(c_t) = c_t$, any program $c \in F_o$ satisfying

$$\sum_{t=0}^{\infty} c_t = 1$$

is obviously optimal.

Now consider the economy above when a production activity is available which provides an output in any period quadruple the previous period's input. Then,

$$F_o = \left\{ c \geqq 0 \,\middle|\, \sum_{t=0}^{\infty} 4^{-t} c_t \leqq 1 \right\} \qquad \text{(cf. [9], p. 39).}$$

A final example for this economy illustrates that an optimal program may exist, but the sum

$$\sum_{t=0}^{\infty} [\omega(c_t) - \bar{\omega}]$$

diverges, for all possible constant "comparison levels" $\bar{\omega}$. Consider

$$\omega(c_t) = -[\log(1 + c_t)]^{-1}.$$

Then,

$$\omega'(c_t) = [\log(1 + c_t)]^{-2} (1 + c_t)^{-1}$$

and

$$\omega''(c_t) = -[\log(1 + c_t)]^{-3} (1 + c_t)^{-2} [2 + \log(1 + c_t)]$$

so that $\omega(c)$ satisfies Assumption 2. Since $\omega(c)$ is bounded above, an optimal plan $\bar{c}$ exists, with $\bar{c}_t \to \infty$, satisfying the condition $\omega'(\bar{c}_t) = k \cdot 4^{-t}$ for some $k > 0$ (cf. [9]; p. 41). Now, $\omega(\bar{c}_t)$ is bounded below any positive $\bar{\omega}$ and is eventually bounded above any negative $\bar{\omega}$. Hence, the only possible comparison level is $\bar{\omega} = 0$. But then we have the inequality

$$\sum_{t=0}^{H} [\omega(\bar{c}_t) - 0] \leqq \int_0^{H+1} \omega(c(t))dt = -\frac{1}{a} \int_{c(0)}^{c(H+1)} \frac{\omega(c)\,\omega''(c)}{\omega'(c)}\, dc \,,$$

where $c(t)$ satisfies $\omega'(c(t)) = ke^{-at}$, $a = \log 4$. But

$$\frac{\omega(c)\,\omega''(c)}{\omega'(c)} \geqq [\log(1 + c)]^{-1} (1 + c)^{-1}$$

implying
$$\sum_{t=0}^{H} \omega(\bar{c}_t) \leq \frac{1}{a} \log \log (1 + c) \Big|_{c(0)}^{c(H+1)}$$

Since $c(H+1) \to +\infty$, the right-hand side of the last inequality approaches $-\infty$, demonstrating that $\bar{\omega} = 0$ is not a possible comparison level.

IV. Existence in the Generalized von Neumann Model

We consider an economy whose production possibility set is described by a generalized von Neumann technology exhibiting the characteristics that (1) all commodities can be produced and (2) when the economy engages in pure accumulation with zero consumption, a program achieving maximal proportional growth can be found such that the economy is expanding, all commodities are actually produced, and all commodities have positive "marginal products". When the unit welfare functions of all consumer units are identical, we obtain the strong result that a necessary and sufficient condition for the existence of a good or optimal plan is that the unit welfare function be bounded from above.

We now describe the detailed structure of the economy in which this result holds.

Assumption 3. There are a constant number of commodities $N_t = N$ in each time period. There are a constant number of consumer units $K_t = K$ coming into existence in each period, and the lifetime of each consumer unit is $\ell + 1$ periods.[10] The unit welfare function $u(c^{kt}, kt)$ can be written in the inter-temporally separable form $u(c^{kt}, kt) = \sum_{\tau=0}^{\ell} \delta^{\tau}\omega(c^{kt}_{t+\tau})$, where the rate of time discount (impatience) δ and the atemporal preference function ω are the same for all consumer units and all periods.

The generalized von Neumann technology is defined by a set Q in the nonnegative orthant of R^{2N} containing all pairs of N-vectors (a_t, b_{t+1}) such that an input of the vector a_t in any period t can yield an output of the vector b_{t+1} in the following period. The technology set Q is assumed to satisfy the following conditions:

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[10] The unrealistic assumption of a constant population of consumer units can be replaced by the less stark, but equally unrealistic, assumption that future welfare, defined in "current" terms, be "discounted" by the expansion factor of population. A somewhat more satisfactory interpretation is that each consumer unit contains a number of individuals which may grow in size over time. This last view requires some tacit "discounting" of the future in the definition of unit welfare.

Assumption 4. The technology Q is a closed convex cone in the nonnegative orthant of R^{2N}, with the properties

(i) zero inputs imply zero outputs: $(0, b_t) \in Q$ implies $b_t = 0$,

(ii) all commodities are producible and the economy is strictly productive: there exists $(a_t, b_{t+1}) \in Q$ such that $b_{n,t+1} > a_{nt}$ for $n = 1, \ldots, N$.

(iii) Free disposal is possible: if $(a_t, b_{t+1}) \in Q$ and $a_t' \geqq a_t$, $0 \leqq b_{t+1}' \leqq b_{t+1}$, then $(a_t', b_{t+1}') \in Q$.

For any point $(a_t, b_{t+1}) \in Q$ with $a_t \neq 0$, define the rate of expansion $\rho(a_t, b_{t+1}) = \max\{\rho \in R \mid b_{t+1} \geqq \rho a_t\}$. When Q satisfies Assumption 4, there exists a scalar $\rho > 1$ and a non-zero vector $\hat{v} \in R^N$ such that $(\hat{v}, \rho\hat{v}) \in Q$ and $\rho \geqq \rho(a_t, b_{t+1})$ for all $(a_t, b_{t+1}) \in Q$ with $a_t \neq 0$. The scalar ρ is the von Neumann growth rate, and $\hat{v}$ lies in a von Neumann ray (see Karlin [5], p. 338).

The following condition ensures that there is sufficiently high substitutability between inputs so that no commodity approaches "overproduction" when overall production approaches the von Neumann expansion rate ρ.

Assumption 5. The technology Q contains no sequence of points $\{(a_t^\nu, b_{t+1}^\nu) \mid \nu = 1, 2, \ldots\}$ such that $b_{t+1} - \rho a_t$ has a nonnegative, non-zero limit point.

The following result is established in [9], pp. 45-47.

Lemma 3. Assumption 4 implies Assumption 1. Assumptions 4 and 5 imply the existence of a strictly positive price vector $\hat{p} \in R^N$ such that $\hat{p} \cdot (b_{t+1} - \rho a_t) \leqq 0$ for all $(a_t, b_{t+1}) \in Q$.

The next assumption ensures that production in von Neumann proportions is feasible, initial resource inputs grow less rapidly than the economy can expand by pure production, and all commodities can be produced in von Neumann proportional growth.

Assumption 6. The initial resource vector $\bar{w}$ has $\bar{w}_0$ strictly positive and $\sum_{t=0}^{\infty} \rho^{-t} \bar{w}_t < +\infty$. The vector $\hat{v}$ can be taken to be strictly positive.

Lemma 4. If Assumptions 4 - 6 hold, then there exist positive scalars m and M such that the function $\gamma(c) = \sum_{t=0}^{\infty} \sum_{n=1}^{N} \rho^{-t} |c_{nt}| \hat{p}_n$

satisfies $m\gamma(c) \leq \phi(c) \leq M\gamma(c)$, where ϕ is the norm on $\mathcal{X}$ given by Lemma 1.[11] Further, the economy is reachable.

Proof: A consequence of Lemma 3 is the inequality $\rho^{-t-1}\hat{r}\cdot b_{t+1} + \sum_{\tau=0}^{t}\rho^{-\tau}\hat{r}\cdot c_\tau \leqq \sum_{\tau=0}^{t}\rho^{-\tau}\hat{r}\cdot\bar{w}_t$, provided $(a_\tau, b_{\tau+1}) \in Q$ and $a_\tau + c_\tau = b_\tau + \bar{w}_\tau$ $(b_o = 0)$. Then for $c \in F_o$, $\gamma(c) \leqq \sum_{\tau=0}^{\infty}\rho^{-\tau}\hat{r}\cdot\bar{w}_t \equiv m^{-1}$. Define constants $\alpha_1 = \max\{\alpha \mid \bar{w}_o = \alpha\hat{v}\}$, $\alpha_2 = \min_n \hat{v}_n$, $\alpha_3 = \min_n \hat{r}_n$ and $M^{-1} = \frac{1}{2}\alpha_1\alpha_2\alpha_3$. If $c \in \Gamma$ satisfies $\gamma(c) \leqq 2M^{-1}$, then

$$\bar{w}_o \geqq \alpha_1\hat{v} \geqq \frac{\gamma(c)\hat{v}}{\alpha_2\alpha_3} \geqq \sum_{t=0}^{\infty}\rho^{-t}\left(\sum_{n=1}^{N}\frac{|c_{nt}|}{\alpha_2}\right)\hat{v}$$

and

$$\left(\sum_{n=1}^{N}\frac{|c_{nt}|}{\alpha_2}\right)\hat{v} \geqq c_t\,.$$

Hence, c can be produced from the resource vector $\bar{w}_o$ employing the von Neumann balanced growth production process, implying $c \in F_o$.

Now suppose $c \in \mathcal{X}$ has $\phi(c) = 1$. By construction, $c = \theta c' - (1-\theta)c''$ for some $0 \leqq \theta \leqq 1$ and $c', c'' \in F_o$. Then, $\gamma(c) \leqq \theta\gamma(c') + (1-\theta)\gamma(c'') \leqq m^{-1}$. Hence, $m\gamma(c) \leqq \phi(c)$ for all $c \in \mathcal{X}$.

Next suppose $c \in \mathcal{X}$ has $\gamma(c) = 2M^{-1}$. By construction, $c = c' - c''$ for some $c', c'' \in \Gamma$. Further, by the free disposal property of F_o, c', c'' can be chosen so that $c'_{nt}\, c''_{nt} = 0$ for all $nt \in N$. Then $\tilde{c}$ defined by $\tilde{c}_{nt} = |c_{nt}|$ has $\gamma(\tilde{c}) = \gamma(c) = 2M^{-1}$, implying $\tilde{c} \in F_o$. Further, $\tilde{c} = c' + c'' \geqq c', c''$ implies $c', c'' \in F_o$, and hence $\phi(c) \leqq \phi(c') + \phi(c'') \leqq 2 = M\gamma(c)$. Hence, for all $c \in \mathcal{X}$, $\phi(c) \leqq M\gamma(c)$.

For any $c \in F_o$, $\lim_{H\to\infty}\gamma(M'_H c) = \lim_{H\to\infty}\sum_{\tau=H}^{\infty}\rho^{-\tau}\hat{r}\cdot c_\tau = 0$, implying $\lim_{H\to\infty}\phi(M'_H c) = 0$. Hence, the economy is reachable. Q. E. D.

[11] The function γ is a norm on $\mathcal{X}$, and this condition implies that the norms γ and ϕ are equivalent (i. e., have the same norm topology).

We can now establish a condition for the existence of good plans.

Theorem 4. Suppose the economy satisfies Assumptions 2 - 6, and suppose that $\omega(c_t)$ is defined and finite for all strictly positive $c_t \in R^N$ (i. e., all strictly positive consumption plans are desired). Then, a necessary and sufficient condition for the existence of a good consumption plan is that ω be bounded from above.

Proof: For $t \geq \ell$, the social welfare accrued by consumption in period t for a desired consumption plan $s = (c^{kt})$ can be computed from the inter-temporally separable welfare function to be $\sum_{k=1}^{K} \sum_{\tau=0}^{\ell} \delta^{\tau} \omega(c_t^{k,t-\tau})$. For a net supply program c which can be distributed to give a desired consumption plan (hereafter, such a c will be termed a desired net supply program), atemporal optimal distribution requires that $\sum_{k=1}^{K} \sum_{\tau=0}^{\ell} \delta^{\tau} \omega(c_t^{k,t-\tau})$ be maximized subject to $\sum_{k=1}^{K} \sum_{\tau=0}^{\ell} c_t^{k,t-\tau} = c_t$. Let $\omega^*(c_t)$ equal the value of this maximum. Then, $\omega^*(c_t)$ is the aggregate social welfare obtained in period t when atemporal distribution is optimal. The function $\omega^*(c_t)$ is concave, non-decreasing, and continuous for desired c_t, and no finite c_t maximizes ω^*. If a sequence of desired $\{c_t^{(\nu)}\}$ converges to a non-desired $c_t^{(0)}$, then $\omega^*(c_t^{(\nu)})$ converges to $-\infty$. The function ω^* is bounded above if and only if ω is bounded above.

Suppose that a good feasible plan $\hat{s}$ exists. We can assume without loss of generality that atemporal distribution is optimal, so that the social welfare accruing in period t ($t \geq \ell$) is $\omega^*(\hat{c}_t)$, where $\hat{c} = c(\hat{s})$. Theorem 3(c) establishes the existence of a price vector $p \in \mathfrak{X}^*$ such that $V_\infty(s, \hat{s}) \leq \bar{v} + p \cdot c(s)$ for all good $s \in \mathcal{S}$. Consider a net supply program $\tilde{c}$ defined by $\tilde{c}_t = \hat{c}_t$ for $t < \ell$ or $t \geq H$ and $\tilde{c}_t = \hat{c}_{t+1}$ for $\ell \leq t < H$. Then, $\gamma(\tilde{c}) \leq (1 + \rho)\gamma(\hat{c})$, implying $\tilde{c} \in \mathfrak{X}$. The program $\tilde{c}$ is good. Hence, one obtains the inequalities

$$\omega^*(\hat{c}_H) - \omega^*(\hat{c}_\ell) = \sum_{t=\ell}^{H-1} [\omega^*(\hat{c}_{t+1}) - \omega^*(\hat{c}_t)] = V_\infty(\tilde{s}, \hat{s})$$

$$\leq \bar{v} + p \cdot \tilde{c} \leq \bar{v} + \lambda(1 + \rho) \frac{M}{m} < +\infty,$$

where $\tilde{s}$ and $\hat{s}$ are the consumption plans obtained from $\tilde{c}$ and $\hat{c}$, respectively, by optimal atemporal distribution. The geometrically expanding program $\check{c}$ defined by $\check{c}_t = \alpha_1(\rho - 1) \frac{1}{2\rho} (\frac{\rho + 1}{2})^t \bar{v}$, where

$\bar{w}_o \geqq \alpha_1 \hat{v} > 0$, is feasible and desirable, implying that $\hat{c}_t$ in the good program cannot remain bounded. Hence, the upper bound on $\omega^*(\hat{c}_H)$ implies that ω^* is bounded from above.

Now suppose that $\omega^*(c_t)$ is bounded from above, and consider the class of programs $F'_o = \{c \in \mathfrak{X} | c_t = \beta_t \hat{v} \text{ and} \sum \rho^{-t} \beta_t \leqq \alpha_1\}$, where $\bar{w}_o \geqq \alpha_1 \hat{v} > 0$. The sub-economy with the feasible set F'_o has the properties of the one-sector linear model analyzed by the author in [9], pp. 39-43, and can be shown to have an optimal program c with $\hat{c}_t = \beta_t \hat{v}$ such that $\omega^*(\beta_t \hat{v}) - \omega^*(\hat{\beta}_t \hat{v}) \leqq k' \rho^{-t}(\beta_t - \hat{\beta}_t)$ for some positive scalar k'. For any desired $c \in F_o$,

$$c_t \leqq \sum_{n=1}^{N} |c_{nt}| \frac{\hat{v}}{\alpha_2} ,$$

where $\alpha_2 = \min_n \hat{v}_n$, implying

$$\omega^*(c_t) - \omega^*(\hat{\beta}_t \hat{v}) \leqq \omega^*\left(\sum_{n=1}^{N} |c_{nt}| \frac{\hat{v}}{\alpha_2} \right) - \omega^*(\hat{\beta}_t \hat{v})$$

$$\leqq \frac{k'}{\alpha_2 \rho^t} \sum_{n=1}^{N} |c_{nt}| .$$

Hence, for $s \in S$, $c = c(s)$ satisfies

$$\sum_{t=\ell}^{H} [\omega^*(c_t) - \omega^*(\hat{\beta}_t \hat{v})] \leqq \frac{k'}{\alpha_2 \alpha_3} \sum_{t=\ell}^{H} \rho^{-t} \sum_{n=1}^{N} \hat{r}_n |c_{nt}| \leqq \frac{k'}{\alpha_2 \alpha_3} \gamma(c)$$

$$\leqq \frac{k'}{\alpha_2 \alpha_3 m} ,$$

where $\alpha_3 = \min_n \hat{r}_n$. Therefore, $\hat{c}$ is good. Q.E.D.

A production possibility set Q satisfying Assumption 4 will admit one or more supporting planes at each point (a_t, b_{t+1}) in its boundary. Q is said to be <u>smooth</u> at this (a_t, b_{t+1}) if its supporting plane is unique. When (a_t, b_{t+1}) is strictly positive, smoothness requires that the surface of Q be differentiable at this point. This condition will certainly hold if Q is defined by a differentiable transformation curve.

<u>Assumption 7.</u> The technology Q is smooth at the von Neumann vector $(\hat{v}, \rho\hat{v}) \in Q$.

For the technology Q, define the <u>dual cone</u> Π as the set of nonnegative vectors $(r_t, r_{t+1}) \in R^{2N}$ such that $r_{t+1} \cdot b_{t+1} - r_t \cdot a_t \leqq 0$ for all $(a_t, b_{t+1}) \in Q$. The set Π is a closed convex cone in the nonnegative orthant of R^{2N} with the properties that (1) $(0, r_{t+1}) \in \Pi$ implies $r_{t+1} = 0$ and (2) $(r_t, r_{t+1}) \in \Pi$, $r_t' \geqq r_t$, and $0 \leqq r_{t+1}' \leqq r_{t+1}$ imply $(r_t', r_{t+1}') \in \Pi$. By Lemma 3, $(\rho \hat{r}, \hat{r}) \in \Pi$.

For a vector $v \in R^N$, let

$$\|v\| = \sum_{n=1}^{N} |v_n|$$

and define the angular distance between two non-zero vectors v', $v'' \in R^N$ by

$$\text{ang}(v', v'') = \left\| \frac{v'}{\|v'\|} - \frac{v''}{\|v''\|} \right\| .$$

We shall employ the following lemma, due originally to Radner [12].

<u>Lemma 5.</u> Suppose Assumptions 4 - 7 hold. Given $\varepsilon > 0$, there exists $\delta > 0$ such that for $(r_t, r_{t+1}) \in \Pi$ with $r_t \neq 0$ and $\text{ang}(r_t, \hat{r}) \geqq \varepsilon$, it follows that $(\rho r_{t+1}(1 + \delta) - r_t) \cdot \hat{v} \leqq 0$.

<u>Proof:</u> Define the set $A = \{(r_t, r_{t+1}) \in \Pi \mid \|r_t\| = 1$ and $\text{ang}(r_t, r) \geqq \varepsilon\}$. Then, A is closed. Further, A is bounded: an unbounded sequence $(r_t^\nu, r_{t+1}^\nu) \in A$ would have $|r_{t+1}^\nu|^{-1}(r_t^\nu, r_{t+1}^\nu) \in \Pi$ with a subsequence converging to a point $(0, r_{t+1}^o) \in \Pi$ with $r_{t+1}^o \neq 0$, contradicting the properties of Π.

By Assumption 7, no point $(r_t, r_{t+1}) \in A$ can be normal to a plane supporting Q through $(\hat{v}, \rho\hat{v})$, implying $(\rho r_{t+1} - r_t) \cdot \hat{v} < 0$ on A. Hence, the function $h(\delta') = (\rho r_{t+1}(1 + \delta') - r_t) \cdot \hat{v}$ achieves a maximum on the compact set A which is negative for $\delta' = \delta$ sufficiently small.

Q. E. D.

We are now prepared to establish an existence for optimal programs.

<u>Theorem 5.</u> Suppose the economy satisfies Assumptions 2 - 7, and suppose that $\omega(c_t)$ is defined and finite for all strictly positive $c_t \in R^N$. Then, a necessary and sufficient condition for the existence of an optimal consumption plan is that ω be bounded from above.

Proof: The necessity of the boundedness condition follows from Theorem 4. We shall now assume ω bounded and prove the existence of an optimal consumption plan.

By Theorem 4, a good feasible consumption plan $\hat{s}$ exists with net supply vector $\hat{c}$, and atemporal distribution can be assumed to be optimal. The hypotheses of Theorem 3 then hold (using Lemma 4), implying the existence of a price vector $p \in \mathscr{X}^*$ and a good consumption plan $\bar{s} = (\bar{c}^{kt}) \in S$ such that

$$\delta^{\tau-t}[\omega(c_\tau) - \omega(\bar{c}_\tau^{kt})] \leqq p_\tau \cdot (c_\tau - c_\tau^{kt}) \text{ for all } kt \in K.$$

These inequalities imply that atemporal distribution in $\bar{s}$ is optimal and $\omega^*(\bar{c}_t)$ gives the social welfare accruing in period t ($t \geqq \ell$) from $\bar{c} = c(\bar{s})$. Further, the inequality $\omega^*(c_t) - \omega^*(\bar{c}_t) \leqq p_t \cdot (c_t - \bar{c}_t)$ is satisfied

The inequality $p \cdot c \leqq \lambda$ for $c \in F_1$ given by Theorem 3 implies that $\tilde{c} \in F_1$, defined by $\tilde{c}_\tau = \bar{c}_\tau$ for $\tau \neq t, t+1$, $\tilde{c}_t = \bar{c}_t - a_t$, $\tilde{c}_{t+1} = \bar{c}_{t+1} + b_{t+1}$ for $(a_t, b_{t+1}) \in Q$, has $p_{t+1} \cdot b_{t+1} - p_t \cdot a_t \leqq \lambda - p \cdot \bar{c}$. Since Q is a cone, this inequality implies that $p_{t+1} \cdot b_{t+1} - p_t \cdot a_t \leqq 0$ for all $(a_t, b_{t+1}) \in Q$. Hence, $(p_t, p_{t+1}) \in \Pi$. The following result, which may be of independent interest, gives an "asymptotic turnpike" property for prices.

Lemma 6. Under the hypotheses of Theorem 5, the price vector $p \in \mathscr{X}^*$ given by Theorem 3 has $\lim_{t \to \infty} \rho^t p_t = \alpha \hat{p}$ for some non-negative scalar α.

Proof of Lemma 6: Since $(\rho p_{t+1} - p_t) \cdot \hat{v} \leqq 0$, it follows that $\rho^t p_t \cdot \hat{v}$ is a non-increasing sequence converging to $\alpha \hat{p} \cdot \hat{v}$ for some scalar $\alpha \geqq 0$. If $p_t = 0$ in some period, it is zero thereafter and the lemma follows trivially. If p_t is always non-zero, but $\lim_{t \to \infty} \text{ang}(p_t, \hat{p}) = 0$, then $\lim_{t \to \infty} \rho^t p_t = \alpha \hat{p}$. Alternately, suppose there exists $\varepsilon > 0$ such that $\text{ang}(p_t, \hat{p}) \geqq \varepsilon$ in infinitely many periods, and let μ_t denote the number of times this inequality holds up to period t. Then, Lemma 5 implies the existence of $\delta > 0$ such that $(\rho^{t+1} p_{t+1}(1 + \delta) - \rho^t p_t) \cdot \hat{v} \leqq 0$ whenever $\text{ang}(p_t, \hat{p}) \geqq \varepsilon$. Then, $\rho^t p_t \cdot \hat{v} \leqq p_o \cdot \hat{v}/(1 + \delta)^{\mu_t} \to 0$, and the lemma is proved.

We continue the proof of Theorem 5. Consider first the case $\lim_{t \to \infty} \rho^t p_t = 0$. Given $\varepsilon > 0$, there exists H_o such that for $H \geqq H_o$, $\rho^t p_t \leqq \varepsilon m \hat{p}$, where $m\gamma(c) \leqq \phi(c)$. Then, for any $c \in F_o$, $p \cdot M_H^! c \leqq \varepsilon m\gamma(c) \leqq \varepsilon$. Hence, the hypothesis of Theorem 3(f) is satisfied, and $\hat{s}$ is optimal.

Consider finally the case $\lim_{t\to\infty} \rho^t p_t = \alpha \hat{r} > 0$. Then there exists H ($H > \ell$) such that for $t \geqq H$, $0 < (\alpha - \beta)\hat{r} \leqq \rho^t p_t \leqq (\alpha + \beta)\hat{r}$, where $\beta = \frac{\alpha}{2(1+\alpha)} \frac{\rho - 1}{\rho + 1}$. Theorem 3(a) and the inequality (1) established in the proof of Theorem 3 imply the inequality

$$V_\infty(s, s^\nu) \leqq p \cdot c(s) - p \cdot c(s^\nu) + [\bar{v} - V_\infty(s^\nu, \hat{s})],$$

where $\hat{s}$ is the good consumption plan, $s \in \mathscr{A}$ is desired, and $\{s^\nu\}$ is an optimizing sequence. Without loss of generality, we can assume that each s^ν in the optimizing sequence has optimal atemporal distribution. Let $c^\nu = c(s^\nu)$, and define $\tilde{c}^\nu$ with $\tilde{c}_t^\nu = c_t^\nu$ for $t \leqq H$ or $t > \tau > H$ and $\tilde{c}_t^\nu = c_{t-1}^\nu$ for $H < t \leqq \tau$. Then, $\gamma(\tilde{c}) \leqq (1 + \frac{1}{\rho})\gamma(c^\nu) \leqq \frac{1}{m}(1 + \frac{1}{\rho})$, implying $\tilde{c} \in \mathscr{K}$. Since $\tilde{c}$ is a good net supply program, we have

$$\omega^*(c_\tau^\nu) - \omega^*(c_H^\nu) = \sum_{t=H+1}^{\tau} [\omega^*(c_t^\nu) - \omega^*(c_{t-1}^\nu)] = V_\infty(s^\nu, \tilde{s}^\nu)$$

$$\geqq p \cdot c^\nu - p \cdot \tilde{c}^\nu - [\bar{v} - V_\infty(s^\nu, \hat{s})].$$

But

$$p \cdot c^\nu - p \cdot \tilde{c}^\nu \geqq (\alpha - \beta)\rho^{-\tau}\, \hat{r} \cdot c_\tau^\nu - (\alpha + \beta)\rho^{-H-1} \hat{r} \cdot c_H^\nu + \theta \sum_{t=H+1} \rho^{-t} \hat{r} \cdot c_t^\nu,$$

where

$$\theta = \left(1 - \frac{1}{\rho}\right) \frac{2 + \alpha}{2(1 + \alpha)} > 0.$$

Letting $\tau \to +\infty$, we obtain the inequality

$$[\hat{\omega} - \omega^*(c_H^\nu)] + [\bar{v} - V_\infty(s^\nu, \hat{s})] + (\alpha + \beta)\rho^{-H-1} \hat{r} \cdot c_H^\nu \geqq \theta \sum_{t=H+1}^{\infty} \rho^{-t} \hat{r} \cdot c_t^\nu,$$

where $\hat{\omega} = \sup\{\omega(c_t) \mid c_t \in R^N,\ c_t \text{ strictly positive}\}$.

For any pair of plans s^ν and s^η in the optimizing sequence, we have for $\tau \geqq H$,

$$\begin{aligned}
\phi(c^\nu - c^\eta) &\leqq \phi(M_\tau(c^\nu - c^\eta)) + \phi(M'_\tau(c^\nu - c^\eta)) \\
&\leqq \phi(M_\tau(c^\nu - c^\eta)) + M\gamma(M'_\tau(c^\nu - c^\eta)) \\
&\leqq \phi(M_\tau(c^\nu - c^\eta)) + \frac{M}{\theta}[2\hat{\omega} - \omega^*(c^\nu) - \omega^*(c_\tau^\eta)] \\
&\quad + \frac{M}{\theta}[2\bar{v} - V_\infty(s^\nu, \hat{s}) - V_\infty(s^\eta, \hat{s})] \\
&\quad + \frac{M}{\theta}(\alpha + \beta)\rho^{-\tau-1}[\hat{r} \cdot c_\tau^\nu + \hat{r} \cdot c_\tau^\eta].
\end{aligned}$$

If the program $\bar{c}$ had $\liminf_{t \to \infty} \|\bar{c}_t\| < +\infty$, then it would be non-good relative to a program c with $c_t = \bar{c}_t + \alpha_1(\rho - 1)\rho^{-2}\left(\frac{\rho + 1}{2}\right)^t \hat{v}$ satisfying $\phi(c) \leq 2$. But this would contradict the inequality of Theorem 3(c) for the good program $\bar{c}$. Hence, $\lim_{t \to \infty} \|\bar{c}_t\| = +\infty$.

Given $\varepsilon > 0$, choose $\tau \geq H$ such that $(M/\theta)[2\hat{\omega} - 2\omega^*(\bar{c}_\tau)] < \varepsilon/9$ and $(2M/\theta)(\alpha + \beta)\rho^{-\tau-1}\hat{r} \cdot \bar{c}_\tau < \varepsilon/9$. Then choose ν_0 such that for $\eta, \nu \geq \nu_0$

$$\phi(M_\tau(c^\nu - c^\eta)) < \varepsilon/9, \quad \frac{M}{\theta}|\omega^*(c^\nu_\tau) - \omega^*(\bar{c}_\tau)| < \varepsilon/9,$$

$$\frac{M}{\theta}[\bar{v} - V_\infty(s^\nu, \bar{s})] < \varepsilon/9, \text{ and } \frac{M}{\theta}(\alpha + \beta)\rho^{-\tau-1}|\hat{r} \cdot c^\nu_\tau - \hat{r} \cdot \bar{c}_\tau| < \varepsilon/$$

Then, $\phi(c^\nu - c^\eta) < \varepsilon$ for $\nu, \eta \geq \nu_0$. Hence, $\{c^\nu\}$ is a ϕ-Cauchy sequence with limit $\bar{c}$, implying that the hypothesis of Theorem 3(e) holds. Hence $\bar{s}$ is optimal. This proves Theorem 5. · Q. E. D.

The results obtained above have employed the restrictive assumptions that (1) the number of commodities and new consumer units in each period is constant, (2) the lifetime of consumer units is constant, (3) all atemporal unit welfare functions are the same, and (4) the technology does not change over time. We now indicate several directions in which these conditions can be relaxed.

First, suppose that an economy satisfies Assumptions 1 and 2 over all time, and suppose that after some very large, but finite, horizon H, the restrictions above hold (precisely, suppose that Assumptions 3 - 5 and 7 hold for $t \geq H$, and suppose that the following condition holds: "Assumption 6': The initial resource vector $\bar{w}$ has $\sum_{t=H}^{\infty} \rho^{-t}\bar{w}_t < +\infty$. The vector $\hat{v}$ can be taken to be strictly positive. There exists a desired consumption plan which has c_H strictly positive. Then, the proofs of Theorems 4 and 5 can be applied for $t \geq H$, establishing the conclusions of these theorems. With this generalization, it is necessary to require only asymptotically the conditions of a stationar list of commodities and a constant population.

Next, we note that the von Neumann technology Q_t may vary from period to period, provided the von Neumann growth rates ρ_t are all contained in a closed bounded subset of $(1, +\infty)$, and the "present value" technologies $Q'_t = \{(a_t, b_{t+1}/\rho_t) | (a_t, b_{t+1}) \in Q_t\}$ have the property that their "core" and "envelope" technologies (defined by $Q^c = \bigcap_{t=0}^{\infty} Q'_t$ and $Q^e = \bigcup_{t=0}^{\infty} Q'_t$, respectively) satisfy Assumptions 4 - 7. In this case, the arguments of Theorems 4 and 5 can be made with only minor modifications (see [9], pp. 45-49).

Finally, we note that the unit atemporal welfare functions need not be identical, provided the "absolute variation" in welfare due to shifts of consumption vectors between temporally adjacent units is finite. This generalization is discussed in greater detail in [9], p. 39.

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