12 On the Existence of Optimal Development Programmes in Infinite-Horizon Economies*

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I. INTRODUCTION

1.1. An economic development programme is a description, over the lifetime of an economy, of the commodity vectors which resource holders, firms and consumer units are required to supply and demand. The objective of development planning is to choose from the set of programmes which are feasible for an economy the one that is best in terms of the planner's imputation of social preferences. In practice, programmes are chosen to maximise an objective function over a relatively short horizon, with terminal conditions established to make this optimisation consistent with optimisation over the full lifetime of the economy.

An important problem in the theory of development planning is to establish the logical relationships which hold among the structure of the social preference ordering, the properties of lifetime optimisation, and the terminal conditions in the practical planning computation.[1] In particular, it is necessary to determine the conditions on social preferences which guarantee the existence of a lifetime optimal development programme.

When the lifetime of an economy is finite and time can be considered as a sequence of short periods, the existence of optimal programmes follows from the mild condition that the set of feasible plans be closed and bounded and that the social preferences be continuous over this set.[2] However, when an economy has an infinite lifetime,

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[1] Important contributions to this problem have been made by Strotz (1956) and Goldman (1969).

[2] When time can be treated as a sequence of short periods, all the results of classical theory of value apply to this problem (Debreu, 1957, 1962). In a continuous time formulation of the problem, the mathematical analysis is more complex, but essentially the same conclusions hold (Yaari, 1964; Bewley, 1969).

existence of an optimal programme will depend on structural properties of its social preferences and technology.[1] Koopmans (1966) has argued for the desirability of conducting 'logical experiments' to establish existence criteria for such economies, and has reviewed in Koopmans (1967) most of the results obtained on this topic through 1966.[2] This paper summarises more recent results of Brock (1970), Brock and Gale (1970) and the author (McFadden (1970)), and provides several multi-sector generalisations.

1.2. We begin by considering Ramsey's classical one-commodity growth model in which aggregate output y_t in period t is divided into consumption c_t and an input x_t to production of the following period's output. The technology is defined by a production function $y_{t+1} = f(x_t)$, where f is assumed to be non-negative, non-decreasing, continuous and concave.[3] The economy begins with a positive endowment y_0. A programme (x_t, y_t, c_t) is *feasible* if it is non-negative and satisfies $x_t + c_t \leqslant y_t$ and $y_{t+1} \leqslant f(x_t)$ for $t = 0, 1, \ldots$.

1.3. It is frequently assumed that the relative social desirability of two feasible programmes (x_t, y_t, c_t) and (x_t', y_t', c_t') can be determined by computing a discounted sum of utility differences

$$\sum_{t=1}^{H} \delta^t [U(c_t) - U(c_t')] \qquad (12.1)$$

where $U(c_t)$ is an atemporal utility of consumption in period t and δ is a discount factor. The stream (c_t) is said to be *no worse than* [resp., *better than*] the stream (c_t') if, as H approaches infinity in (12.1), all the limit points of the partial sums are non-negative [resp., positive]. (Note that if the partial sums have both positive and negative limit points, then the two streams are not comparable.)

[1] A fundamental difficulty in the infinite-horizon economy is that reasonable axioms on social preferences which are consistent for a finite horizon may be inconsistent for an infinite horizon (Koopmans (1960); Diamond (1965)). Moreover, reasonable preference orderings will frequently fail to admit a continuous numerical representation in any topology in which the set of feasible programmes is closed and bounded (McFadden (1967)).

[2] Several authors, including this one, have argued rather unconvincingly that infinite-horizon models are a reasonable representation of reality, and are thus worthy of a scholastic examination of internal consistency. A better case is this: to the extent that infinite-horizon models allow one to simplify the description of an economy by eliminating terminal conditions, such models will be useful approximations to reality (in the spirit of frictionless planes and point masses). It is then important to know the internal logical structure of these approximations.

[3] In growth theory, the one-commodity model is usually given the formulation $c_t + i_t = g(k_{t-1})$ and $k_t = i_t + (1-d)k_{t-1}$, where c_t, i_t, k_t are consumption, gross investment and capital stock, respectively, and d is a depreciation rate for capital stock. In the terminology of this paper, $x_{t-1} = k_{t-1}$, $y_t = c_t + k_t$ and $f(x_{t-1}) = g(x_{t-1}) + (1-d)x_{t-1}$. Hence, $f'(+\infty) = g'(+\infty) + (1-d) = 1-d$ is positive if capital is not completely perishable.

This is called the overtaking criterion for optimality, introduced by Weizsäcker (1965) as a generalisation of Ramsey's notion of a 'bliss' comparison utility level.[1]

A feasible programme is *optimal* if it is comparable to and no worse than any other feasible programme, and is *maximal* if it is no worse than any other feasible programme to which it is comparable. Any programme which is optimal is also maximal, but an economy may have many non-comparable maximal programmes (one example is given by Brock (1970), and a second is given in section 2.13 below).

We assume that the atemporal utility function U is concave and twice continuously differentiable for positive c, with $U'(c)$ positive and $U(0) = \lim_{c \to 0} U(c)$.

1.4. We wish to discover the conditions on the production function, atemporal utility function and discount rate which imply the existence or non-existence of an optimal programme. An elementary result due to Ramsey (1928) provides a prototype existence criterion for a much broader class of economies.

1.5. *Lemma. In the one-commodity Ramsey growth model, assume a linear production function* $y_{t+1} = f(x_t) = \rho_0 x_t$ *with* $\rho_0 > 0$, *and a constant elasticity utility function* $U(c) = c^{1-\alpha}/(1-\alpha)$ *with* $\alpha > 0$, $\alpha \neq 1$. *Then, an optimal programme exists if and only if*

$$\delta \rho_0^{1-\alpha} < 1. \tag{12.2}$$

Further, an optimal programme has c_t *decreasing [resp., constant, increasing] if* $\delta\rho_0 < 1$ *[resp.,* $\delta\rho_0 = 1$, $\delta\rho_0 > 1$*].*

1.6. Note that the constant elasticity utility function is bounded below for $0 < \alpha < 1$ and bounded above for $\alpha > 1$. Hence in the no-discounting case $\delta = 1$, the inequality (12.2) holds if and only if the constant elasticity utility stream is bounded for all feasible programmes. This property again generalises to a broader class of economies.

1.7. Several generalisations of the existence criterion (12.2) can be made for the one-commodity model. Retaining the assumption of a linear production function, but removing the assumption of a constant elasticity utility function, the author (McFadden (1967), Theorem 4 and Lemma 10) has established the following result.

1.8. *Lemma. In the one-commodity Ramsey growth model, assume a linear production function* $y_{t+1} = f(x_t) = \rho_0 x_t$ *with* $\rho_0 > 0$. *Then the following existence conditions hold (Note: the remarks relate these conditions to the criterion (12.2) for the constant elasticity case):*

[1] The use of this formulation of social preferences is discussed by Gale (1967) and McFadden (1967).

(i) *Suppose* $\delta = 1, \rho_0 > 1$. *An optimal programme exists if and only if* $U(c)$ *is bounded above.* [*Remark:* $\delta\rho_0^{1-\alpha} < 1$ *if and only if* $\alpha > 1$.]

(ii) *Suppose* $\delta = 1, \rho_0 < 1$. *An optimal programme exists if and only if* $U(c)$ *is bounded below.* [*Remark:* $\delta\rho_0^{1-\alpha} < 1$ *if and only if* $\alpha < 1$.]

(iii) *Suppose* $\delta\rho_0 \leqslant 1, \rho_0 > 1$. *An optimal programme exists for any* $U(c)$. [*Remark:* $\delta\rho_0^{1-\alpha} < 1$ *if* $\alpha > 0$.]

(iv) *Suppose* $\delta\rho_0 > 1, \rho_0 > 1, \delta \leqslant 1$. $U(c)$ *bounded above implies an optimal programme exists.* [*Remark:* $\delta\rho_0^{1-\alpha} < 1$ *if* $\alpha > 1$.]

(v) *Suppose* $\rho_0 > 1, \delta \geqslant 1$. *The existence of an optimal programme implies* $U(c)$ *bounded above.* [*Remark:* $\delta\rho_0^{1-\alpha} > 1$ *if* $\alpha < 1$.]

(vi) *Suppose* $\delta\rho_0 \geqslant 1, \rho_0 \leqslant 1$. *No optimal programme exists for any* $U(c)$, *unless* $\delta = \rho_0 = 1$ *and* $U(c)$ *is linear.* [*Remark:* $\delta\rho_0^{1-\alpha} \geqslant 1$ *if* $\alpha \geqslant 0$.]

(vii) *Suppose* $\delta\rho_0 < 1, \rho_0 < 1, \delta \geqslant 1$. *The existence of an optimal programme implies* $U(c)$ *bounded below.* [*Remark:* $\delta\rho_0^{1-\alpha} > 1$ *if* $\alpha > 1$.]

(viii) *Suppose* $\delta\rho_0 < 1, \rho_0 \leqslant 1, \delta \leqslant 1$. $U(c)$ *bounded below implies the existence of an optimal programme.* [*Remark:* $\delta\rho_0^{1-\alpha} < 1$ *if* $\alpha < 1$.]

1.9. A second generalisation of the criterion (12.2) to the case where both the production function f and the utility function U are arbitrary has been made by Brock and Gale (1970). This result introduces two concepts, the *asymptotic elasticity* of the utility function $U(c)$, and the *asymptotic average productivity* of the production function $f(x)$. Define an elasticity of marginal utility with respect to consumption at any $c > 0$ by

$$\alpha(c) = -d \log U'(c)/d \log c = -cU''(c)/U'(c). \qquad (12.3)$$

(Note that in the case of the constant elasticity utility function in 1.5, one has $\alpha(c) = \alpha$.) Define asymptotic elasticities,

$$\alpha_0 = \lim_{c\to 0} \alpha(c) \quad \text{and} \quad \alpha_1 = \lim_{c\to +\infty} \alpha(c) \qquad (12.4)$$

assuming that these limits exist.[1] The average productivity of a production function $f(x)$ at a positive input level x is equal to $\rho(x) = f(x)/x$. Define asymptotic average productivities

$$\rho_0 = \lim_{x\to 0} \rho(x) \quad \text{and} \quad \rho_1 = \lim_{x\to +\infty} \rho(x). \qquad (12.5)$$

[1] When these limits fail to exist, the existence conditions given below continue to hold with appropriate lim inf and lim sup operations replacing lim operations. *Note that the elasticity* α_1 *in the terminology of this paper is equal to* $1-\alpha$ *in the terminology* of Brock and Gale.

Because the production function is concave and non-decreasing, these limits will always exist, and for a non-trivial technology (i.e. $f \neq 0$) satisfy $0 < \rho_0 \leqslant +\infty$, $0 \leqslant \rho_1 < +\infty$, and $\rho_1 \leqslant \rho_0$. The results obtained by Brock and Gale can readily be shown to imply the following.[1]

1.10. *Lemma. In the one-commodity Ramsey growth model, assume that the asymptotic elasticities of the utility function (12.4) and the asymptotic average productivities of the production function (12.5) are given. Then, any one of the following three conditions is sufficient for the existence of an optimal programme:*

(a) $\rho_1 > 1$ *and* $\delta\rho_1^{1-\alpha_1} < 1$;

(b) $\rho_0 < 1$ *and* $\delta\rho_0^{1-\alpha_0} < 1$;

(c) $\rho_0 > 1 > \rho_1$ *and* $\delta < 1$.

Further, any one of the following three conditions is sufficient for the non-existence of a maximal programme:

(d) $\rho_1 > 1$ *and* $\delta\rho_1^{1-\alpha_1} > 1$;

(e) $\rho_0 < 1$ *and* $\delta\rho_0^{1-\alpha_0} > 1$;

(f) $\rho_0 > 1 > \rho_1$ *and* $\delta\rho_1 > 1$.

[1] The existence criteria in Brock and Gale (1970) formulae (I) and (II)) are defined only for the special case of completely perishable capital, $\rho_1 = 0$, but allow commodity-augmenting technical change. This lemma is an easy modification of their result in the case of no technical change. Alternately, one can generalise the Brock–Gale model in the case technical change is present as follows. Suppose one has $c_{t+1}+i_{t+1} = A^t g((B/A)^t k_t)$ and $k_{t+1} = i_{t+1}+(1-d)k_t$, with $g'(+\infty) = \rho \geqslant 0$, where A and B are interpreted as rates of 'labour' and 'capital' augmentation, respectively. Define $h(x) = g(x)-\rho x$. Then, $h'(+\infty) = 0$. Suppose $\beta = \lim_{x\to+\infty} xh'(x)/h(x)$, termed the asymptotic elasticity of h, exists and satisfies $\beta < 1$, and suppose α_1 is defined as in the text of this paper. The gross production of the economy then satisfies $y_{t+1} = A^t h((B/A)^t x_t)+(\rho B^t+1-d)x_t$ with $x_t = k_t$ and $y_t = k_t+c_t$. If $\rho = 0$, one finds that an optimal programme exists for $\delta < \bar{\delta}$, and fails to exist for $\delta > \bar{\delta}$, where $\bar{\delta}g^{1-\alpha_1} = 1$ and $g = AB^{\beta/(1-\beta)} > 1$, and finds further that an optimal programme grows asymptotically at rate g. (This is precisely the Brock–Gale result. Hence, that conclusion derived under the assumption $d = 1$ actually holds under the more general depreciation condition $0 < d \leqslant 1$.) If $\rho > 0$ and $B = 1$, the critical discount rate again satisfies $\bar{\delta}g^{1-\alpha_1} = 1$, but $g = \text{Max}\{A^\beta, 1+\rho-d\}$. (The optimal programme will grow asymptotically at the rate g only if $A^\beta > 1+\rho-d$.) Finally, if $\rho > 0$ and $B > 1$, an optimal programme exists for $\alpha_1 > 1$ and any δ, and grows asymptotically at a faster than geometric rate, whereas no optimal programme exists for $0 < \alpha_1 < 1$.

One final generalisation of these formulae may deserve a note. If the partial utility sums in (12.1) have the form $\sum_{t=0}^{\nu} \delta^t U(\lambda^t c_t)$, where λ is a discount factor 'inside' the utility function, then the critical discount rate is given by $\bar{\delta}(\lambda g)^{1-\alpha_1} = 1$.

1.11. An interpretation of conditions (a)–(c) in this lemma is that they establish critical levels of the discount factor below which the distant future is insignificant and optimal programmes exist, and above which no maximal programmes exist. Note that this lemma is exhaustive except for 'borderline' cases. Unfortunately, two of these cases, which require a detailed analysis of the structure of the economy to establish existence criteria, correspond to commonly used economic models. The first is a model arising in neo-classical growth theory of a productive, primary resource-limited economy with no discounting or with some negative discounting (i.e. $\rho_0 > 1 > \rho_1$ with $\delta = 1$ or with $\delta > 1$, $\delta\rho_1 \leqslant 1$). With mild additional differentiability assumptions, Koopmans (1966) has established that optimal programmes exist in the no-discounting case, and that maximal programmes fail to exist in the case of negative discounting. The second borderline case, arising in the study of Leontief and von Neumann models, is a productive linear economy without resource constraints and with no discounting (i.e. $\rho_1 > 1$ and $\delta = 1$). Existence criteria sharpening 1.10 (a) and (d) have been established for this case by the author (1967), (1970).

1.12. The non-triviality of the question of existence in the borderline cases above can be illustrated with several examples. For the resource-limited, no-discounting economy with $\delta = 1$, $U(c) = \log c$, and $y = f(x) = x^{\beta}$, $1/2 < \beta < 1$, and with $y_0 = 1/2$, one has $\rho_0 = +\infty$, $\rho_1 = 0$, and the existence of an optimal programme satisfying $c_t = (1-\beta)2^{-\beta^t}\beta^{\beta(1-\beta^t)/(1-\beta)}$. However, in the limit $\beta = 1$, one has the case in 1.8 (vi) in which no optimal programme exists.

In the next example, consider a productive linear technology $y = f(x) = \rho_0 x$, $\rho_0 > 1$, with no discounting ($\delta = 1$), and consider the utility functions $U(c) = \log(1+c)$ and $U(c) = -1/\log(1+c)$. The first of these functions is unbounded above, and no optimal programme exists, by 1.8 (i), while the second function is bounded above and an optimal programme exists. However, both functions have the asymptotic elasticity $\alpha_1 = 1$, and 1.10 (a) or (d) do not apply. Further, one can show that for $U(c) = -1/\log(1+c)$, the sum $\sum_{t=0}^{\infty} [U(c_t) - \bar{u}]$, where (c_t) is the optimal programme, diverges for every constant $\bar{u}$. This is in contrast to any economy satisfying (a), (b) or (c) in 1.10, for which the sum $\sum_{t=0}^{\infty} \delta^t U(c_t)$ converges for the optimal programme when the zero level of U is defined appropriately. This convergence property plays a crucial role in the proof of 1.10. Hence, this example shows that the Brock–Gale approach cannot be extended directly to all the borderline cases. This example also shows that the overtaking criterion applies to a broader class of economies than the Ramsey comparison with bliss.

For a final example, consider an economy with the linear utility function $U(c) = c$ and a discount factor $\delta > 1$. In the first case, suppose the economy has a production function

$$y = f(x) = \begin{cases} \rho_0 x & \text{for} \quad x \leqslant 1 \\ \rho_0 + \rho_1 x & \text{for} \quad x > 1 \end{cases}$$

with $\rho_0 > 1 > \delta\rho_1$, and has $y_0 = \rho_0$. Then, the programme $x_t = 1$, $c_t = \rho_0 - 1$ can be shown to be optimal. This example shows that differentiability is essential to Koopman's conclusion that no maximal programmes will exist in the resource-limited economy with negative discounting.

1.13. Result 1.10 and the two major borderline cases discussed in 1.11 provide a useful taxonomy of existence criteria: (1) the case in which the distant future is insignificant and one of the conditions 1.10 (a)–(c) is satisfied; (2) the resource-limited economy with no discounting; and (3) the productive linear economy with no discounting and no resource limits. The following sections of this paper will discuss each of these cases in turn for multi-commodity economies.

II. A MODEL OF A MULTI-COMMODITY ECONOMY

2.1. Consider time as an infinite sequence of short periods $t = 0, 1, \ldots$, and assume that there are a finite number of commodities N in each period. Let $\mathbf{x}_t$, $\mathbf{y}_t$ and $\mathbf{c}_t$ denote commodity vectors specifying the inputs to production, outputs from production, and consumption, respectively, in period t. Assume that the production possibilities of the economy are defined by a set $\mathbf{T}$ of non-negative input–output vectors $(\mathbf{x}_t, \mathbf{y}_{t+1})$ with the property that the output vector $\mathbf{y}_{t+1}$ can be attained when the input vector $\mathbf{x}_t$ is utilised in the preceding period.[1] Define an output correspondence $\mathrm{Q}(\mathbf{x}) = \{\mathbf{y} \mid (\mathbf{x}, \mathbf{y}) \in \mathrm{T}\}$. A feasible programme will be a non-negative sequence $(\mathbf{x}_t, \mathbf{y}_t, \mathbf{c}_t)$ satisfying $\mathbf{x}_t + \mathbf{c}_t = \mathbf{y}_t$ and $\mathbf{y}_{t+1} \in \mathrm{Q}(\mathbf{x}_t)$ for $t = 0, 1, \ldots$, where $\mathbf{y}_0$ is a given initial endowment.

[1] Hereafter we ignore the possibility of technical change, either via the introduction of new commodities or improvement in the technique of making old ones. There is no difficulty in principle in modifying the existence criteria below when technical change is present. However, there seems to be no consensus on the most appropriate way to introduce a structure on technical change in the multi-commodity model.

2.2. The following assumptions will be imposed on the production possibility set **T** and its output correspondence **Q**(**x**):

A.1. **T** is a closed convex set in the non-negative orthant of a $2N$-dimensional Euclidean space.

A.2. **T** allows free disposal of inputs and outputs [$(\mathbf{x}, \mathbf{y}) \in \mathbf{T}$, $\mathbf{x}' \geqslant \mathbf{x}$, $\mathbf{0} \leqslant \mathbf{y}' \leqslant \mathbf{y}$ imply $(\mathbf{x}', \mathbf{y}') \in \mathbf{T}$].

A.3. $\mathbf{Q}(\mathbf{0})$ is bounded.

A.4. Every commodity is producible [there exists $(\mathbf{x}, \mathbf{y}) \in \mathbf{T}$ with **y** positive].

These assumptions encompass both von Neumann and neo-classical models of the technology, provided in the latter case that endowments of primary and non-producible commodities grow at a common geometric rate, and the production possibility set is defined only over producible commodities, deflated by the growth rate of primary resources. In the case of a von Neumann technology, assumptions A.1 to A.4 are imposed directly on the production possibility set, along with the requirement that **T** be a cone and that $\mathbf{Q}(\mathbf{0}) = \{\mathbf{0}\}$. In the case of a neo-classical technology, we may think of an underlying production possibility cone $\mathbf{T}'$ containing triples $(\mathbf{z}_t, \mathbf{x}_t', \mathbf{y}'_{t+1})$ composed of a vector of endowments of commodities $\mathbf{z}_t$, including possibly both producible and primary commodities, a vector of inputs $\mathbf{x}_t$ attained from the output just produced, and a vector of outputs $\mathbf{y}_{t+1}$ in the following period. If $\mathbf{z}_t$ grows at a geometric rate g, so that $\mathbf{z}_t = \mathbf{z}_0 g^t$, define deflated commodity vectors $\mathbf{x}_t = \mathbf{x}_t'/g^t$, $\mathbf{y}_t = \mathbf{y}_t'/g^t$, and a stationary technology $\mathbf{T} = \{\mathbf{x}, \mathbf{y}) | (\mathbf{z}_0, \mathbf{x}, g\mathbf{y}) \in \mathbf{T}'\}$ expressed in 'per unit of primary commodity' terms. This technology will satisfy A.1 to A.4.[1]

2.3. It is convenient to summarise the asymptotic structure of the technology by defining the following two sets (illustrated in Fig. 12.1). Let $\mathbf{T}_0$ denote the closed cone spanned by the production possibility set **T**, i.e.

$$\mathbf{T}_0 = \text{Closure}\,(\{\lambda(\mathbf{x}, \mathbf{y}) | (\mathbf{x}, \mathbf{y}) \in \mathbf{T}, \quad \lambda \geqslant 0\}). \qquad (12.6)$$

Let $\mathbf{T}_1$ denote the asymptotic cone of **T**, i.e.

$$\mathbf{T}_1 = \{(\mathbf{x}, \mathbf{y}) | \lambda(\mathbf{x}, \mathbf{y}) \in \mathbf{T} \quad \text{for all} \quad \lambda \geqslant 0\}. \qquad (12.7)$$

Clearly $\mathbf{T}_1 \subseteq \mathbf{T} \subseteq \mathbf{T}_0$, with $\mathbf{T}_1 = \mathbf{T}_0$ if and only if **T** is a cone. We shall employ the following standard result on a *linear* technology.

2.4. *Theorem.*[2] *If* $\mathbf{T}^*$ *is a cone satisfying A.1, A.2, A.4 and* $\mathbf{Q}(\mathbf{0}) = \{\mathbf{0}\}$, *then there exist semi-positive vectors* $\hat{\mathbf{x}}$ *and* $\hat{\mathbf{p}}$ *and a positive scalar* ρ *(termed the maximal expansion rate or von Neumann growth rate of the technology) such that* $\hat{\mathbf{x}}$ *can be expanded at the*

[1] Gale (1967) gives the details of this construction, and discusses its properties.

[2] Karlin (1959) or Gale (1956).

rate ρ [*i.e.* $(\hat{\mathbf{x}}, \rho\hat{\mathbf{x}}) \in \mathbf{T}^*$], *profits* $\hat{\mathbf{p}}.\mathbf{y} - \rho\hat{\mathbf{p}}.\mathbf{x}$ *are non-positive for all* $(\mathbf{x}, \mathbf{y}) \in \mathbf{T}^*$, *and for all* $(\mathbf{x}, \mathbf{y}) \in \mathbf{T}^*$, $\mathbf{x} \neq \mathbf{0}$, ρ *is at least as great as any scalar* λ *satisfying* $\mathbf{y} \geqslant \lambda\mathbf{x}$.

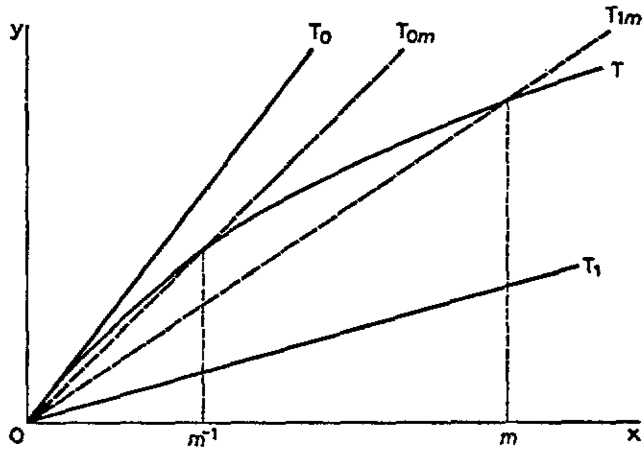

FIG. 12.1

2.5. The cone $\mathbf{T}_0$ either satisfies 2.4 and has a maximal expansion rate $\rho_0 < +\infty$, or contains a point $(0,\mathbf{y})$ with $\mathbf{y} \neq \mathbf{0}$ and can be defined to have a maximal expansion rate $\rho_0 = +\infty$. Similarly, the cone $\mathbf{T}_1$ either satisfies 2.4 and has a maximal expansion rate $\rho_1 > 0$, or has some non-producible commodity which is essential to production and can be defined to have a maximal expansion rate $\rho_1 = 0$. The expansion rates ρ_0 and ρ_1 will play the same role as did the asymptotic average productivities in the one-commodity model.

2.6. If maximal expansion in a linear technology $\mathbf{T}^*$ is achieved at an input vector $\hat{\mathbf{x}}$ which is not strictly positive, it may be impossible to produce a positive output vector starting from $\hat{\mathbf{x}}$. Alternately, if there exists a sequence $(\hat{\mathbf{x}}, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1})$ with $(\mathbf{x}_n, \mathbf{x}_{n+1}) \in \mathbf{T}^*$ and $\mathbf{x}_{N-1}$ positive, the technology is said to have the *recovery property*. This property will be required on the cones $\mathbf{T}_0$ or $\mathbf{T}_1$ of the general technology $\mathbf{T}$ for some existence criteria.

2.7. We shall assume that the overtaking criterion defined in section 1.3 is used to order feasible programmes, with the atemporal utility function $U(\mathbf{c})$ now assumed to have the following properties:

B.1. $U(\mathbf{c})$ is a continuous, concave, non-decreasing function of positive $\mathbf{c}$.

B.2. $U(\mathbf{c})$ is closed; i.e. if $\mathbf{c}$ is non-negative with some zero components, then

$$\overline{\lim_{\substack{\mathbf{c}'>0\\ \mathbf{c}'\to\mathbf{c}}}}\ U(\mathbf{c}') = U(\mathbf{c}) \geqslant -\infty.$$

B.3. $U(\mathbf{c})$ is non-satiated; i.e. $\mathbf{c}'-\mathbf{c}$ positive implies $U(\mathbf{c}') > U(\mathbf{c})$.

2.8. We now define asymptotic elasticities for the utility function $U(\mathbf{c})$, following rather closely the construction of Brock and Gale. Define a scalar $\bar{u}$ equal to the least upper bound of $U(\mathbf{c})$ provided this bound is finite, and equal to zero otherwise. For any positive vector $\mathbf{c}$ and scalar γ with $U(\gamma\mathbf{c}) \neq \bar{u}$ and $\gamma \neq 1$, define the *exponent* of $|U(\gamma\mathbf{c})-\bar{u}|$ as

$$1-\alpha(\mathbf{c}, \gamma) = \log_\gamma |U(\gamma\mathbf{c})-\bar{u}|. \tag{12.8}$$

Then, one has

$$|U(\gamma\mathbf{c})-\bar{u}| = \gamma^{1-\alpha(\mathbf{c},\gamma)} \tag{12.9}$$

revealing the relation of the exponent to the parameter α in the one-commodity, constant elasticity utility function. On the ray through a positive $\mathbf{c}$, define an asymptotic elasticity $\bar{\alpha}_1(\mathbf{c}) = \overline{\lim}_{\gamma\to+\infty}\ \alpha(\mathbf{c}, \gamma)$. We next show that $\bar{\alpha}_1 = \bar{\alpha}_1(\mathbf{c})$ is independent of $\mathbf{c}$.

For a positive scalar λ and a large positive scalar γ, one has from (12.8) the relation $1-\alpha(\mathbf{c}, \lambda\gamma) = [1-\alpha(\lambda\mathbf{c}, \gamma)](1-\log_{\lambda\gamma}\lambda)$ and $\lim_{\gamma\to+\infty} \log_{\lambda\gamma}\lambda = 0$. Hence, $\bar{\alpha}_1(\mathbf{c})$ is homogeneous of degree zero in $\mathbf{c}$. Consider two positive vectors $\mathbf{c}'$ and $\mathbf{c}''$, and positive scalars λ, μ such that $\lambda\mathbf{c}' \leqslant \mathbf{c}'' \leqslant \mu\mathbf{c}'$. By B.3, we have $U(\gamma\lambda\mathbf{c}') \leqslant U(\gamma\mathbf{c}'') \leqslant U(\gamma\mu\mathbf{c}')$, implying that $\alpha(\mathbf{c}'', \gamma)$ is bracketed by $\alpha(\lambda\mathbf{c}', \gamma)$ and $\alpha(\mu\mathbf{c}', \gamma)$. Hence, $\bar{\alpha}_1(\mathbf{c}'')$ is bracketed by $\bar{\alpha}_1(\lambda\mathbf{c}') = \bar{\alpha}_1(\mathbf{c}')$ and $\bar{\alpha}_1(\mu\mathbf{c}') = \bar{\alpha}_1(\mathbf{c}')$, implying $\bar{\alpha}_1(\mathbf{c}'') = \bar{\alpha}_1(\mathbf{c}')$. Hence, $\bar{\alpha}_1(\mathbf{c})$ is independent of $\mathbf{c}$. Similarly, $\underline{\alpha}_1 = \underline{\alpha}_1(\mathbf{c}) = \underline{\lim}_{\gamma\to+\infty}\ \alpha(\mathbf{c}, \gamma)$ is independent of $\mathbf{c}$.

With this result, we define the asymptotic elasticity of the utility function

$$\alpha_1 = \lim_{\gamma\to+\infty} \alpha(\mathbf{c}, \gamma) \quad \text{(for any positive } \mathbf{c}) \tag{12.10A}$$

where we impose the assumption

B.4A. The limit defining α_1 exists (i.e. $\bar{\alpha}_1 = \alpha_1$).

A similar construction will give an asymptotic elasticity α_0. Define $\bar{u}$ in (12.8) to be the greatest lower bound of $U(\mathbf{c})$ provided this bound is finite, and set $\bar{u}$ equal to zero otherwise. Then define the asymptotic elasticity of the utility function

$$\alpha_0 = \lim_{\gamma\to 0} \alpha(\mathbf{c}, \gamma) \quad \text{(for any positive } \mathbf{c}) \tag{12.10B}$$

where this limit is again independent of **c**, and exists under the assumption

B.4B. The limit defining α_0 exists (i.e. $\bar{\alpha}_0 = \alpha_0$).

Note that the following implications hold between the asymptotic elasticities and boundedness of the utility function:

$$\alpha_1 > 1 \rightarrow U(\mathbf{c}) \text{ bounded above} \rightarrow \alpha_1 \geqslant 1$$
$$0 \leqslant \alpha_1 < 1 \rightarrow U(\mathbf{c}) \text{ unbounded above} \rightarrow 0 \leqslant \alpha_1 \leqslant 1$$
$$0 \leqslant \alpha_0 < 1 \rightarrow U(\mathbf{c}) \text{ bounded below} \rightarrow 0 \leqslant \alpha_0 \leqslant 1$$
$$\alpha_0 > 1 \rightarrow U(\mathbf{c}) \text{ unbounded below} \rightarrow \alpha_0 \geqslant 1$$

2.9. An important property of a maximal programme $(\bar{\mathbf{x}}_t, \bar{\mathbf{y}}_t, \bar{\mathbf{c}}_t)$ in an economy is that it can normally be sustained by a 'decentralised' price system $(\bar{\mathbf{p}}_t)$ satisfying

$$\delta^t[U(\mathbf{c})-U(\bar{\mathbf{c}}_t)] \leqslant \bar{\mathbf{p}}_t.(\mathbf{c}-\bar{\mathbf{c}}_t) \quad \text{for all positive } \mathbf{c} \qquad (12.11)$$

and

$$\bar{\mathbf{p}}_{t+1}.\bar{\mathbf{y}}_{t+1}-\bar{\mathbf{p}}_t.\bar{\mathbf{x}}_t \geqslant \bar{\mathbf{p}}_{t+1}.\mathbf{y}-\bar{\mathbf{p}}_t.\mathbf{x} \quad \text{for all} \quad (\mathbf{x}, \mathbf{y})\varepsilon\mathbf{T}. \qquad (12.12)$$

A price system for which (12.12) is satisfied has been shown to exist under very general conditions (see Malinvaud, 1953; Radner, 1967). Price systems satisfying both (12.11) and (12.12) have been shown by Gale (1967) to exist for the resource-limited economy with no discounting, and by McFadden (1967) to exist for the non-resource-limited linear economy with no discounting. These constructions hold for much more general economies. We have the following result, in which the hypotheses are still unnecessarily strong:

2.10. *Theorem. Suppose an economy has a technology satisfying A.1 to A.4 and social preference satisfying B.1 to B.3. Suppose that the asymptotic cone of the technology,* $\mathbf{T}_1$, *has the recovery property. If* $(\bar{\mathbf{x}}_t, \bar{\mathbf{y}}_t, \bar{\mathbf{c}}_t)$ *is a maximal programme, then there exists a price system* $(\bar{\mathbf{p}}_t)$, *not identically zero, such that (12.12) holds, and*

$$U(\mathbf{c}) \geqslant U(\bar{\mathbf{c}}_t) \quad \text{implies} \quad \bar{\mathbf{p}}_t.\mathbf{c} \geqslant \bar{\mathbf{p}}_t.\mathbf{c}_t. \qquad (12.13)$$

If $\bar{\mathbf{p}}_t.\bar{\mathbf{c}}_t > 0$ *for any t, then the price system* $(\bar{\mathbf{p}}_t)$ *can be scaled so that both (12.11) and (12.12) hold.*

Proof: Define $\mathbf{c}^\nu = (\mathbf{c}_0, \ldots, \mathbf{c}_\nu)$, and define the set

$$\mathbf{C}^\nu = \{\mathbf{c}^\nu | \mathbf{c}_t = \mathbf{y}_t - \mathbf{x}_t, \quad (\mathbf{x}_t, \mathbf{y}_{t+1})\varepsilon\mathbf{T}, \quad \mathbf{y}_0 = \bar{\mathbf{y}}_0, \quad \mathbf{x}_\nu = \bar{\mathbf{x}}_\nu\}$$

and the function

$$W^\nu(\mathbf{c}^\nu) = \sum_{t=0}^{\nu} \delta^t U(\mathbf{c}_t).$$

Define the set

$$\mathbf{A} = \{(\mu, \mathbf{c}^\nu) | \mathbf{c}^\nu \geqslant \mathbf{c}'^\nu - \mathbf{c}''^\nu, \quad \mu \leqslant W^\nu(\mathbf{c}'^\nu), \quad \mathbf{c}''^\nu \varepsilon \mathbf{C}^\nu\}.$$

One can show that $\mathbf{A}$ is closed and convex, with a non-empty interior, and that $[W^{\nu}(\bar{\mathbf{c}}^{\nu}), 0]$ is a boundary point of $\mathbf{A}$. Then, there exists a vector $(\lambda, -\mathbf{p}^{\nu}) \neq 0$ such that

$$\lambda W^{\nu}(\bar{\mathbf{c}}^{\nu}) \geqslant \lambda\mu - \mathbf{p}^{\nu}.\mathbf{c}^{\nu}$$

for all $(\mu, \mathbf{c}^{\nu})\varepsilon\mathbf{A}$. From the construction of $\mathbf{A}$, one has $\lambda \geqslant 0$ and $\mathbf{p}^{\nu} \geqslant 0$. If one had $\lambda = 0$, then one would obtain the inequality $\mathbf{p}^{\nu}.\mathbf{c}^{\nu} \geqslant 0$, which is contradicted for some negative $\mathbf{c}^{\nu}$. Hence, one can normalise $\lambda = 1$. Taking $\mathbf{c}'^{\nu} = \bar{\mathbf{c}}^{\nu}$, $\mathbf{c}''^{\nu} = \mathbf{y}^{\nu} - \mathbf{x}^{\nu}$, $\mathbf{x}^{\nu} = (\bar{\mathbf{x}}_1, \ldots, \bar{\mathbf{x}}_{t-2}, \mathbf{x}, \bar{\mathbf{x}}_t, \ldots, \bar{\mathbf{x}}_{\nu})$ and $\mathbf{y}^{\nu} = (\bar{\mathbf{y}}_0, \ldots, \bar{\mathbf{y}}_{t-1}, \mathbf{y}, \bar{\mathbf{y}}_{t+1}, \ldots, \bar{\mathbf{y}}_{\nu})$ with $(\mathbf{x}, \mathbf{y})\varepsilon\mathbf{T}$, one obtains the condition

$$\mathbf{p}^{\nu}_{t+1}.\bar{\mathbf{y}}_{t+1} - \mathbf{p}_t^{\nu}.\bar{\mathbf{x}}_t \geqslant \mathbf{p}^{\nu}_{t+1}.\mathbf{y} - \mathbf{p}_t^{\nu}.\mathbf{x} \quad \text{for all} \quad (\mathbf{x}, \mathbf{y})\epsilon\mathbf{T}.$$

Since $\mathbf{T}_1$ has the recovery property, one has $(\mathbf{x}^*, \theta\mathbf{x}^*)\epsilon\mathbf{T}_1$ for some positive θ and $\mathbf{x}^*$, and hence $(\bar{\mathbf{x}}_t + \mathbf{x}^*, \bar{\mathbf{y}}_{t+1} + \theta\mathbf{x}^*)\epsilon\mathbf{T}$ implies in the inequality above that $\mathbf{p}_t^{\nu}.\mathbf{x}^* \leqslant \theta^{-t}\mathbf{p}_0^{\nu}.\mathbf{x}^*$. Next, letting $\mathbf{c}'^{\nu} = (\bar{\mathbf{c}}_1, \ldots, \bar{\mathbf{c}}_{t-1}, \mathbf{c}, \bar{\mathbf{c}}_{t+1}, \ldots, \bar{\mathbf{c}}_{\nu})$ and $\mathbf{c}''^{\nu} = \bar{\mathbf{c}}^{\nu}$ define a point in $\mathbf{A}$, one obtains the condition $\delta^t[U(\mathbf{c}) - U(\bar{\mathbf{c}}_t)] \leqslant \mathbf{p}_t^{\nu}.(\mathbf{c} - \bar{\mathbf{c}}_t)$. Now consider the sequence $\{\mathbf{p}_0^{\nu}\}$ as $\nu \to +\infty$. By B.3 and the last inequality, $\mathbf{p}_0^{\nu}$ is bounded positive as $\nu \to +\infty$. If $\{\mathbf{p}_0\}$ has a bounded sequence converging to a point $\bar{\mathbf{p}}_0$, then one can construct by the diagonal process and the bound $\mathbf{p}_t^{\nu}.\mathbf{x}^* \leqslant \theta^{-t}\mathbf{p}_0^{\nu}.\mathbf{x}^*$ a subsequence of $\mathbf{p}^{\nu}$ as $\nu \to +\infty$ converging pointwise to a sequence $(\bar{\mathbf{p}}_t)$ satisfying (12.11) and (12.12). Alternately, if $\{\mathbf{p}_0^{\nu}\}$ is unbounded, then a diagonal subsequence of $\{\mathbf{p}^{\nu}/|\mathbf{p}_0^{\nu}|\}$ converges to a sequence $(\bar{\mathbf{p}}_t)$ satisfying (12.12). Further, $U(\mathbf{c}) \geqslant U(\bar{\mathbf{c}}_t)$ implies $\mathbf{p}_t^{\nu}.(\mathbf{c} - \bar{\mathbf{c}}_t) \geqslant 0$, and hence $(\bar{\mathbf{p}}_t)$ satisfies (12.13). Taking $\mathbf{c} = \bar{\mathbf{c}}_t/2$ implies $\mathbf{p}_t^{\nu}.\bar{\mathbf{c}}_t \leqslant 2\delta^t[U(\bar{\mathbf{c}}_t) - U(\bar{\mathbf{c}}_t/2)]$, and hence in this case one has $\bar{\mathbf{p}}_t.\bar{\mathbf{c}}_t = 0$ for all t.

2.11. A feasible programme $(\bar{\mathbf{x}}_t, \bar{\mathbf{y}}_t, \bar{\mathbf{c}}_t)$ satisfying (12.11) and (12.12) is termed a *finitely competitive programme*. It would be most useful if every finitely competitive programme could be shown to be optimal, or even maximal. Combining (12.11) and (12.12), one can show that a finitely competitive programme satisfies

$$\sum_{t=0}^{\nu} \delta^t[U(\mathbf{c}_t) - U(\bar{\mathbf{c}}_t)] \leqslant \bar{\mathbf{p}}_{\nu}.(\bar{\mathbf{x}}_{\nu} - \mathbf{x}_{\nu}) \tag{12.14}$$

for any feasible programme $(\mathbf{x}_t, \mathbf{y}_t, \mathbf{c}_t)$. If one can establish that $\bar{\mathbf{p}}_{\nu}.\bar{\mathbf{x}}_{\nu} \to 0$, or that $|\bar{\mathbf{p}}_{\nu}(\bar{\mathbf{x}}_{\nu} - \mathbf{x}_{\nu})| \to 0$ for any programme that is not 'infinitely worse' than the finitely competitive programme, then one can attain this desired conclusion. This is the case, for example, in some models studied by Gale (1967) and the author (1967). However, in general, a finitely competitive programme need not be maximal, and a maximal programme need not be optimal. We give two examples:

2.12. First, consider a one-commodity economy with a production function $y_{t+1} = f(x_t) = x_t$, a utility function $U(c) = c/(1+c)$, a discount factor $\delta = 1$, and an initial endowment $y_0 = 1$. Then, $\bar{x}_t = \bar{y}_t = 1$, $\bar{c}_t = 0$, $\bar{p}_t = 1$ is a finitely competitive programme satisfying (12.11) and (12.12), but is clearly not maximal. (In this example, due to Gale, no maximal programme exists.)

2.13. Second, consider a six-commodity economy with a utility function $U(\mathbf{c}) = c_6$ which is linear in the sixth commodity and independent of the remaining commodities, a discount factor $\delta = 1$, and a von Neumann technology of the form $\mathbf{T} = \{(\mathbf{x}, \mathbf{y}) | \mathbf{x} \geqslant \mathbf{A}v, \mathbf{y} \leqslant \mathbf{B}v, v \geqslant \mathbf{0}\}$, where $\mathbf{A}$ and $\mathbf{B}$ are matrices satisfying

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 \end{bmatrix}$$

Suppose the economy has the initial endowment $\mathbf{y}_0 = (1, 0, 0, 0, 0, 0)$. A pure accumulation programme using activity 1 in period zero, followed by activities 2 and 3 alternated in succeeding periods, yields a consumption stream $(c_{6t}') = (0, 2, 0, 8, 0, 32, \ldots)$. Alternately, a pure accumulation programme using activity 4 in period zero, followed by activities 5 and 6 alternated in succeeding periods, yields a consumption stream $(c_{6t}'') = (0, 0, 4, 0, 16, 0, \ldots)$. These two streams and their convex combinations are the only efficient consumption programmes. However, one has for $\nu > 1$:

$$\sum_{t=0}^{\nu} [U(\mathbf{c}_t') - U(\mathbf{c}_t'')] = \begin{cases} 2(2^\nu - 1)/3 & \text{for} \quad \nu \text{ even} \\ -2(2^\nu + 1)/3 & \text{for} \quad \nu \text{ odd.} \end{cases}$$

Hence, for any $0 \leqslant \theta \leqslant 1$, the programme $(\theta\mathbf{c}_t' + (1-\theta)\mathbf{c}_t'')$ is maximal, but none of these maximal programmes are optimal.

III. EXISTENCE CRITERIA FOR ECONOMIES WITH AN INSIGNIFICANT FUTURE

3.1. We are now prepared to state a multi-sector generalisation of the existence criteria for that 'insignificant future' case treated by Brock and Gale.

3.2. *Theorem. Suppose a multi-sector economy has a technology satisfying A.1 to A.4 and social preferences satisfying B.1 to B.4. Suppose that the overtaking criterion (I) is used to define optimal* *programmes. Suppose further that the initial endowment* $\mathbf{y}_0$ *is positive. Then, any one of the following three conditions is sufficient for the existence of an optimal programme:*

(a) $\rho_1 > 1$, $\delta\rho_1^{1-\alpha_1} < 1$, *and* $\mathbf{T}_1$ *has the recovery property.*

(b) $\rho_0 < 1$, $\delta\rho_0^{1-\alpha_0} < 1$, *and* $\mathbf{T}_0$ *has the recovery property.*

(c) $\rho_0 > 1 > \rho_1$ *and* $\delta < 1$.

3.3. The remainder of this section will be taken up with the proof of this theorem and the statement and proof of the converse non-existence theorem. We begin with a series of preliminary lemmas.

3.4. *Lemma. If A.1 to A.4 hold, then, given* $\varepsilon > 0$, *there exists* $m > 0$ *such that the cones*

$$\mathbf{T}_{0m} = \{\lambda(\mathbf{x}, \mathbf{y}) | (\mathbf{x}, \mathbf{y}) \epsilon \mathbf{T}, \ |\mathbf{x}| \geqslant m^{-1}, \quad \lambda \geqslant 0\}$$
$$\mathbf{T}_{1m} = \{\lambda(\mathbf{x}, \mathbf{y}) | (\mathbf{x}, \mathbf{y}) \epsilon \mathbf{T}, \ |\mathbf{x}| \geqslant m, \quad \lambda \geqslant 0\}$$

satisfy $|\rho_{0m}^{-1} - \rho_0^{-1}| < \varepsilon$ *and* $|\rho_{1m} - \rho_1| < \varepsilon$, *where* ρ_{im} *is the maximal expansion rate for* $\mathbf{T}_{im}$, $i = 0, 1$ *(see Fig. 12.1).*

3.5. *Lemma.*[1] *Suppose* $\mathbf{T}^*$ *is a linear technology with a maximal expansion rate* ρ'. *Then, for any* $\rho > \rho'$, *there exists* $\eta > 1$ *such that for any sequence* $(\mathbf{x}_0, \ldots, \mathbf{x}_t)$ *with* $(\mathbf{x}_{\tau-1}, \mathbf{x}_\tau) \epsilon \mathbf{T}^*$, $\tau = 1, \ldots, t$, *it follows that* $|\mathbf{x}_t|/\rho^t \leqslant \eta|\mathbf{x}_0|$.

3.6. *Lemma. If* $\mathbf{T}$ *satisfies A.1 to A.4 and has* $\rho_1 \geqslant 1$, *then for any* $\rho > \rho_1$, *there exists* $\eta_2 > 0$ *such that* $|\mathbf{c}_t|/\rho^t \leqslant \eta_2$ *for any feasible programme* $(\mathbf{x}_t, \mathbf{y}_t, \mathbf{c}_t)$.

Proof: Given ρ, choose $\varepsilon = (\rho - \rho_1)/2$ in 3.4. Consider the cone $\mathbf{T}_{1m}$, and let η be the bound given by 3.5. Note that $(\mathbf{x}, \mathbf{y}) \epsilon \mathbf{T}$, $|\mathbf{x}| \leqslant m$ implies $|\mathbf{y}| \leqslant m\eta$. Consider any $\mathbf{y}_t$. If $|\mathbf{x}_{t-1}| \leqslant m$, then $|\mathbf{y}_t| \leqslant m\eta \leqslant m\eta\rho^t$. Alternately, if one has $|\mathbf{x}_s| \leqslant m$ and $|\mathbf{x}_\tau| > m$ for $s < \tau < t$, then 3.5 implies $|\mathbf{y}_t|/\rho^{t-s} \leqslant \eta m \leqslant \eta m \rho^s$. Finally, if one has $|\mathbf{x}_\tau| > m$ for $0 \leqslant \tau < t$, then 3.5 implies $|\mathbf{y}_t|/\rho^t \leqslant \eta|\mathbf{y}_0|$. Hence, taking $\eta_2 = \eta \max(m, |\mathbf{y}_0|)$ yields the result.

3.7. *Lemma. If* $\mathbf{T}$ *satisfies A.1 to A.4 and has* $\rho_1 < 1$, *then there exists* $\eta_2 > 0$ *such that* $|\mathbf{c}_t| \leqslant \eta_2$ *for any feasible programme* $(\mathbf{x}_t, \mathbf{y}_t, \mathbf{c}_t)$.

Proof: In the proof of 3.6, choose $\varepsilon = (1 - \rho_1)/2$. Then, that argument implies $|\mathbf{y}_t| \leqslant \eta \max(m, |\mathbf{y}_0|) = \eta_2$.

3.8. *Lemma. If* $\mathbf{T}$ *satisfies A.1 to A.4 and has* $\rho_0 \leqslant 1$, *then for any* $\rho > \rho_0$ *there exists* $\eta > 0$ *such that* $|\mathbf{c}_t|/\rho^t \leqslant \eta$ *for any feasible programme* $(\mathbf{x}_t, \mathbf{y}_t, \mathbf{c}_t)$.

Proof: Since $(\mathbf{x}_t, \mathbf{y}_{t+1}) \epsilon \mathbf{T}_0$, 3.5 implies the result.

3.9. *Lemma. Suppose* $\mathbf{T}^*$ *is a linear technology with a maximal*

[1] Winter (1965) theorem 2.

expansion rate ρ' which has the recovery property. Then, for any $\rho'' < \rho'$, there exists a feasible programme $(\mathbf{x}_t, \mathbf{y}_t, \mathbf{c}_t)$ for $\mathbf{y}_0$ positive such that $\lim_{t\to+\infty} \mathbf{c}_t/(\rho'')^t = +\infty$.

Proof: Let $(\hat{\mathbf{x}}, \hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_{N-1})$ be the sequence defined in the recovery property which has $\hat{\mathbf{x}}_{N-1}$ positive. Choose $\theta > 0$ such that $\theta^{-1}\mathbf{y}_0 \geqslant 2\hat{\mathbf{x}} + \sum_{t=1}^{N-1} \hat{\mathbf{x}}_t$. Computation shows that a programme based on accumulation at the maximal rate can yield $\mathbf{c}_t = \theta\hat{\mathbf{x}}_{N-1}$ for $t = 0, \ldots, N-1$ and $c_t = \theta(\rho'-\rho)\rho^{t-N}\hat{\mathbf{x}}_{N-1}$ for $t = N, N+1, \ldots$, and $\rho'' < \rho < \rho'$, establishing the result.

3.10. *Lemma. Consider a utility function $U(\mathbf{c})$ satisfying B.1 to B.4 with asymptotic elasticities α_0 and α_1. Given $\varepsilon > 0$ and a closed, bounded set $\mathbf{C}$ of positive vectors $\mathbf{c}$, there exists $\gamma_1 > 0$ such that*

$$\gamma^{1-\alpha_1-\varepsilon} < |U(\gamma\mathbf{c})-\bar{u}| < \gamma^{1-\alpha_1+\varepsilon} \qquad (12.14\text{A})$$

for $\mathbf{c}\epsilon\mathbf{C}$ and $\gamma \geqslant \gamma_1$, where $\bar{u}$ is defined as in equation (12.8). Similarly, there exists $\gamma_0 > 0$ such that

$$\gamma^{1-\alpha_0-\varepsilon} > |U(\gamma\mathbf{c})-\bar{u}| > \gamma^{1-\alpha_0+\varepsilon} \qquad (12.14\text{B})$$

for $\mathbf{c}\epsilon\mathbf{C}$ and $\gamma \leqslant \gamma_0$.

Proof: Given $\mathbf{C}$, choose any $\mathbf{c}'\epsilon\mathbf{C}$ and positive scalars λ, μ such that $\lambda\mathbf{c}' \leqslant \mathbf{c}'' \leqslant \mu\mathbf{c}'$ for all $\mathbf{c}''\epsilon\mathbf{C}$. From the definition of α_1, there exists γ_1 such that (12.14A) holds for $\mathbf{c} = \lambda\mathbf{c}'$ and $\mathbf{c} = \mu\mathbf{c}'$, and such that $U(\gamma\lambda\mathbf{c}')$ is univalent for $\gamma \geqslant \gamma_1$. Then $|U(\gamma\mathbf{c})-\bar{u}|$ defined on $\mathbf{C}$ is bracketed by the values of this expression for $\mathbf{c} = \lambda\mathbf{c}'$ and $\mathbf{c} = \mu\mathbf{c}'$, implying the stated result. A similar argument establishes (12.14B).

3.11. *Lemma.*[1] *Let $\mathbf{U}$ be a family of non-negative sequences (u_t) which is closed under pointwise convergence and such that $\sum_{t=0}^{+\infty} u_t < +\infty$ for at least one member of $\mathbf{U}$. Then, there is a member $(\bar{u}_t)$ such that $\sum_{t=0}^{+\infty} \bar{u}_t$ is a minimum.*

3.12. We are now prepared to prove the theorem 3.2. The argument follows closely that of Brock and Gale.

(a) Suppose $\rho_1 > 1$ and $\delta\rho_1^{1-\alpha_1} < 1$. Choose $1 < \underline{\rho} < \rho_1 < \bar{\rho}$ and $\underline{\alpha} < \alpha_1 < \bar{\alpha}$ such that $\delta\underline{\rho}^{1-\underline{\alpha}} < 1$ and $\delta\bar{\rho}^{1-\underline{\alpha}} < 1$. First consider the case of $U(\mathbf{c})$ unbounded above, implying $0 \leqslant \alpha_1 \leqslant 1$. By 3.6 and 3.10, $U(\mathbf{c}_t) = U[\bar{\rho}^t(\mathbf{c}_t/\bar{\rho}^t)] \leqslant U(\bar{\rho}^t\eta_2\hat{\mathbf{c}}) < (\bar{\rho}^t)^{1-\underline{\alpha}}$ for sufficiently large t, where $\hat{\mathbf{c}}$ is a vector of ones and $(\mathbf{c}_t)$ is any feasible programme. Hence, for η_5 sufficiently large, the sequence $\{\eta_5(\delta\bar{\rho}^{1-\underline{\alpha}})^t - \delta^t U(\mathbf{c}_t)\}$

[1] Brock and Gale (1970).

is non-negative for any feasible programme $(\mathbf{c}_t)$. By 3.9, there exists a feasible programme $(\bar{\mathbf{c}}_t)$ with $\bar{\mathbf{c}}_t/\rho^t \to +\infty$. For t large, $\bar{\mathbf{c}}_t/\rho^t \geqslant \hat{\mathbf{c}}$, and 3.10 implies $U(\bar{\mathbf{c}}_t) \geqslant U(\rho^t\hat{\mathbf{c}}) \geqslant (\rho^t)^{1-\bar{\alpha}}$. Hence, $\sum_{t=0}^{\infty} \delta^t U(\bar{\mathbf{c}}_t)$ is bounded below. Since U is continuous and the set of feasible programmes is pointwise closed and bounded, the family of sequences $\{\eta_5(\delta\bar{\rho}^{1-\alpha})^t - \delta^t U(\mathbf{c}_t)\}$ satisfies 3.11. This result then establishes that an optimal utility stream exists and is achieved by some feasible programme which is consequently optimal.

Next consider the case of $U(\mathbf{c})$ bounded above, implying $\alpha_1 \geqslant 1$. Without loss of generality, take $\bar{u} = 0$. Then, $-\sum_{t=0}^{+\infty} \delta^t U(\mathbf{c}_t)$ is non-negative, and by 3.9 and 3.10 there exists a programme $(\bar{\mathbf{c}}_t)$ such that for large t, $-U(\bar{\mathbf{c}}_t) \leqslant (\rho^{1-\alpha})^t$, or $-\sum_{t=0}^{+\infty} \delta^t U(\bar{\mathbf{c}}_t) \leqslant \eta_6/(1-\delta\rho^{1-\alpha})$, for some η_6. Then 3.11 implies the existence of an optimal programme.

(b) and (c). Suppose $\rho_1 < 1$. By 3.7, there exists a bound η_2 such that $|\mathbf{c}_t| \leqslant \eta_2$ for any feasible programme $(\mathbf{c}_t)$. Hence, without loss of generality, we can define the zero level of $U(\mathbf{c})$ so that $\{-U(\mathbf{c}_t)\}$ is a non-negative sequence for all feasible programmes. Consider any ρ satisfying $0 < \rho < \rho_0$. By 3.4, there exists $m > 0$ such that $\mathbf{T}_{0m}$ has the recovery property, and has a maximum expansion rate ρ with $\rho < \rho < \rho_0$. Further, $(\mathbf{x}, \mathbf{y}) \epsilon \mathbf{T}_{0m}$ and $|\mathbf{x}| < m^{-1}$ implies $(\mathbf{x}, \mathbf{y}) \epsilon \mathbf{T}$. In case (b) with $\delta\rho_0^{1-\alpha_0} < 1$, choose $\rho < \rho_0$ and $\bar{\alpha} > \alpha_0$ such that $\delta\rho^{1-\bar{\alpha}} < 1$. We can apply 3.9 to $\mathbf{T}_{0m}$ to establish the existence of a feasible programme $(\bar{\mathbf{c}}_t)$ with $\bar{\mathbf{c}}_t/\rho^t \to +\infty$. Then, using 3.10, one has $-U(\bar{\mathbf{c}}_t) \leqslant (\rho^{1-\bar{\alpha}})^t$ for t large, and $-\sum_{t=0}^{\infty} \delta^t U(\bar{\mathbf{c}}_t)$ is bounded. Then, 3.11 can be applied to establish the existence of an optimal programme. In case (c) with $\rho_0 > 1$, choose $\rho = 1$. Then using the same arguments as in the proof of 3.9, we can establish the existence of $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \epsilon \mathbf{T}_{0m}$ with $|\tilde{\mathbf{x}}| < m^{-1}$ and $\bar{\mathbf{c}} = \tilde{\mathbf{y}} - \tilde{\mathbf{x}}$ positive. Then, $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \epsilon \mathbf{T}$, and we can assume $\tilde{\mathbf{y}} \leqslant \mathbf{y}_0$. Hence, the steady-state programme $(\bar{\mathbf{c}})$ is feasible, and $-\sum_{t=0}^{\infty} \delta^t U(\bar{\mathbf{c}})$ is bounded. The existence of an optimal programme is then established using 3.11.

3.13. One would like to establish a multi-commodity analogue of the non-existence criteria (d)–(f) in 1.10, corresponding to the result 3.2. That some further assumption is required to establish such a theorem is shown by the following example. Consider a two-commodity economy with a constant elasticity utility function $U(c_1, c_2) = c_2^{1-\alpha}/(1-\alpha)$ and a linear technology with a single efficient activity $(\mathbf{x}, \mathbf{y}) \epsilon \mathbf{T}$ satisfying $\mathbf{x} = (1, 0)$ and $\mathbf{y} = (\rho, \rho)$ with

$\rho > 0$. Then, given $\mathbf{y}_0 = (1, 1)$, the programme $\mathbf{c}_t = (0, \rho^t)$ is optimal for any values of the parameters δ, ρ and α. We next introduce several conditions which will be sufficient to establish criteria for non-existence.

We shall call a utility function $U(\mathbf{c})$ which satisfies B.1 to B.4 *asymptotically homothetic* at infinity (resp., at zero) if it can be written as the sum of two functions $U(\mathbf{c}) = u[H(\mathbf{c})] + V(\mathbf{c})$, where H and V are concave non-decreasing functions of positive $\mathbf{c}$, with H linear homogeneous and u a concave increasing function on the positive real line, and where $U(\mathbf{c})$ and $u[H(\mathbf{c})]$ have the same asymptotic elasticity α_1 (resp., α_0).[1] Without loss of generality, one can assume in the definition of an asymptotically homothetic utility function that

$$\operatorname*{Max}_{|\mathbf{c}|=1} H(\mathbf{c}) = 1.$$

Then,

$$\operatorname*{Max}_{|\mathbf{c}|=\gamma} u[H(\mathbf{c})] = u(\gamma).$$

If $U(\mathbf{c})$ is asymptotically homothetic at infinity, $u(\gamma)$ and $H(\mathbf{c})$ are continuously differentiable, and

$$\lim_{\gamma \to +\infty} \log_\gamma u'(\gamma)$$

exists, then $U(\mathbf{c})$ will be called *asymptotically smooth*. For this case, one has

$$\alpha_1 = -\lim_{\gamma \to +\infty} \log_\gamma u'(\gamma).^{2}$$

A similar definition can be made at zero. The following condition guarantees that maximal programmes will be strictly positive:

B.5. $U(\mathbf{c})$ is continuously differentiable for $\mathbf{c}$ positive, and if a non-negative $\mathbf{c}'$ has some zero components, then the corresponding components of $U'(\mathbf{c})$ are unbounded for positive $\mathbf{c}$ converging to $\mathbf{c}'$.

We are now prepared to state criteria for non-existence of maximal programmes.

3.16. *Theorem. Suppose a multi-commodity economy has a technology satisfying A.1 to A.4, and social preferences satisfying B.1 to B.5. Suppose that the initial endowment* $\mathbf{y}_0$ *is positive. Then, any one of the following three conditions is sufficient for the non-existence of a maximal programme:*

[1] Suppose the function $u(\gamma)$ on the positive real line has an asymptotic elasticity α_1' defined as in (12.10A), and that the function $V(\mathbf{c})$ has asymptotic elasticities $\underline{\alpha}_1''$ and $\overline{\alpha}_1''$ defined as in the argument preceding (12.10A). If $\alpha_1' \leqslant \underline{\alpha}_1''$ and $H(c)$ is not identically zero, then $U(c)$ has the same asymptotic elasticity $\alpha_1 = \alpha_1'$ as $u(H(c))$. Analogously, if $\alpha_0' \geqslant \overline{\alpha}_0''$ and $H(c)$ is not identically zero, then $\alpha_0 = \alpha_0'$.

[2] Brock and Gale (1970) Appendix.

(d) $\rho_1 > 1$, $\delta\rho_1^{1-\alpha_1} > 1$, T_1 *has the recovery property, and* $U(\mathbf{c})$ *is asymptotically homothetic and smooth at infinity.*

(e) $\rho_0 < 1$, $\delta\rho_0^{1-\alpha_0} > 1$, T_0 *contains a point* $(\hat{\mathbf{x}}, \rho_0\hat{\mathbf{x}})$ *with* $\hat{\mathbf{x}}$ *positive, and* $U(\mathbf{c})$ *is asymptotically homothetic and smooth at zero.*

(f) $\rho_0 > 1 > \rho_1$, $\delta\rho_1 > 1$, *and* T_1 *contains a point* $(\hat{\mathbf{x}}, \rho_1\hat{\mathbf{x}})$ *with* $\hat{\mathbf{x}}$ *positive.*

Proof: (d) Suppose $\rho_1 > 1$ and $\delta\rho_1^{1-\alpha_1} > 1$, but suppose that a *maximal programme* $(\bar{\mathbf{c}}_t)$ exists. By B.5, $\bar{\mathbf{c}}_0$ is positive. Choose $\theta > 0$ such that $\bar{\mathbf{c}}_0 - \theta\hat{\mathbf{x}}$ is positive, where $\hat{\mathbf{x}}$ is a semi-positive vector with $(\hat{\mathbf{x}}, \rho_1\hat{\mathbf{x}}) \in T_1$. Since $T = T + T_1$, a programme $(\tilde{\mathbf{c}}_t)$ with $\tilde{\mathbf{c}}_0 = \bar{\mathbf{c}}_0 - \theta\hat{\mathbf{x}}$, $\tilde{\mathbf{c}}_\tau = \bar{\mathbf{c}}_\tau + \theta\rho_1^{\tau-N}\mathbf{x}_{N-1}$, and $\tilde{\mathbf{c}}_t = \bar{\mathbf{c}}_t$ for $t \neq 0, \tau$ is feasible for $\tau > N$, where $\mathbf{x}_{N-1}$ is a positive vector which can be produced from $\hat{\mathbf{x}}$ in N periods. Let $\lambda = U(\bar{\mathbf{c}}_0) - U(\tilde{\mathbf{c}}_0)$. Note that $0 < H(\mathbf{x}_{N-1}) \leqslant H'(\mathbf{c}) . \mathbf{x}_{N-1}$ for all positive $\mathbf{c}$, and hence that

$$u[H(\mathbf{c}+\mathbf{c}')] - u[H(\mathbf{c})] \geqslant u'[H(\mathbf{c}+\mathbf{c}')]H'(\mathbf{c}).\mathbf{c}' \geqslant u'[H(\mathbf{c}+\mathbf{c}')]H(\mathbf{c}').$$

Then,

$$\sum_{t=0}^{+\infty} \delta^t[U(\tilde{\mathbf{c}}_t) - U(\bar{\mathbf{c}}_t)] \geqslant -\lambda + \delta^\tau\{u[H(\tilde{\mathbf{c}}_\tau)] - u[H(\bar{\mathbf{c}}_\tau)]\}$$
$$\geqslant -\lambda + \delta^\tau u'[H(\tilde{\mathbf{c}}_\tau)]H(\theta\rho_1^{\tau-N}\mathbf{x}_{N-1}).$$

Choose $\underline{\alpha} < \alpha_1$ and $\bar{\rho} > \rho_1$ such that $\delta\rho_1\bar{\rho}^{-\underline{\alpha}} > 1$. From the properties of the asymptotic elasticity, one has $u'(\gamma) > \gamma^{-\bar{\alpha}}$ for γ sufficiently large. By 3.6, one has $|\bar{\mathbf{c}}_t|/\rho^{-t} \leqslant \eta$. Hence,

$$H(\tilde{\mathbf{c}}_\tau) \leqslant \eta\rho^{-\tau} \quad \text{and} \quad u'[H(\tilde{\mathbf{c}}_\tau)] > (\eta)^{-\bar{\alpha}}\bar{\rho}^{-\bar{\alpha}\tau}.$$

Let

$$\lambda' = (\eta)^{-\bar{\alpha}}\theta\rho_1^{-N}H(\mathbf{x}_{N-1}).$$

Then,

$$\sum_{t=0}^{+\infty} \delta^t[U(\tilde{\mathbf{c}}_t) - U(\bar{\mathbf{c}}_t)] \geqslant -\lambda + \lambda'(\delta\rho_1\bar{\rho}^{-\underline{\alpha}})^\tau.$$

For τ sufficiently large, the right-hand side of this expression is positive, contradicting the supposition that $(\bar{\mathbf{c}}_t)$ was maximal.

(e) Suppose $\rho_0 < 1$ and $\delta\rho_0^{1-\alpha_0} > 1$, but suppose that a maximal programme $(\bar{\mathbf{c}}_t)$ exists. Choose $\rho < \rho_0$ such that $\delta\rho^{1-\alpha_0} > 1$. From the construction of T_0, there exists a positive scalar γ such that $\gamma(\hat{\mathbf{x}}, \rho\hat{\mathbf{x}})$ is the interior of T. Then, there exists $\eta > 0$ such that $(\mathbf{x}, \mathbf{y}) \in T$ and $|\mathbf{x}| \leqslant \eta$ *imply* $\gamma(\hat{\mathbf{x}}, \rho\hat{\mathbf{x}}) + (\mathbf{x}, \mathbf{y}) \in T$. *By 3.8, the maximal* programme $(\bar{\mathbf{x}}_t, \bar{\mathbf{y}}_t, \bar{\mathbf{c}}_t)$ has $|\bar{\mathbf{x}}_t| \leqslant \eta/2$ for $t \geqslant \nu$, some $\nu > 0$. Choose $\theta > 0$ such that $\bar{\mathbf{c}}_\nu - \theta\hat{\mathbf{x}}$ is positive. Then, the programme $(\tilde{\mathbf{c}}_t)$ with $\tilde{\mathbf{c}}_\nu = \bar{\mathbf{c}}_\nu - \theta\hat{\mathbf{x}}$, $\tilde{\mathbf{c}}_\tau = \bar{\mathbf{c}}_\tau + \theta\rho^{\tau-\nu}\hat{\mathbf{x}}$, and $\tilde{\mathbf{c}}_t = \bar{\mathbf{c}}_t$ for $t \neq \nu, \tau$ is feasible, and can be shown by an argument paralleling that of (d) to be better than $(\bar{\mathbf{c}}_t)$ for τ sufficiently large. Hence, $(\bar{\mathbf{c}}_t)$ cannot be maximal.

(f) Suppose $\rho_0 > 1 > \rho_1$ and $\delta\rho_1 > 1$, but suppose that a maximal programme $(\bar{\mathbf{c}}_t)$ exists. Since $(\bar{\mathbf{x}}_t+\hat{\mathbf{x}}, \bar{\mathbf{y}}_{t+1}+\rho_1\hat{\mathbf{x}})\epsilon\mathbf{T}$ for all t, one must have $U(\bar{\mathbf{c}}_t)+\delta U(\bar{\mathbf{c}}_{t+1}) \geqslant U(\bar{\mathbf{c}}_t-\theta\hat{\mathbf{x}})+\delta U(\bar{\mathbf{c}}_{t+1}+\theta\rho_1\hat{\mathbf{x}})$ for small θ, implying $0 \geqslant -\delta^t U'(\bar{\mathbf{c}}_t).\hat{\mathbf{x}}+\delta^{t+1}\rho_1 U'(\bar{\mathbf{c}}_{t+1}).\hat{\mathbf{x}}$. Hence, $U'(\bar{\mathbf{c}}_t).\hat{\mathbf{x}} \leqslant U'(\bar{\mathbf{c}}_0).\hat{\mathbf{x}}/(\delta\rho_1)^t$. But the right-hand side of this expression converges to zero, implying that $\bar{\mathbf{c}}_t$ is unbounded, and contradicting 3.7. Hence, $(\bar{\mathbf{c}}_t)$ cannot be maximal.

IV. EXISTENCE CRITERIA FOR THE RESOURCE-LIMITED, NO-DISCOUNTING ECONOMY

4.1. We next summarise existence criteria for an important 'borderline' case, the economy with no discounting in which outputs of produced commodities are limited by the availability of primary resources. This problem has been solved for the multi-commodity case by Gale (1967). A slight weakening of Gale's assumptions and a considerable simplification in analysis have been made by Brock (1970). In stating this result, we require one additional assumption (a somewhat weaker condition is used by Brock):

B.6. $U(\mathbf{c})$ is strictly concave and continuously differentiable, with $U'(\mathbf{c})$ bounded, for all positive $\mathbf{c}$.

Note that assumption B.6 is inconsistent with assumption B.5.

4.2. *Theorem. Suppose a multi-commodity economy has a technology satisfying A.1 to A.4, with* $\rho_0 > 1 > \rho_1$, *and social preferences satisfying B.1 to B.3 and B.6, and* $\delta = 1$. *Suppose that the initial endowment vector* $\mathbf{y}_0$ *is positive. Then, an optimal programme exists.*

Proof: By 3.7, if $\rho_1 < 1$, then all feasible programmes are bounded. Hence, replacing the original technology $\mathbf{T}$ with the technology $\mathbf{T}' = \{(\mathbf{x}, \mathbf{y})\epsilon\mathbf{T}\,|\,|\mathbf{x}| \leqslant \eta\}$ for a large scalar η leaves the problem unchanged except that the technology $\mathbf{T}'$ is closed and bounded. Then, Brock's proof applies.

V. EXISTENCE CRITERIA FOR THE NON-RESOURCE-LIMITED NO-DISCOUNTING ECONOMY

5.1. The final 'borderline' case we shall consider is a productive linear technology (i.e. T is a cone with $\rho_0 = \rho_1 > 1$) in which outputs are not limited by the availability of primary resources, for the case of no discounting. For this case, results 3.2 (a) and 3.16 (d) establish (1) if the asymptotic elasticity α_1 is greater than one (implying $U(\mathbf{c})$ bounded above), then an optimal programme exists; and (2) if α_1 is less than one and $U(\mathbf{c})$ is asymptotically homothetic and smooth (implying $U(\mathbf{c})$ unbounded above), then no optimal programme exists. With several additional restrictions on the technology, the author

(1970) has sharpened this result to establish that $U(\mathbf{c})$ bounded above is necessary and sufficient to imply the existence of an optimal programme. To the assumptions A.1 to A.4, we first add the condition:

A.5. The technology $\mathbf{T}$ is a cone with $\rho_1 > 1$, and the vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ in 2.4, satisfying $(\hat{\mathbf{x}}, \rho_0\hat{\mathbf{x}})\epsilon\mathbf{T}$ and $\hat{\mathbf{p}}.\mathbf{y} \leqslant \rho_0\hat{\mathbf{p}}.\mathbf{x}$ for all $(\mathbf{x}, \mathbf{y})\epsilon\mathbf{T}$, can be taken to be positive.

This assumption will be satisfied if the economy is irreducible (i.e. all commodities are needed, directly or indirectly, to produce any given commodity) and has sufficient output substitutability to avoid 'over-production' of some commodities in attaining maximal growth.

A feasible programme $(\bar{\mathbf{x}}_t, \bar{\mathbf{y}}_t, \bar{\mathbf{c}}_t)$ is *good* if there is a scalar $M > 0$ such that for any other feasible programme $(\mathbf{x}_t, \mathbf{y}_t, \mathbf{c}_t)$, one has

$$\sum_{t=0}^{\nu} [U(\mathbf{c}_t) - U(\bar{\mathbf{c}}_t)] \leqslant M, \quad \nu = 1, 2, \ldots.$$

The first result is a condition for the existence of good programmes:

5.2. *Theorem. Suppose a multi-commodity economy has a linear technology satisfying A.1 to A.5. Suppose that social preferences satisfy B.1 to B.3 and $\delta = 1$. Suppose that the initial endowment $\mathbf{y}_0$ is positive. Then, a good programme exists if and only if $U(\mathbf{c})$ is bounded above.*

Proof: McFadden (1967), Theorem 6.

5.3. A sequence of programmes $(\mathbf{x}_t^j, \mathbf{y}_t^j, \mathbf{c}_t^j)$ for $j = 1, 2, \ldots$, is termed an *optimising sequence* if each of these programmes is comparable to all other feasible programmes; i.e.

$$\lim_{\nu\to\infty} \sum_{t=0}^{\nu} [U(\mathbf{c}_t) - U(\mathbf{c}_t^j)]$$

exists for all feasible programmes $(\mathbf{c}_t)$, and one has

$$\lim_{j\to+\infty} \sum_{t=0}^{\infty} [U(\mathbf{c}_t) - U(\mathbf{c}_t^j)] \leqslant 0.$$

A result established by the author for a very general class of economies with linear technologies can be specialised to give a relationship between good programmes and optimising sequences of programmes:

5.4. *Theorem. Suppose an economy satisfies A.1 to A.5, B.1 to B.3, and $\delta = 1$. Suppose that $\mathbf{y}_0$ is positive, and that a good programme $(\bar{\mathbf{x}}_t, \bar{\mathbf{y}}_t, \bar{\mathbf{c}}_t)$ exists. Then, the following results hold:*

(1) *All good feasible plans are comparable, and if a programme $(\mathbf{x}_t, \mathbf{y}_t, \mathbf{c}_t)$ is not good, then*

$$\lim_{\nu\to\infty} \sum_{t=0}^{\nu} [U(\mathbf{c}_t) - U(\bar{\mathbf{c}}_t)] = -\infty.$$

(2) *There exists an optimising sequence* $(\mathbf{x}_t^j, \mathbf{y}_t^j, \mathbf{c}_t^j)$, $j = 1, 2, \ldots$.

(3) *The optimising sequence has a subsequence converging pointwise to a programme* $(\bar{\mathbf{x}}_t, \bar{\mathbf{y}}_t, \bar{\mathbf{c}}_t)$, *which is good.*

(4) *There exists a price system* $(\bar{\mathbf{p}}_t)$, *not identically zero, such that* $(\bar{\mathbf{x}}_t, \bar{\mathbf{y}}_t, \bar{\mathbf{c}}_t)$ *is a finitely competitive programme (i.e. (12.11) and (12.12) hold). Further,*

$$\sum_{t=0}^{\infty} \bar{\mathbf{p}}_t \cdot \mathbf{c}_t$$

exists for all feasible $(\mathbf{c}_t)$, *and one has*

$$\sum_{t=0}^{\infty} [U(\mathbf{c}_t) - U(\bar{\mathbf{c}}_t)] - \bar{M} \leqslant \sum_{t=0}^{\infty} \bar{\mathbf{p}}_t \cdot (\mathbf{c}_t - \mathbf{c}_t^j),$$

where

$$\bar{M} = \sup \{ \sum_{t=0}^{\infty} [U(\mathbf{c}_t') - U(\bar{\mathbf{c}}_t)] \,|\, (\mathbf{c}_t') \textit{ feasible} \}$$

and $(\mathbf{c}_t^j)$ *is any member of the optimising sequence.*

(5) *If*

$$\sum_{t=0}^{\infty} \bar{\mathbf{p}}_t \cdot (\bar{\mathbf{c}}_t - \mathbf{c}_t) \geqslant 0$$

for all feasible $(\mathbf{c}_t)$, *then* $(\bar{\mathbf{c}}_t)$ *is optimal.*

(6) *If*

$$\lim_{j \to \infty} \sum_{t=1}^{\infty} |\mathbf{c}_t^j - \bar{\mathbf{c}}_t| / \rho_0^t = 0,$$

then $(\bar{\mathbf{c}}_t)$ *is optimal.*

(7) *If*

$$\lim_{\nu \to = \infty} \sup \{ \sum_{t=\nu}^{\infty} \bar{\mathbf{p}}_t \cdot \mathbf{c}_t \,|\, (\mathbf{c}_t) \textit{ feasible} \} = 0,$$

then $(\bar{\mathbf{c}}_t)$ *is optimal.*

Proof: McFadden (1970), Theorem 3.

5.5. The technology T will admit one or more supporting planes at each point $(\mathbf{x}, \mathbf{y})$ in its boundary. The technology is *smooth* at $(\mathbf{x}, \mathbf{y})$ if the supporting plane there is unique. We make one additional assumption:

A.6. The technology is smooth at the maximal expansion path $(\hat{\mathbf{x}}, \rho_0 \hat{\mathbf{x}})$.

This condition is satisfied if production possibilities are representable by a collection of production and transformation functions which are differentiable at the maximal expansion path, or is satisfied by a finite von Neumann technology in which $2N-1$ linearly independent activities are operated at non-zero levels at the maximal expansion path.[1] Under this assumption, the price system $(\bar{\mathbf{p}}_t)$ given in 5.4 (4) has a 'turnpike' property that $[\bar{\mathbf{p}}_{t+1}(1+\varepsilon)-\bar{\mathbf{p}}_t].\hat{\mathbf{x}} \leqslant 0$ when the angle between $(\bar{\mathbf{p}}_t, \bar{\mathbf{p}}_{t+1})$ and $(\rho_0\hat{\mathbf{p}}, \hat{\mathbf{p}})$ is sufficiently large (McFadden, 1970, Lemma 5). Hence, one has

$$\lim_{t\to+\infty} \rho_0{}^t\bar{\mathbf{p}}_t = \theta\hat{\mathbf{p}}$$

for some non-negative scalar θ (McFadden, 1970, Lemma 6). We are now able to state the main result:

5.6. *Theorem. Suppose a multi-commodity economy has a linear technology satisfying A.1 to A.6. Suppose that social preferences satisfy B.1 to B.3 and $\delta = 1$. Suppose that the initial endowment $\mathbf{y}_0$ is positive. Then, an optimal programme exists if and only if $U(\mathbf{c})$ is bounded above.*

Proof: McFadden (1970), Theorem 5.

[1] Since A.6 allows non-joint production, it is less objectionable economically than the dual proposition frequently assumed in turnpike theory that the maximal expansion path is the only 'break-even' programme at von Neumann prices.

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