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On Hicksian Stability

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1. *Introduction*

J. R. Hicks first raised the question of the stability of a general equilibrium system in *Value and Capital*, and suggested as an answer his now-classic stability conditions:

> What do we mean by stability in multiple exchange ? Clearly . . . that a fall in the price of X in terms of the [numéraire] commodity will make the demand for X greater than the supply. But are we to suppose that it must have this effect (a) when the prices of other commodities are given, or (b) when other prices are adjusted so as to preserve equilibrium in the other markets ? . . . Strictly, we should distinguish a series of conditions: that a rise in the price of X will make supply greater than demand, (a) all other prices given, (b) allowing for the price of Y being adjusted to maintain equilibrium in the Y-market, (c) allowing for the prices of Y and Z being adjusted, and so on, until all prices have been adjusted. . . . I propose to call a system in which all conditions of stability are satisfied *perfectly stable* (Hicks, 1946, p. 66).

However, formulating the price adjustment mechanism in the multiple exchange model as a system of ordinary differential equations, P. A. Samuelson (1941, 1942, 1944) showed that in general perfect stability was neither a necessary nor sufficient condition for dynamic stability. Samuelson went on to argue: 'Why any system should be expected to possess *perfect* stability, or why an economist should be interested in this property is by no means clear.'

Commenting on Samuelson's results in a note to the second edition of *Value and Capital*, Hicks writes:

> My discussion of static equilibrium in this book was intended as no more than a preliminary to what I called economic dynamics; thus

the discussion of static stability was deliberately and explicitly timeless. And when I passed on to my dynamics, the discussion of stability remained timeless, at least in this sense: that I assumed the process of adjustment to a temporary equilibrium to be completed within a short period (a 'week'), while I neglected the movement of prices within the week, so that my economic system could be thought of as taking up a series of temporary equilibria. In adopting this device, I was following in the tradition of Marshall, though I was of course aware that the assumption of an 'easy passage to temporary equilibrium' required more justification when it was applied to my problem of many markets than it did when applied to Marshall's case of a single market. . . .

Professor Samuelson has turned some much heavier mathematical artillery than mine onto this precise issue, and has undoubtedly made important progress with it. He drops the assumption of a quick and easy passage to temporary equilibrium, assuming instead that rates of price-change are functions of differences between demands and supplies. His whole theory thus becomes dynamic in a different sense than mine, but one which is perhaps more acceptable to mathematicians. . . .

In terms of this new technique, my static theory can be 'dynamized'; it is possible to inquire into the stability of the static system in the sense of investigating whether the movements set up when a system is initially out of equilibrium will converge upon an equilibrium position. Since Professor Samuelson's system has a new degree of freedom, it is not surprising that his stability conditions are different from mine and more elaborate than mine; his system may fail of stability, not only for my reasons, but because of a lack of adjustment between rates of adaption in different markets, or rates of response by persons trading. All this opens a most promising line of investigation, which is clearly by no means exhausted by the work hitherto done on it.

Professor Samuelson's work thus represents an important advance in our knowledge of the mechanics of related markets; his 'dynamizing' of static theory is a notable achievement. But I still feel that something is wanted which is parallel to *my* dynamic theory, and I miss this in Professor Samuelson's work. By my hypothesis of

> essentially instantaneous adjustment, I reduced the purely mechanical parts of my dynamic theory to the simplest terms–it is now quite evident that I oversimplified it. But in so doing I did leave myself free to make some progress with the less mechanical parts–expectations and so on. I still feel that this procedure has its uses, and I should be sorry to abandon it altogether in favor of a pure concentration on mechanism (Hicks, 1946, p. 336).

In the quotation above, Sir John Hicks alludes to the intimate connection between his conditions of perfect stability and the basic assumptions of Marshallian partial equilibrium analysis, and suggests that the sequential adjustments to temporary equilibria associated with partial equilibrium analysis capture the spirit of his 'dynamics'. He then asks whether this view of stability has a parallel within the framework of Samuelson's 'dynamic stability'. An affirmative answer is given in this paper. The assumptions of partial equilibrium analysis are first formalized and shown to be essentially equivalent to Hicks's perfect stability conditions. Then, for any economy satisfying these assumptions, a class of dynamic processes is shown to be dynamically stable in the sense of Samuelson. These dynamic processes are found to be 'close' to the sequential adjustments to temporary equilibria which Hicks envisioned. Thus, a synthesis of Hicksian and dynamic stability conditions can be achieved, providing a justification, within the framework of dynamic stability theory, for the application of Hicksian stability conditions in economic problems.

The assumptions of partial equilibrium analysis are formalized in Section II of the paper. Section III relates the partial equilibrium assumptions to the properties of Hicksian matrices. Sections IV and V give, respectively, local and global stability theorems.

II. *Hicksian Stability and Partial Equilibrium Analysis*

Although the issue of Hicksian versus dynamic stability can be raised in any dynamic system, we shall for concreteness consider the multiple exchange model. Consider an economy with n commodities, labeled $1, 2, \ldots, n$. The price of commodity i, assumed to be non-negative, is denoted by p_i.[1] The aggregate excess demand for commodity i is denoted by x_i, and is given by an excess demand function $x_i = h_i(p_1, \ldots, p_n)$. In vector notation, $\underline{p} = (p_1, \ldots, p_n)$, $\underline{x} = (x_1, \ldots, x_n)$, and

$$\underline{x} = \underline{h}(\underline{p}) = (h_1(\underline{p}), \ldots, h_n(\underline{p})). \tag{1}$$

The ith market is in *equilibrium* if it has zero excess demand or if commodity i is a free good in excess supply.

The dynamic price adjustment mechanism formulated by Samuelson states that the rate of change in the price of the ith commodity is positive when that commodity is in excess demand, and negative when it is non-free and in excess supply. This dynamic process can then be described by a series of difference equations of the form [2]

$$\Delta p_i(t) = p_i(t+1) - p_i(t) = c_i H_i(\underline{p}(t)) \qquad (i = 1, \ldots, n), \quad (2)$$

where $\underline{p}(t) = (p_1(t), \ldots, p_m(t))$ denotes the price vector prevailing at the point of time t; c_i is a positive *rate of accommodation* in market i; and $H_i(\underline{p}(t))$ satisfies the conditions:

(a) $x_i = h_i(\underline{p}(t)) > 0$ implies $H_i(\underline{p}(t)) > 0$;

(b) $x_i = h_i(\underline{p}(t)) < 0$ and $p_i(t) > 0$ imply $0 > H_i(\underline{p}(t))$ [and $p_i(t+1) = c_i H_i(\underline{p}(t)) - p_i(t) \geqslant 0$]; and

(c) if market i is in equilibrium, then $H_i(\underline{p}(t)) = 0$.

We shall term $H_i(\underline{p})$ the *market demand index* for commodity i. An often-analyzed case is one in which the market demand index for a commodity equals its excess demand or is a sign-preserving function of its excess demand. In vector notation, the system (2) will be written

$$\Delta \underline{p}(t) = \underline{H}(\underline{p}(t))\underline{C}, \quad (3)$$

where $\underline{C}$ is a diagonal matrix of the rates of accommodation c_i, and $\underline{H}$ is a row vector of the functions H_i. In the case that the market demand indices are continuously differentiable, an analysis of the stability of this system in a neighborhood of the general equilibrium is conveniently carried out using a Taylor's expansion of (3),

$$\Delta \underline{p}(t) = (\underline{p}(t) - \bar{\underline{p}})\underline{A}(\bar{\underline{p}})\underline{C} + \text{remainder}, \quad (4)$$

where $\bar{\underline{p}}$ denotes a general equilibrium price vector satisfying $\underline{H}(\bar{\underline{p}}) = 0$, and $\underline{A}(\underline{p})$ denotes the Jacobian matrix of $\underline{H}(\underline{p})$ evaluated at $\underline{p}$:

$$a_{ij}(\underline{p}) = \partial H_i(\underline{p}) / \partial p_j \qquad (i,j = 1, \ldots, n). \quad (5)$$

A *partial equilibrium price vector* for a given subset of markets is one which achieves equilibrium in the subset markets, given fixed prices in the remaining markets. Consider a single primary market i, and suppose that the remaining markets can be divided into two subsets such that prices are fixed for the markets in one subset, and are adjusted to maintain partial equilibrium in the other subset for each primary market price. The resulting excess demand in the primary market depends only on its own price and the fixed prices, and is termed the *compensated excess demand* for the primary market i, conditioned on the subset of markets which adjust to partial equilibrium.

Marshallian partial equilibrium analysis studies price behavior in a single primary market, postulating that price behavior in remaining markets can be ignored. Two types of external market behavior are consistent with this postulate:

(1) The structure of the primary market varies smoothly with the external market price, and this external price does not change significantly over the period of analysis.

(2) The external price effectively adjusts to maintain partial equilibrium in its market, so that the *compensated* primary market can be analyzed.

Even more basic to the Marshallian analysis are the assumptions that a unique partial equilibrium price exists in the compensated primary market, and that excess demand in this market is positive when its price falls below its partial equilibrium level. The conditions for perfect stability by Hicks correspond precisely to these assumptions.

A dynamic price adjustment mechanism which is consistent with the partial equilbrium assumptions above for a given primary market i must have, relative to the rate of price adjustment in this market, a very rapid rate of adjustment in markets where 'partial equilibrium is maintained', and a very slow rate of adjustment in markets where prices are 'fixed'. Hence, if we require that partial equilibrium analysis be applicable to *each* market $i = 1, \ldots, n$, then we should be able to rank these markets in order of decreasing 'rates of accommodation'. Then, the dynamic adjustment process would be essentially 'sequential': the first market would be brought into approximate equilibrium, maintained there while the second market was adjusted, and so forth.

The geometry of the Marshallian assumptions in the case of two commodities is illustrated in Figure 1. The contours of zero excess demand in each of the markets 1 and 2 are plotted, and the sign of excess demand on each side of each of these curves is indicated. For each value of p_2,

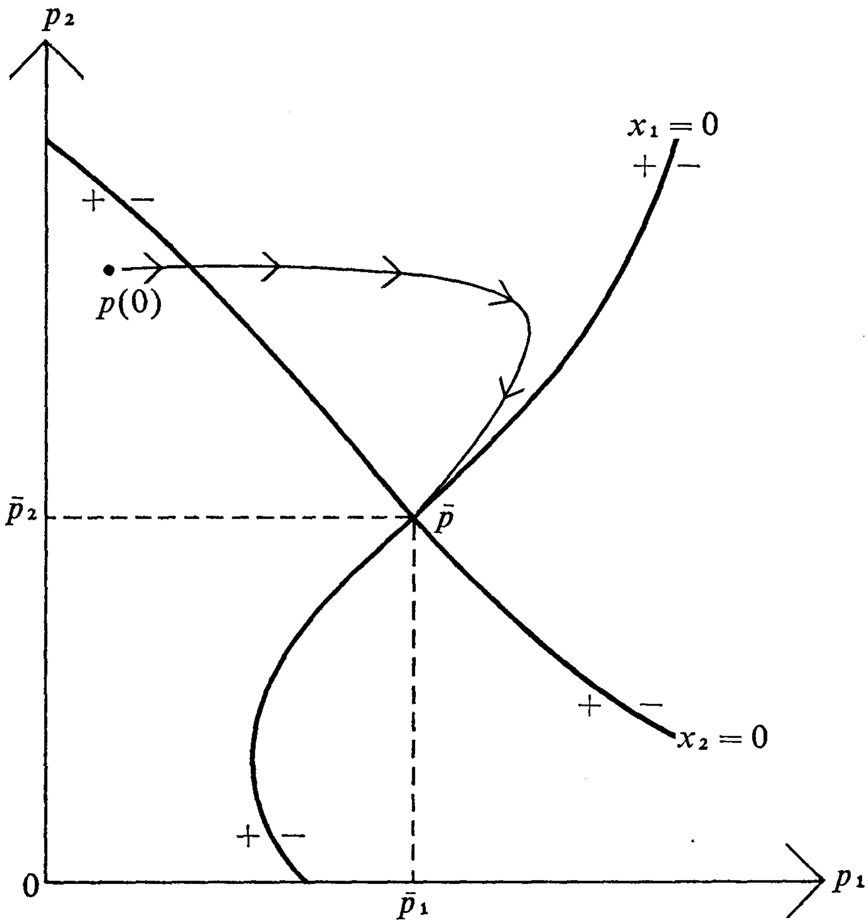

Figure 1 Structure of Partial Equilibrium Analysis

the locus $x_1 = 0$ determines a unique partial equilibrium price in market 1. The compensated excess demand for market 2, conditioned on partial equilibrium in market 1, is given by values of x_2 for prices in the locus $x_1 = 0$, and has the property that this excess demand is negative when p_2 is above the market 2 equilibrium price $\bar{p}_2$. One can see clearly in this diagram that the dynamic price adjustment mechanism suggested above should be stable under the Marshallian assumptions: Starting from an initial price vector $p(0)$, the price in market 1 will adjust rapidly toward the locus $x_1 = 0$. After p_1 is close to this locus, prices will move, much more slowly, along the locus to equilibrium. In the remainder of this paper, we shall verify formally this intuitive result.

A formalization of the assumptions of partial equilibrium analysis which will be shown to admit stable dynamic processes of the type just described is summarized in the following postulate:

Assumption 1 (*Hicks's Perfect Stability*) *The markets of the economy can be ranked* (*and this ranking can be taken, without loss of generality, to be their labeling* 1, 2, . . ., *n*) *such that for each market i* ($i = 1, \ldots, n$), *the following conditions hold:*
(1) *given any fixed prices in markets* $i+1, \ldots, n$, *there exists a unique partial equilibrium price vector for markets* $1, \ldots, i$. *Further, these partial equilibrium prices for the first i markets can be written as continuously differentiable functions of the remaining prices.*
(2) *the compensated excess demand function for commodity i, conditioned on the subset of markets* $1, \ldots, i-1$ *adjusting to partial equilibrium, is negative when its price is above its partial equilibrium level and is positive when its price is below its partial equilibrium level.*

Several features of our form of the Hicks conditions should be emphasized. First, the assumption is on the *static* structure of the economy and involves no dynamics, even though it was motivated by dynamic considerations. Second, the assumption imposes conditions on the *sign* of excess demands and the regularity of the equilibrium, but imposes no conditions of continuity or differentiability on the excess demand functions or demand indices away from partial equilibria. Third, the conditions are assumed to hold only for a single ranking of the markets, rather than for all possible rankings of markets, as in Hicks's original formulation.

III. *Perfect Stability and Hicksian Matrices*

Consider a square matrix $\underline{A}$ of order n, and let A_i denote the upper left-hand principal minor of $\underline{A}$ with order i:

$$A_i = \begin{vmatrix} a_{11} & \ldots & a_{1i} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{i1} & \ldots & a_{ii} \end{vmatrix} \qquad (i = 1, \ldots, n).$$

The matrix $\underline{A}$ is called *Hicksian* if A_1 is negative and the principal minors alternate in sign; i.e. $(-1)^i A_i$ is positive for $i = 1, \ldots, n$.

In the case that the market demand indices are continuously differentiable in a neighborhood of the general equilibrium price vector $\bar{p}$, Assumption 1 will be shown to imply that the Jacobian matrix $\underline{A}(p)$ of $\underline{H}$ is Hicksian at some price vector p^0 which is as close as one pleases to $\bar{p}$. The upper left-hand principal minor of $\underline{A}(p)$ of order i will be denoted by $A_i(p)$. Throughout the remainder of the paper, the notation '[i' will be used as a subscript on a vector to denote the subvector with components from i through n; e.g., $p_{[i} = (p_i, \ldots, p_n)$. Similarly, the subscripts 'i]', 'i[', and ']i' will be used to denote subvectors with components $i] = (1, \ldots, i)$, $i[= (i+1, \ldots, n)$, and $]i = (1, \ldots, i-1)$, respectively. Under Assumption 1, there exist unique partial equilibrium prices in the first i markets which can be written as continuously differentiable functions of the prices in the remaining markets. Denote these functions by

$$p_j = f_{i,j}(p_{i+1}, \ldots, p_n) \qquad j = 1, \ldots, i\,, \tag{6}$$

or more compactly, $p_{i]} = f_i(p_{i[})$. By assumption, the system (6) is a unique solution for $(p_1, \ldots, p_i)$ in the system of equations

$$0 = H_j(p) \qquad j = 1, \ldots, i\,. \tag{7}$$

Suppose $\underline{H}(p)$ is continuously differentiable in a neighborhood $\underline{N}$ of $\bar{p}$. Then, for any solution $(f_i(p_{i[}), p_{i[})$ of (7) which is contained in $\underline{N}$, a converse of the implicit function theorem due to Bernstein and Toupin (1962) can be applied to establish that $A_i(p)$, the Jacobian of (7) with respect to $p_{i[}$, assumes non-singular values for some price vector $(p'_{i]}, p_{i[})$ with $p'_{i[}$ as close as one pleases to $f_i(p_{i[})$. Starting with $i = n$, a recursive argument then establishes the existence of points p^0 arbitrarily close to $\bar{p}$ where $A(p^0)$ has all principal minors $A_i(p^0)$ non-zero. With little loss of economic generality, Assumption 1 can then be strengthened to

Assumption 1* *The conditions of Assumption* 1 *hold, the market demand index functions* $\underline{H}$ *are continuously differentiable in a neighborhood of the general equilibrium price vector* $\bar{p}$, *and all the principal minors* $A_i(p)$ *of the Jacobian of* $\underline{H}$ *are non-zero at* $\bar{p}$.

Under this assumption, the own price derivative of the compensated market demand index function for commodity i, conditioned on achievement of partial equilibrium in markets $1, \ldots, i-1$, is negative. Further, when the price vector $p_{i[}$ attains its general equilibrium level $\bar{p}_{i[}$, this derivative, evaluated at $\bar{p}_i$, equals $A_{i-1}(\bar{p})/A_i(\bar{p})$, with $A_0(p) = 1$ by convention. Hence, $\underline{A}(\bar{p})$ is Hicksian under Assumption 1*.

IV. *A Local Stability Theorem*

A remarkable theorem by Fisher and Fuller (1958) on the stabilization of matrices allows us to establish immediately the local stability of the dynamic process (2), provided the rates of accommodation c_i are small and are ranked in size in the manner suggested in our discussion of

Marshallian analysis. The relevance of the Fisher-Fuller theorem for stability analysis was first noted by P. Newman (1959). The local stability theorem given below was communicated to me in essentially its present form by J. Quirk.

Theorem 1. *Suppose Assumption* 1* *holds. Then, there exists a positive scalar* ε_0 *such that if* (a) *the positive rates of accommodation* c_i *satisfy* $c_1 < \varepsilon_0$ *and* $c_i / c_{i-1} < \varepsilon_0$ *for* $i = 2, \ldots, n$, *and* (b) *the initial price vector satisfies* $| p(0) - \bar{p} | < \varepsilon_0$,[3] *then the dynamic process* (2) *is stable, and all the characteristic roots of the matrix* $\underline{A}(\bar{p})\underline{C}$ *in the Taylor's expansion* (4) *of the dynamic process* (2) *are real, negative, distinct, and less than one in modulus.*

Proof. Since $\underline{A}(\bar{p})$ is Hicksian and $\underline{A}(p)$ is continuous in a neighborhood of $\bar{p}$, there exists $\varepsilon_1 > 0$ such that $\underline{A}(p)$ is Hicksian for $| p - \bar{p} | \leqslant \varepsilon_1$. The corollary to the Fisher-Fuller theorem given in Appendix A then establishes the existence of a scalar $\varepsilon_2 > 0$ such that for $\underline{C}$ satisfying $c_i / c_{i-1} < \varepsilon_2$, $\underline{A}(p)\underline{C}$ has real, distinct characteristic roots which are bounded negative. Choosing $c_1 < 1/n \| \underline{A}(p) \|$ for all p in the ε_1-neighborhood ensures that the roots will have modulus less than one. Take $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$.

For $| p(t) - \bar{p} | \leqslant \varepsilon$, the dynamic process (2) can be written in the form

$$\Delta p(t) = \underline{H}(p(t))\underline{C} = (p(t) - \bar{p})\underline{A}(p^{\Delta}(t))\underline{C},$$

where $p^{\Delta}(t) = \theta\bar{p} + (1-\theta)p(t)$ for some scalar θ satisfying $0 < \theta < 1$. The norm $| p(t) - \bar{p} |$ then satisfies

$$| p(t+1) - \bar{p} |^2 - | p(t) - \bar{p} |^2 = (p(t) - \bar{p})[2\underline{A}(p^{\Delta}(t))\underline{C} + \underline{A}(p^{\Delta})\underline{C}\underline{C}'\underline{A}(p^{\Delta})'](p(t) - \bar{p})' < 0$$

for $p(t) \neq \bar{p}$, since the matrix in square brackets is negative definite. Hence, the solution is contained in the neighborhood $| p - \bar{p} | \leqslant \varepsilon$ and converges to $\bar{p}$. Q.E.D.

When the system of market demand index functions $\underline{H}(p)$ is linear, Theorem 1 establishes stability globally (i.e., for arbitrary $p(0)$).

v. *A Global Stability Theorem*

A global stability result analogous to the local stability theorem proved above can be established using the geometry of the partial equilibrium system. The class of dynamic processes which are proved stable again are of the type illustrated in Figure 1, with the markets adjusting to partial equilibrium essentially in a sequential fashion.

Theorem 2. *Suppose Assumption* 1* *holds, and assume that the market demand index functions* $\underline{H}(\underline{p})$ *are continuous for all non-negative p. Given a positive bound M, there exists a positive scalar* ε (*which depends in general on M*) *such that if* (a) *the positive rates of accommodation* c_i *satisfy* $c_1 < \varepsilon$ *and* $c_i/c_{i-1} < \varepsilon$ *for* $i = 2, \ldots, n$, *and* (b) *the initial price vector satisfies* $|\underline{p}(0) - \bar{\underline{p}}| \leqslant M$, *then the dynamic process* (2) *is stable.*

Discussion of the Proof. The formal proof of this theorem will follow closely the geometric argument for stability given in Figure 1. The first market price in the solution is shown to monotonically approach its corresponding partial equilibrium value until the solution is trapped in a neighborhood of the locus of partial equilibria for market 1. Then, the second market price vector approaches its compensated partial equilibrium value, and so forth, until the price vector is trapped in a neighborhood of general equilibrium.

The theorem will be proved in five steps: *Step* 1 constructs a rectangle which is later shown to contain the solution when hypothesis (a) of the theorem is satisfied. *Step* 2 introduces a system of notation which is used in the proof, and *Step* 3 establishes several geometric implications of the partial equilibrium assumptions. *Step* 4 establishes the value of ε required in the hypothesis of the theorem. *Step* 5 establishes a basic induction step which guarantees that after some time t_n, the solution will be trapped in a small neighborhood of the general equilibrium. Theorem 1 is then applied to complete the proof that the system is stable.

Proof. Step 1. First, a rectangle $\underline{R}^*$ is constructed in such a way that it contains the partial equilibrium values of any component which might result when succeeding components are in $\underline{R}^*$. In particular, all non-negative prices satisfying $|\underline{p} - \bar{\underline{p}}| \leqslant M$ will be contained in $\underline{R}^*$. Define the functions

$$b_i(\theta) = \text{Max}\{\, |\, f_{i,i}(\underline{p}_{i\text{l}}) - \bar{\underline{p}}_i \,|\; \big|\; |\, \underline{p}_{i\text{l}} - \bar{\underline{p}}_{i\text{l}} \,| \leqslant \theta\}$$

for $\theta \geqslant 0$ and $i = 1, \ldots, n-1$. Define a scalar $\theta_{n+1} = nM + \sum_{i=1}^{n-1} b_i(M) + 1$. Then define scalars $\theta_n, \theta_{n-1}, \ldots, \theta_1$ by the recursion formula $\theta_i = \theta_{i+1} + 4\theta_{n+1} + 3b_i(\theta_{i+1})$. Define a box $\underline{R}^* = \{\underline{p} \geqslant 0 \,|\, |\, p_i - \bar{p}_i \,| \leqslant \theta_i,\ i = 1, \ldots, n\}$ and let $\underline{R} = \{\underline{p} \geqslant 0 \,|\, |\, p_i - \bar{p}_i \,| \leqslant 2\theta_i,\ i = 1, \ldots, n\}$ be a larger box containing $\underline{R}^*$.

Step 2. We now introduce a system of notation for neighborhoods of partial equilibrium price vectors. The structure of these neighborhoods is illustrated in Figures 2 and 3. Let α denote a positive scalar. The sets

$$\underline{S}_i(\alpha) = \{\underline{p} \text{ in } \underline{R} \mid H_i(\underline{p}) \leqslant -\alpha\},$$
$$\underline{D}_i(\alpha) = \{\underline{p} \text{ in } \underline{R} \mid H_i(\geqslant \underline{p})\alpha\},$$
$$\underline{E}_i(\alpha) = \{\underline{p} \text{ in } \underline{R} \mid |H_i(\underline{p})| \leqslant \alpha\}$$

denote, respectively, points in $\underline{R}$ where excess demand in market i is bounded negative, bounded positive, and near zero. The set

$$\underline{T}_i(\alpha) = \{\underline{p} \text{ in } \underline{R} \mid |p_{i]} - \underline{f}_i(\underline{p}_{i[})| \leqslant \alpha\}$$

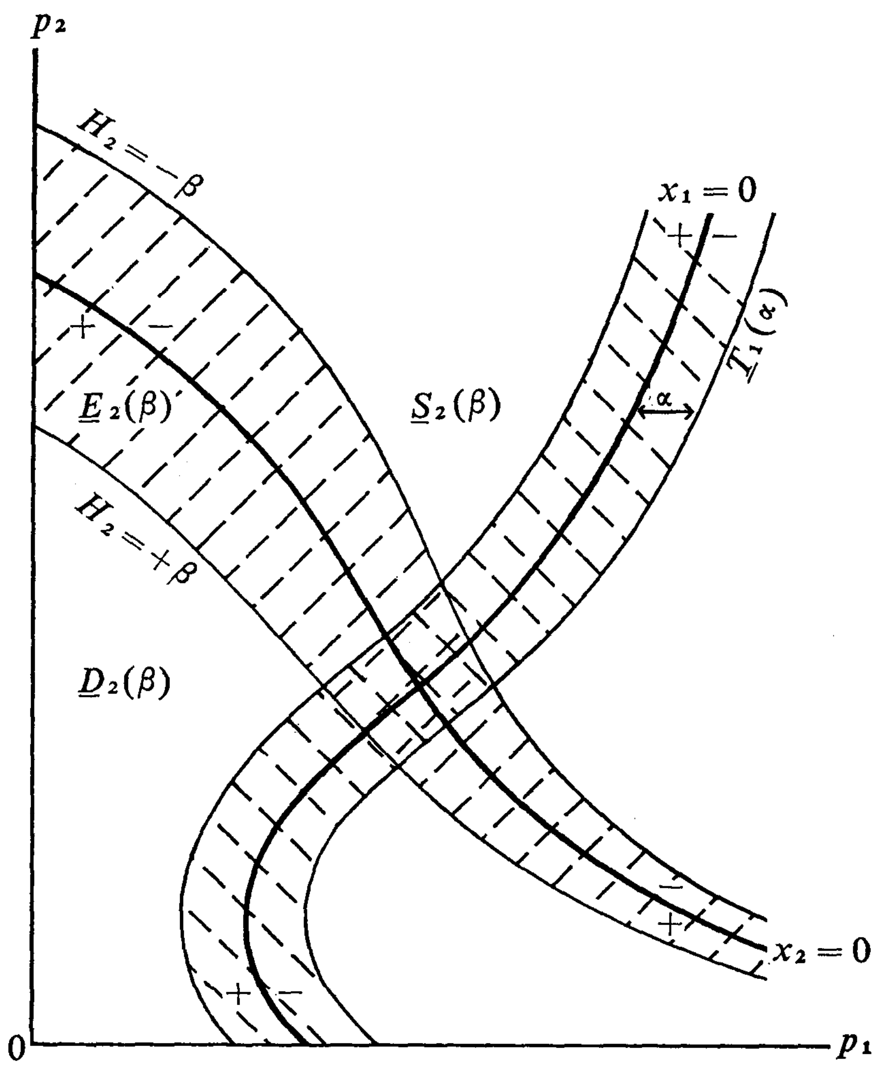

Figure 2 Sets $\underline{E}_2(\beta)$ (shaded northeast–southwest), $\underline{T}_1(\alpha)$ (shaded northwest–southeast), $\underline{S}_2(\beta)$ (northeast of the $H_2 = -\beta$ contour), and $\underline{D}_2(\beta)$ (southwest of the $H_2 = +\beta$ contour)

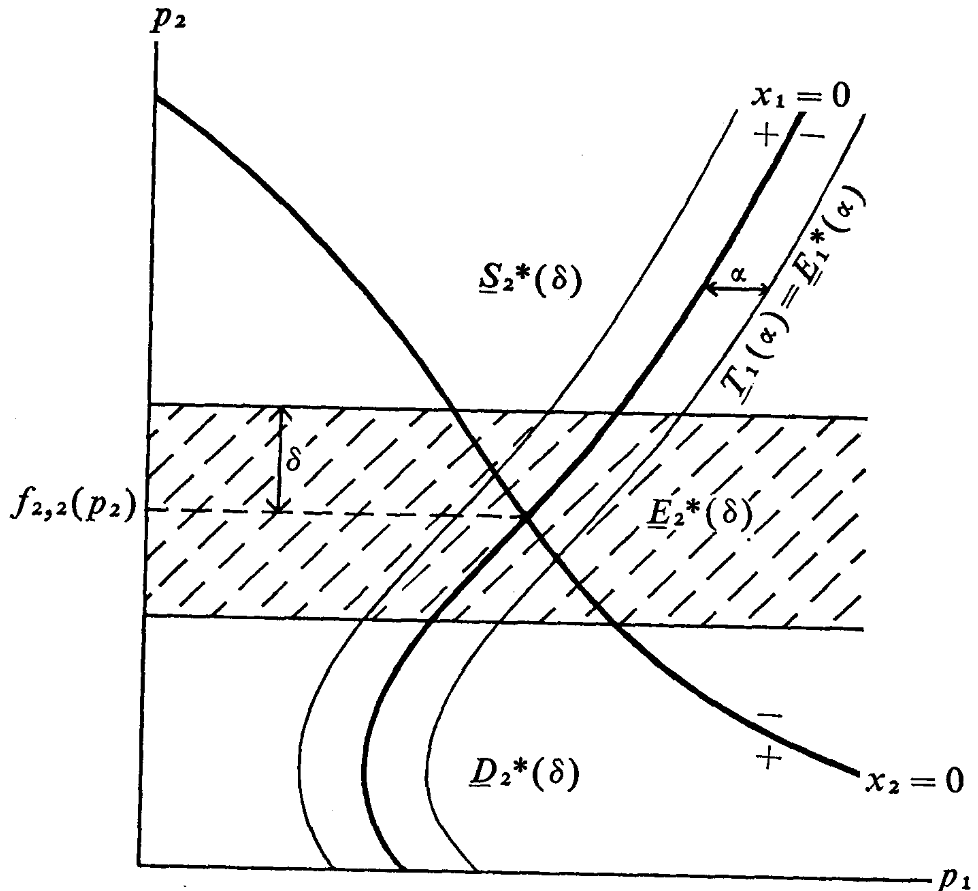

Figure 3 Sets $\underline{E}_2^*(\delta)$ (shaded), $\underline{S}_2^*(\delta)$ (above the shaded area), and $\underline{D}_2^*(\delta)$ (below the shaded area)

denotes a neighborhood of the locus of joint partial equilibria in the first i markets. ($\underline{T}_n(\alpha)$ is a α-neighborhood of the general equilibrium price vector $\bar{p}$.) The sets

$$\underline{S}_i^*(\alpha)=\{\underline{p} \text{ in } \underline{R} \mid p_i \geqslant f_{i,i}(\underline{p}_{i\Gamma})+\alpha\},$$

$$\underline{D}_i^*(\alpha)=\{\underline{p} \text{ in } \underline{R} \mid p_i \leqslant f_{i,i}(\underline{p}_{i\Gamma})-\alpha\},$$

$$\underline{E}_i^*(\alpha)=\{\underline{p} \text{ in } \underline{R} \mid |p_i-f_{i,i}(\underline{p}_{i\Gamma})| \leqslant \alpha\}$$

denote, respectively, price vectors in which the price of commodity i is bounded above, bounded below, or near its partial equilibrium price which would prevail if the first i markets were in partial equilibrium.

Step 3. Several 'geometrically obvious' relations between the various sets defined above will now be established. By hypothesis, $\underline{H}_i(\underline{p})$ is continuous, and if the first $i-1$ markets are in partial equilibrium and p_i is bounded above its partial equilibrium level, then $\underline{H}_i(\underline{p})$ is negative (by Assumption 1). Then, $\underline{H}_i(\underline{p})$ is uniformly continuous on the compact set $\underline{R}$, and is bounded negative on the set $\underline{S}_i^*(\varepsilon) \cap \underline{T}_{i-1}(\delta)$ for sufficiently small δ. Hence, we can conclude

Given $\varepsilon > 0$, *there exist* $\delta, \gamma > 0\,(\delta \leqslant \varepsilon)$ *such that* $\underline{S}_i^*(\varepsilon) \cap \underline{T}_{i-1}(\delta)$ *is contained in* $\underline{S}_i(\gamma)$. *The result holds when* $\underline{S}$ *is replaced by* $\underline{D}$. (8)

By Assumption 1, the partial equilibrium prices $f_i(p_{i[})$ are continuously differentiable on the compact set $\underline{R}$. Hence, there exists a scalar $\mu(\mu > 1)$ such that [Note: $f_{i-1}(f_{ii}(p_{i[}), p_{i[}) \equiv f_{i],i}(p_{i[})$]

$$| f_{i-1}(p_i, p_{i[}) - f_{i,]i}(p_{i[}) | \leqslant \mu \, | p_i - f_{i,i}(p_{i[}) |$$

for p in $\underline{R}$. If p is in $\underline{T}_{i-1}(\varepsilon) \cap \underline{E}_i^*(v)$, then

$$| p_{]i} - f_{i,]i}(p_{i[}) | \leqslant | p_{]i} - f_{i-1}(p_i, p_{i[}) | + | f_{i-1}(p_i, p_{i[}) - f_{i,]i}(p_{i[}) |$$
$$\leqslant \varepsilon + \mu \, | p_i - f_{i,i}(p_{i[}) | \leqslant \varepsilon + \mu v \,,$$

and we conclude

$\underline{T}_{i-1}(\varepsilon) \cap \underline{E}_i^*(v)$ *is contained in* $\underline{T}_i(\varepsilon + \mu v)$. (9)

Step 4. A series of bounds will now be established, among them the value of ε required in the hypothesis of Theorem 2. Choose a positive scalar $\varepsilon_n(\varepsilon_n < 1$ and $\varepsilon_n \leqslant \theta_{n+1}/4)$ which satisfies Theorem 1. Then, define a series of scalars $v_n, \varepsilon_{n-1}, v_{n-1}, \ldots, v_2, \varepsilon_1$ recursively as follows:

Given $\varepsilon_i > 0$, choose $v_i(\varepsilon_i/2\mu \leqslant v_i < \varepsilon_i/\mu)$ such that $\underline{S}_i^*(v_i/2) \cap \underline{T}_{i-1}(\varepsilon_i - \mu v_i)$ is contained in $\underline{S}_i(\delta)$ and $\underline{D}_i^*(v_i/2) \cap \underline{T}_{i-1}(\varepsilon_i - \mu v_i)$ is contained in $\underline{D}_i(\delta_i)$ for some $\delta_i > 0$ and then choose $\varepsilon_{i-1} = \varepsilon_i - \mu v_i$. (The scalar μ is given in (9). That this recursive definition is possible follows from (8) and the observation that $\varepsilon_i - \mu v_i$ can be made as close to zero as we please.) Finally, define $v_1 = \varepsilon_1/2\mu$, and choose δ_1 such that $\underline{S}_1^*(v_1)$ and $\underline{D}_1^*(v_1)$ are contained in $\underline{S}_1(\delta_1)$ and $\underline{D}_1(\delta_1)$, respectively. Define $\delta = \min\{\delta_1, \ldots, \delta_n\}$, where the δ_i are given by the recursive procedure above.

Let $G \geqslant 1$ be a upper bound on $| \underline{H}_i(p) |$ for p in $\underline{R}$ and $i = 1, \ldots, n$. Now, choose ε to be the *smallest* of the numbers $v_1/8nG\mu$, $\delta/2nG\mu$, $\theta_{n+1}/8G$, and $v_1\delta/6nG\theta_1$. The matrix $\underline{C}$ will now be assumed to satisfy the hypotheses of the theorem for this ε.

Using (9) and the condition $\varepsilon_i = \varepsilon_{i-1} + \mu v_i$ given by the recursion above, one obtains the useful condition

$\underline{T}_{i-1}(\varepsilon_{i-1}) \cap \underline{E}_i^*(v_i)$ *is contained in* $\underline{T}_i(\varepsilon_i)$. (10)

Step 5. Suppose a given matrix $\underline{C}$ satisfies the hypotheses of Theorem 2 for the ε given in the previous step. Define a sequence of times t_i, $i = 1, \ldots n$, by the following recursive procedure (define $t_0 = 0$ and $t_{n+1} = +\infty$): Given t_{i-1}, define t_i as the largest integer which is less than $t_{i-1}+3\theta_1/\delta c_i$.

We shall now give an induction argument which shows that (a) up until time t_{i-1}, market i exhibits 'insignificant' price changes, so that the solution remains in the rectangle $\underline{R}^*$, (b) after time t_{i-1}, the price in market i approaches its partial equilibrium value monotonically until, by time t_i or before, it is trapped in a neighborhood $\underline{E}_i^*(v_i)$ of this equilibrium value, and (c) after time t_i, the price vector is trapped in a neighborhood $\underline{T}_i(\varepsilon_i)$ of the locus of partial equilibrium price vectors for the first i markets.

We shall require the following bound: for $k > i$

$$t_i c_k \leqslant c_k \sum_{j=1}^{i} \frac{3\theta_1}{\delta c_j} \leqslant \frac{3\theta_1 i}{\delta} \frac{v_1 \delta}{6nG\theta_1} \leqslant \frac{v_1}{G}. \tag{11}$$

Define a norm $V_i(\underline{p}) = |p_i - f_{i,i}(\underline{p}_{i\Gamma})|$ for $i = 1, \ldots, n$, and note that $\underline{E}_i^*(v_i) = \{\underline{p} \text{ in } \underline{R} \mid V_i(\underline{p}) \leqslant v_i\}$. The basic induction step can now be stated:

Lemma. If the Induction Hypothesis below holds for time t', then it remains valid when t' is replaced by $t'+1$.

Induction Hypothesis. At the time t (with i defined so that $t_{i-1} \leqslant t < t_i$), the following conditions hold:

(a) $\underline{p}(\tau)$ *is in* $\underline{R}^*$ *for* $\tau \leqslant t$;

(b) $\underline{p}(\tau)$ *is in* $\bigcap_{k=1}^{j} \underline{E}_k^*(v_k) \subseteq \underline{T}_j(\varepsilon_j)$ *for* $t_j \leqslant \tau \leqslant t_{j+1}$

for $j = 1, \ldots, i-2$ *and is in* $\bigcap_{k=1}^{i-1} \underline{E}_k^*(v_k) \subseteq \underline{T}_{i-1}(\varepsilon_{i-1})$ *for* $t_{i-1} \leqslant \tau \leqslant t$;

(c) $V_i(\underline{p}(t_{i-1})) \leqslant \theta_i - 2\theta_{n+1} - b_i(\theta_{i+1})$;

(d) $V_j(\underline{p}(t)) \leqslant \text{Max}\left\{v_j, V_j(\underline{p}(t_{j-1})) - \frac{c_j\delta}{2}(t - t_{j-1})\right\}$

for $j = 1, \ldots, i$.

Proof of the lemma. Suppose the induction hypothesis holds for t. Then $|p_i(t+1) - p_i(t)| \leqslant Gc_i \leqslant \theta_{n+1}/2 < \theta_i$ for each i, and $\underline{p}(t+1)$ is in $\underline{R}$.

The proof will now be carried out in three phases. In Phase 1, (b), (c), (d) will be shown to hold for $t+1 \leqslant t_i$ with the index i held fixed. Phase 2 will verify condition (a). Finally, Phase 3 will verify that (b), (c), (d) continue to hold when $t+1 = t_i$ and the index i is advanced in (b), (c), (d). This will prove the lemma.

Phase 1. Consider the case where $t+1 \leqslant t_i$, and (b), (c), (d) are considered without advancing the subscript i for $t+1 = t_i$. Condition (c) continues to hold without induction. The next chain of arguments will establish that condition (d) holds in this case.

Suppose first that for some $j \leqslant i$, we have $\underline{p}(t)$ in $\underline{S}_j^*(v_j/2) \cap \underline{T}_{j-1}(\varepsilon_{j-1})$. From Step 4, $\underline{p}(t)$ is then in $\underline{S}_j(\delta_j)$, implying $\underline{H}_j(\underline{p}(t)) \leqslant -\delta_j$. Further, $p_j(t+1)-f_{j,j}(\underline{p}_{j\complement}(t+1)) \geqslant v_j/2-|p_j(t+1)-p_j(t)|-\mu|\underline{p}_{j\complement}(t+1)-\underline{p}_{j\complement}(t)| \geqslant v_j/2-Gc_j-\mu Gnc_{j+1} \geqslant v_j/4 > 0$. Hence,

$$
\begin{aligned}
V_j(\underline{p}(t+1))-V_j(\underline{p}(t)) = p_j(t+1)-p_j(t)-& \\
-[f_{j,j}(\underline{p}_{j\complement}(t+1))-f_{j,j}(\underline{p}_{j\complement}(t))]& \qquad (12)\\
\leqslant -\delta_j c_j+\mu\,|\underline{p}_{j\complement}(t+1)-\underline{p}_{j\complement}(t)|& \\
\leqslant -\delta_j c_j+\mu Gnc_{j+1} \leqslant -\frac{\delta}{2}c_j,&
\end{aligned}
$$

using the inequalities satisfied by ε. If $\underline{p}(t)$ is in $\underline{D}_j^*(v_j/2) \cap \underline{T}_{j-1}(\varepsilon_{j-1})$, a similar argument again establishes that $V_j(\underline{p}(t+1))-V_j(\underline{p}(t)) \leqslant -\delta c_j/2$.

Next suppose that for some $j \leqslant i$, we have $\underline{p}(t)$ in $\underline{E}_j^*(v_j/2) \cap \underline{T}_{j-1}(\varepsilon_{j-1})$. Then,

$$
\begin{aligned}
V_j(\underline{p}(t+1)) \leqslant V_j(\underline{p}(t))+|p_j(t+1)-p_j(t)|+& \\
+|f_{j,j}(\underline{p}_{j\complement}(t+1))-f_{j,j}(\underline{p}_{j\complement}(t))|& \\
\leqslant v_j/2+v_1/4+\mu\,|\underline{p}_{j\complement}(t+1)-\underline{p}_{j\complement}(t)| \leqslant v_j,&
\end{aligned}
$$

where the bounds on ε have again been utilized.

But under the induction hypothesis, $\underline{p}(t)$ is in one of the cases we have just considered. Hence, for each $j \leqslant i$, either $V_j(\underline{p}(t+1)) \leqslant v_j$ or $V_j(\underline{p}(t+1))-V_j(\underline{p}(t)) \leqslant -\delta_j c_j/2$. Therefore, condition (d) of the induction hypothesis holds.

From condition (d), one then has $\underline{p}(t+1)$ in the intersection of $\underline{E}_j^*(v_j)$ for $j = 1, \ldots, i-1$, which implies by an application of (10) that $\underline{p}(t+1)$ is in $\underline{T}_{i-1}(\varepsilon_{i-1})$. Hence, condition (b) holds.

Phase 2. Condition (a) will now be verified. Consider first a market k for $k > i$. Then, by (11),

$$| p_k(t+1) - \bar{p}_k | \leqslant | p_k(t+1) - p_k(0) | + | p_k(0) - \bar{p}_k | \leqslant Gc_k t_i + \theta_{n+1} \leqslant v_1 + \theta_{n+1} \leqslant 2\theta_{n+1} \leqslant \theta_k , \tag{14}$$

and the components $i+1, \ldots, n$ of $\underline{p}(t+1)$ are in $\underline{R}^*$.

To establish (a) for the remaining markets, we shall employ the inequality

$$| p_j - \bar{p}_j | \leqslant | p_j - f_{j,j}(\underline{p}_{j[}) | + | f_{j,j}(\underline{p}_{j[}) - \bar{p}_j | \leqslant V_j(\underline{p}) + b_j(| \underline{p}_{j[} - \bar{\underline{p}}_{j[} |) \tag{15}$$

From condition (d) established for $t+1$ in Phase 1 above, $V_i(\underline{p}(t+1)) \leqslant \text{Max } \{v_i, \ V_i(\underline{p}(t_{i-1}))\} \leqslant \theta_i - b_i(\theta_{i+1}) - 2\theta_{n+1}$. Then, (15) establishes $| p_i(t+1) - \bar{p}_i | \leqslant \theta_i$. An induction argument completes the demonstration: Suppose $| p_j(t+1) - \bar{p}_j | \leqslant \theta_j$ has been established for markets $k+1$ to n $(k < i)$. Then, condition (b) implies $V_k(\underline{p}(t+1)) \leqslant v_k$, and $| p_k(t+1) - \bar{p}_k | \leqslant v_k + b_k(\theta_{k+1}) \leqslant \theta_k$. Hence, (a) holds.

Phase 3. Consider the case where $t+1 = t_i$. Phase 1 established conditions (c) and (d) at t_i when the subscript $i-1$ is maintained. The second term in the bound on the right-hand side of (d) then satisfies

$$V_i(\underline{p}(t_{i-1})) - \frac{c_i\delta}{2}(t_i - t_{i-1}) \leqslant \theta_i - \frac{c_i\delta}{2}\left(\frac{3\theta_1}{\delta c_i} - 1\right) \leqslant \theta_i - \tfrac{3}{2}\theta_1 + \frac{c_i\delta}{2} < 0$$

by the construction of the t_i and the bounds on ε. Hence $V_i(\underline{p}(t_i)) \leqslant v_i$, and (b) holds with i replaced by $i+1$.

To show (c) in this case, we make the expansion

$$V_{i+1}(\underline{p}(t_i)) \leqslant V_{i+1}(\underline{p}(0)) + | f_{i+1,i+1}(\underline{p}_{i+1,[}(0)) - \bar{p}_{i+1} | + + | p_{i+1}(t_i) - p_{i+1}(0) | + | f_{i+1,i+1}(\underline{p}_{i+1,[}(t_i)) - \bar{p}_{i+1} | . \tag{16}$$

From Step 1 of the theorem proof, the first two terms on the right-hand-side of (16) are bounded by θ_{n+1}. By (11), the term $| p_{i+1}(t_i) - p_{i+1}(0) |$ is bounded by v_1. By (a), established for t_i in Phase 2 above, the last term is bounded by $b_{i+1}(\theta_{i+2})$. Hence,

$$V_{i+1}(\underline{p}(t_i)) \leqslant \theta_{n+1} + v_1 + b_{i+1}(\theta_{i+2}) \leqslant \theta_{i+1} - 2\theta_{n+1} - b_{i+1}(\theta_{i+2}) ,$$

since $\theta_{i+1}-2\theta_{n+1}-b_{i+1}(\theta_{i+2}) = 2\theta_{n+1}+2b_{i+1}(\theta_{i+2})$ and $v_i \leqslant \theta_{n+1}$. Hence, (c) holds when i is advanced to $i+1$, $t+1 = t_i$.

Finally, note that (d) holds at $t+1 = t_i$ and i not advanced by the results of Phase 1. Then, it holds by definition when i is advanced to $i+1$. Q.E.D. Lemma.

From condition (b) of the induction hypothesis, the solution will be contained in $\underline{T}_n(\varepsilon_n)$ after time t_n. Hence, Theorem 1 can be applied to establish that the solution converges to the general equilibrium price vector $\bar{p}$. Q.E.D. Theorem 2.

Theorem 2 may be generalized in several directions:

(1) If there are sub-groups of markets which are stable for more general dynamic processes than the essentially sequential processes we have considered, then Theorem 2 can be generalized to establish the stability of a process which is essentially sequential between subgroups.

(2) In more general dynamic systems than the multiple market model, transformations of the dynamic system, $\Delta p(t) = \underline{H}(p(t))\underline{BC}$, where $\underline{B}$ is an $n \times n$ matrix, may be possible and may result in a system which is Hicksian. Theorem 2 can then be applied to establish the stability of this modified system. For example, in the case that $\underline{H}(p(t)) = (p(t)-\bar{p})\underline{A}(\bar{p})$ is linear and $\underline{A}(\bar{p})$ is non-singular, there always exists a sequence of column permutations and sign changes which reduces $\underline{A}(\bar{p})$ to a Hicksian matrix. Then Theorem 1 would establish the stability of the transformed system.

(3) If the bounds (8) and (9) on the structure of the neighborhoods of partial equilibria and the upper bound G on the market demand index functions hold uniformly in M, then Theorem 2 holds for a value of ε independent of M.

(4) If no continuity assumptions are imposed on the market demand index functions H_i other than the condition that they be bounded away from zero when their own prices are bounded away from their partial equilibrium values, the proof of Theorem 2 still establishes that, given any small neighborhood of the general equilibrium, there exists $\varepsilon > 0$ such that for $\underline{C}$ satisfying the hypotheses of Theorem 2, the dynamic process (2) will converge to the given neighborhood. This result could be applied, for example, to the case where the levels of excess demand for some commodities are discrete, leading always to finite jumps in some prices.

(5) Theorem 2 continues to hold if the market demand index functions and rates of accommodation are no longer autonomous, but depend on

time, provided the bounds used on these functions hold uniformly in time. In particular, if the rates of accommodation converge to zero, but converge at a slow enough rate so that their partial sums diverge, then the dynamic system (2) can be made stable even under the relaxed assumptions on the H_i suggested in the previous paragraph.

The assumption of the existence of partial equilibria employed in obtaining the stability results of this paper is intuitively reasonable in many economic models. Further, it is an implication of some of the conditions commonly imposed in the analysis of multi-market stability. In particular, if a condition of strong gross substitutability holds globally or if the Jacobian matrix of $\underline{H}$ has a negative dominant diagonal,[4] then the partial equilibrium conditions will be satisfied. We now demonstrate these propositions rigorously:

Theorem 3. *Suppose an economy satisfies the following conditions (for commodities $i = 1, \ldots, n$):*

(a) *the excess demand functions are homogeneous of degree zero and continuously differentiable,*

(b) *the differential form of the gross substitutability condition holds,*[5] *and*

(c) *Walras's law holds and the supplies of commodities are bounded.*

Then, Assumption 1 *holds.*

Proof. Assume $p_n > 0$. The theorem is proved by induction. Suppose we have established that partial equilibria exist in markets $1, \ldots, i-1$, and satisfy Assumption 1. Let $\underline{p}_{]i} = f_{i-1}(\underline{p}_{[i})$ denote these equilibrium prices, as before, and let

$$z_j = h_j(f_{i-1}(\underline{p}_{[i}), \underline{p}_{[i}) \equiv \xi_{i-1,j}(\underline{p}_{[i}), \quad j = i, \ldots, n$$

denote the compensated excess demand functions in the remaining markets. Suppose further that the compensated demand functions $\xi_{i-1,j}(\underline{p}_{[i})$ satisfy the differential gross substitutability assumption, are homogeneous of degree zero in prices, and satisfy the 'modified' Walras's law,

$$\sum_{j=i}^{n} p_j \xi_{i-1,j}(\underline{p}_{[i}) \equiv 0 .$$

We now demonstrate that these conditions hold for markets $1, \ldots, i$. Applying a lemma established by Arrow and Hurwicz (1959, Lemma 1, p. 89), one has $\xi_{i-1,i}(0, \underline{p}_{i[}) = +\infty$. The homogeneity and gross substitutability properties imply that $\partial\xi_{i-1,i}(p_i, \underline{p}_{i\lceil})/\partial p_i < 0$. By the modified Walras's law,

$$\xi_{i-1,i}(1,\underline{p}_{i\lceil}/p_i) = -\sum_{j=i+1}^{n}(p_j/p_i)\xi_{i-1,j}(1,\underline{p}_{i\lceil}/p_i)\,.$$

But the Arrow-Hurwicz lemma implies $\xi_{i-1,j}(1, \underline{p}_{i\lceil}/p_i) > 0$ for $\underline{p}_{i\lceil}/p_i$ sufficiently small, $j = i+1, \ldots, n$. Hence, $\xi_{i-1,i}(+\infty, \underline{p}_{i\lceil}) < 0$. By continuity, there then exists $p_i = f_{i,i}(\underline{p}_{i\lceil})$ such that $\xi_{i-1,i}(f_{i,i}(\underline{p}_{i\lceil}), \underline{p}_{i\lceil}) = 0$. Applying an implicit function theorem, we establish that $f_{i,i}(\underline{p}_{i\lceil})$ is continuously differentiable, and

$$\partial f_{i,i}/\partial p_j = -\frac{\partial\xi_{i-1,i}}{\partial p_j}\Big/\frac{\partial\xi_{i-1,i}}{\partial p_i} > 0 \text{ for } j = i+1, \ldots, n\,,$$

where $f_{i,i}$ is evaluated at $\underline{p}_{i\lceil}$ and $\xi_{i-1,i}$ is evaluated at $(f_{i,i}, \underline{p}_{i\lceil})$. Now consider the compensated excess demand functions

$$z_j = \xi_{i-1,j}(f_{i,i}(\underline{p}_{i\lceil}), \underline{p}_{i\lceil}) = \xi_{i,j}(\underline{p}_{i\lceil}), \quad j = i+1, \ldots, n\,.$$

The modified Walras's law and homogeneity of degree zero are easily verified for these functions. Further, for j, $k = i+1, \ldots, n$ and $j \neq k$,

$$\frac{\partial\xi_{i,j}}{\partial p_k} = \frac{\partial\xi_{i-1,j}}{\partial p_i}\frac{\partial f_{i,i}}{\partial p_k} + \frac{\partial\xi_{i-1,j}}{\partial p_k} > 0\,,$$

where ξ_{ij} is evaluated at $\underline{p}_{i\lceil}$ and $\xi_{i-1,j}$ is evaluated at $(f_{i,i}(\underline{p}_{i\lceil}), \underline{p}_{i\lceil})$, establishing the gross substitutability property for market i, and proving the induction step. Q.E.D. Theorem 3.

Theorem 4. *Suppose an economy has market demand index functions* $\underline{H}(\underline{p})$ *which are continuously differentiable and suppose that a general equilibrium price vector* $\bar{\underline{p}}$ *exists. Suppose further that the Jacobian matrix* $\underline{A}(\underline{p})$ *of* $\underline{H}(\underline{p})$ *has a negative dominant diagonal for all non-negative prices.*[6] *Then, the system satisfies Assumption* 1.

Proof. Under the hypotheses, every principal sub-matrix of $\underline{A}(\underline{p})$ also has a negative dominant diagonal. Two theorems of McKenzie (1960, Theorems 1 and 2, p. 49) establish that these principal submatrices are non-singular and negative definite. Hence, the matrix $\underline{A}(\underline{p})$ is Hicksian for all $\underline{p}$. A strong implicit function theorem (the global univalence theorem of Gale and Nikaido (1965)) then establishes the existence of a unique solution for a partial equilibrium price vector in any subset of markets. Q.E.D. Theorem 4.

VI. *Conclusions*

Theorems 1 and 2 above verify formally the suggestion made by Sir John Hicks that his perfect stability conditions should be sufficient for the dynamic stability of a price adjustment process which is 'close' to the Marshallian concept of movement through a series of temporary equilibria. The class of essentially sequential price adjustment mechanisms we have studied cannot be put forth seriously as a model of empirical price behavior (except possibly in certain systems where one market is known to adjust much more rapidly than another: as, for example, in the case of a rapidly adjusting bond market and a slowly adjusting durables market, or in the case of a rapidly adjusting domestic market and slowly adjusting international market). On the other hand, the results have theoretical value in that they provide a dynamic framework in which partial equilibrium analysis can be rigorously justified. Further, they allow the model builder to verify satisfaction of stability conditions (and meet the requirements of Samuelson's correspondence principle) with minimum effort, freeing him to pursue more complex representations of the static structure. While the cost in terms of realism is high in restricting one's analysis to models satisfying the partial equilibrium assumptions, the literature of economics attests to the value of added descriptive detail for particular markets. Finally, the possibility of stabilization of a variety of dynamic systems is suggested by Theorems 1 and 2 in the case that the matrix $\underline{C}$ is an instrument of the economic planner or programmer.

APPENDIX : STABILIZATION OF MATRICES

A fundamental theorem due to Fisher and Fuller (1958) forms the basis for our analysis of local stability. A slightly strengthened version of this result is the

> *Fisher-Fuller Theorem (Strong Form)–Suppose $\underline{A}$ is a real $n \times n$ matrix with the property that the upper left-hand principal minor A_i of each order $i = 1, \ldots, n$ is non-zero. Then, there exists a positive scalar ε such that if the real diagonal matrix $\underline{C} = diag.\ (c_i)$ satisfies $c_i A_i / A_{i-1} < 0$ (where $A_0 = 1$) for $i = 1, \ldots, n$ and $| c_i | / | c_{i-1} | < \varepsilon$ for $i = 2, \ldots, n$, then the characteristic roots of $\underline{AC}$ are real, negative, and distinct.*

This result is stronger than the original Fisher-Fuller theorem in that lower bounds on the c_i of the form $\alpha\varepsilon < | c_i | / | c_{i-1} |$, where α is a positive scalar, $\alpha < 1$, are no longer required in the proof. The method of proof used by Fisher and Fuller is to show that $\underline{AC}$ has real characteristic roots which lie close to the values $r_i = c_i A_i / A_{i-1}$, provided ε is sufficiently small.

First, choose ε small enough so that $|r_i| > 4|r_{i+1}|$, and choose c_n so that $r_n = -1$ (the matrix $\underline{C}$ can be rescaled to any desired level of r_n). Employing a lemma on the approximation of roots of polynomials, Fisher and Fuller establish that a sufficient condition for $\underline{AC}$ to have real characteristic roots λ_k satisfying $|\lambda_k - r_k| < \frac{1}{2}$ is that the following inequality be satisfied for each k.[8]

$$1 > \frac{\sum_{s=1}^{n-1} |r_k|^{n-s} \cdot |r_1||r_2| \ldots |r_{s-1}||r_{s+1}| m_s}{\frac{1}{2}|r_k| \cdot \prod_{i=1}^{k-1} |r_i - \frac{3}{2}r_k| \cdot \prod_{i=k+1}^{n} |\frac{1}{2}r_k - r_i|} = (*),$$

where the m_s are positive constants determined by the matrix A. This expression can be rewritten in the form

$$(*) = \frac{2^{1+k-n}\left[\sum_{s=1}^{k-1} \left|\frac{r_k}{r_s}\right| \cdot \prod_{i=s+2}^{k} \left|\frac{r_k}{r_i}\right| m_s + \sum_{s=k}^{n-1} \left|\frac{r_{s+1}}{r_k}\right| \cdot \prod_{i=k}^{s-1} \left|\frac{r_i}{r_k}\right| m_s\right]}{\prod_{i=1}^{k-1} \left|1 - \frac{3}{2}\frac{r_k}{r_i}\right| \cdot \prod_{i=k+1}^{n} \left|1 - 2\frac{r_i}{r_k}\right|}$$

(products Π over empty index sets are taken to equal one). Using the condition $|r_i| > 4|r_{i+1}|$, one can establish the inequality

$$(*) < 2\left(\frac{8}{5}\right)^{k-1}\left[\left|\frac{r_k}{r_{k-1}}\right| \sum_{s=1}^{k-1} m_s + \left|\frac{r_{k+1}}{r_k}\right| \sum_{s=k}^{n-1} m_s\right].$$

For $|c_{i+1}/c_i| < \varepsilon = 2^{-n-1}\left(1 + \sum_{s=1}^{n-1} m_s\right)^{-1} \cdot \min_{k=1,\ldots,n-1} |A_k^2 / A_{k+1}A_{k-1}|$,

the right-hand side of this expression will be less than one, and the sufficient condition for the validity of the theorem holds.

The following corollary establishes that the Fisher-Fuller theorem above holds uniformly when $\underline{A}$ varies continuously over a compact set.

Corollary. Suppose the matrix $\underline{A}(p)$ is a continuous function over p contained in a compact set $\underline{N}$, and has $A_i(p)$ non-zero on $\underline{N}$. Then there exists $\varepsilon > 0$ such that if $\underline{C}$ satisfies the conditions of the Fisher-Fuller theorem, then the characteristic roots of $\underline{A}(p)\underline{C}$ are real, distinct, finite, and bounded negative uniformly for p in $\underline{N}$.

The bound established for $|c_{i+1}/c_i|$ in the proof outline of the theorem above is found to depend on the terms m_s and A_k, which are continuous functions of determinants of various sub-matrices of $\underline{A}(\underline{p})$. Further, the $A_k(\underline{p})$ are bounded away from zero on $\underline{N}$. Hence, the bound for $|c_{i+1}/c_i|$ is continuous in $\underline{p}$, and achieves a positive minimum on $\underline{N}$.

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NOTES AND REFERENCES

[1] The prices p_i may be thought of as 'normalized' prices, and a numéraire commodity $n+1$ with price one may also be present in the economy. In our analysis, it is *unnecessary* to assume that such a 'non-normalized' system is behind our model and related to it by homogeneity and Walras's law. However, it is a special property of the class of dynamic processes we consider that, when this relationship does hold, the solutions of the 'normalized' ($p_{n+1} = 1$) and 'non-normalized' (p_{n+1} modified by the dynamic process) systems are almost identical. The only additional restriction required is that the rate of accommodation in market $n+1$ be low enough so that p_{n+1} remains positive.

[2] The results of this paper are unchanged if the difference equations (2) are replaced by differential equations $dp_i(t)/dt = c_i H_i(\underline{p}(t))$.

[3] The notation $|\underline{p}| = \left(\sum_{i=1}^{n} |p_i|^2\right)^{\frac{1}{2}}$ and $\|\underline{A}\| = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}$ is used for the Euclidean norms of a vector $\underline{p}$ and a matrix $\underline{A}$.

[4] A matrix $\underline{A} = (a_{ij})$ is said to have a negative dominant diagonal if there exists a positive diagonal matrix $\underline{C}$ such that $-a_{ii}c_i > \sum_{j \neq i} |a_{ij}| c_j$ for all i. For a full discussion, see McKenzie (1960).

[5] Differential gross substitutability requires that $\partial h_i(\underline{p})/\partial p_j$ be positive for $i \neq j$, while Walras's law requires that $\sum_{i=1}^{n} p_i h_i(\underline{p})$ be identically zero.

[6] The normalized price system, where commodity $n+1$ has price $p_{n+1} = 1$, will tacitly be assumed in this theorem.

[7] The expression (*) is a bound on the ratio $|g((1\pm\delta)r_k)/|f((1\pm\delta)r_k)|$ used by Fisher and Fuller (1958, equations (14)–(20), p. 442), taken with $\delta = \frac{1}{2}$.