Journal of Mathematical Economics 6 (1979) 15–16. © North-Holland Publishing Company

A NOTE ON THE COMPUTABILITY OF TESTS OF THE STRONG AXIOM OF REVEALED PREFERENCE

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Received May 1977, final version received August 1978

Let R^l_+ denote the non-negative orthant; and $P=\{p\in R^l_+ | p\gg 0\}$, the positive orthant. Let $d:p\to R^l_+$ be a continuous demand function satisfying $p\cdot d(p)=1$ for every $p\in P$. Define a binary relation W [resp., $\bar{W}$] on $d(P)$ by xWy [*resp.*, $x\bar{W}y$] if and only if there exist $p,p'\in P$ such that $x=d(p)$, $y=d(p')$, and $p\cdot d(p')<1$ [resp., $x\neq y$ and $p\cdot d(p')\leqq 1$]. Note that $\bar{W}$ is the usual direct revealed preference relation, and W is a non-tight direct revealed preference relation satisfying $W\subseteq\bar{W}$. Define a binary relation S [resp., $\bar{S}$] on $d(P)$ by xSy [resp., $x\bar{S}y$] if there exist $z^1,\ldots,z^n$ in $d(P)$ such that $xWz^1W\ldots Wz^nWy$ [resp., $x\bar{W}z^1\bar{W}\ldots\bar{W}z^n\bar{W}y$]. Note that $\bar{S}$ is the usual indirect revealed preference relation, and that $S\subseteq\bar{S}$.

A demand function satisfies the Strong Axiom of Revealed Preference (SARP) if and only if $\bar{S}$ is acyclic. From the theory of revealed preference, satisfaction of SARP is necessary and sufficient for a single-valued demand function to be consistent with maximization of a locally non-satiated preference preorder.

This note establishes that satisfaction of SARP for *continuous* demand functions can be tested in principle by a recursive computational algorithm, and a failure can be found in a *finite* number of steps. Let P^* be any countable dense subset of P; e.g., the set of $p\in P$ with rational coordinates. Let Q^* denote the countable set of all finite sequences of points from P^*. Define the following:

Algorithm. Let $i=1,2,\ldots$ index the elements of Q^*. For each element $(p^{1i},\ldots,p^{n_i i})$ of Q^* in this sequence, terminate the algorithm if $\max_{k=1,\ldots,n_i} p^{ki}\cdot d(p^{k+1,i})<1$, where $p^{1i}=p^{n_i+1,i}$; otherwise continue.

The main result is:

*Research was supported in part by National Science Foundation Grant No. SOC75-22657 to the University of California, Berkeley.

Theorem. *Suppose $d: P \to R^l_+$ is continuous. Then, d fails to satisfy SARP if and only if the Algorithm terminates in a finite number of steps.*

The following three lemmas prove the theorem:

Lemma 1. $x\bar{W}y$ *implies* xSy.

Proof. Suppose $x=d(p)$, $y=d(p')$, $p \cdot y \leqq 1$ for some $p, p' \in P$. If $p \cdot y < 1$, then xWy, implying xSy. Alternately, suppose $p \cdot y = 1$. Since $x \neq y$, we can construct a hyperplane with normal $q \neq 0$ such that $q \cdot x > q \cdot y$. For $\alpha > 0$ such that $p + \alpha q \in P$, define $p^\alpha = (p + \alpha q)/(1 + \alpha\{q \cdot (x+y)/2\})$. Then $\lim_{\alpha \to 0} p^\alpha = p$, implying $\lim_{\alpha \to 0} d(p^\alpha) = x$. Now, $p^\alpha \cdot y = (1 + \alpha q \cdot y)/(1 + \alpha\{q \cdot (x+y)/2\}) < 1$. Hence, $x^\alpha = d(p^\alpha)$ satisfies $x^\alpha W y$. Also, the formula for p^α implies $p = (1 + \alpha\{q \cdot (x+y)/2\})p^\alpha - \alpha q$. Hence, $p \cdot x^\alpha = 1 + \alpha[\{q \cdot (x+y)/2\} - q \cdot x^\alpha]$. Since $\lim_{\alpha \to 0} q \cdot x^\alpha = q \cdot x > q \cdot y$, one has $p \cdot x^\alpha < 1$ for α sufficiently small, implying xWx^α. Hence $xWx^\alpha Wy$ implies xSy. Q.E.D.

Lemma 2.[1] $S = \bar{S}$.

Proof. We only need to show $\bar{S} \subseteq S$. Suppose $x\bar{S}y$, or $x\bar{W}z^1\bar{W} \ldots \bar{W}z^n\bar{W}y$. By Lemma 1, $xSz^1s \ldots Sz^nSy$. Since S is transitive by construction, xSy. Q.E.D.

Lemma 3. *If $d: P \to R^l_+$ is continuous, then it satisfies SARP if and only if its restriction to P^* satisfies SARP.*

Proof. The 'only if' direction is trivial. To show the 'if' direction, suppose there exist $x, y \in d(P)$ such that $x\bar{S}y\bar{S}x$, so that SARP fails. By Lemma 2, $xSySx$. Writing out the conditions for these relations, there exist $p^i \in P$ such that $p^n = p^1$ and $d(p^i)Wd(p^{i+1})$ for $i = 1, \ldots, n-1$ [where $x = d(p^1)$ and $y = d(p^k)$ for some k], or $p^i \cdot d(p^{i+1}) < 1$. By continuity of d, there exist $\hat{p}^i \in P^*$ sufficiently close to p^i to satisfy $\hat{p}^i \cdot d(\hat{p}^{i+1}) < 1$ for $i = 1, \ldots, n-1$. Hence, $d(\hat{p}^1)Sd(\hat{p}^{n-1})Sd(\hat{p}^1)$, implying that the restriction of d to P^* fails to satisfy SARP. Q.E.D.

[1] A result along the lines of Lemma 2 is contained in the unpublished paper by A. Mas-Colell, 'Preferences and income Lipschitzian demand: Continuity and compactness properties', IP-173, Center for Research in Management Science, University of California, Berkeley, CA, October 1972.