A NOTE ON THE COMPUTABILITY OF TESTS OF THE STRONG AXIOM OF REVEALED PREFERENCE

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Let $R_+^1$ denote the non-negative orthant; and $P = \{ p \in R_+^1 | p \gg 0 \}$, the positive orthant. Let $d: p \rightarrow R_+^1$ be a continuous demand function satisfying $p \cdot d(p) = 1$ for every $p \in P$. Define a binary relation $W$ [resp., $\bar{W}$] on $d(P)$ by $xW y$ [resp., $x\bar{W} y$] if and only if there exist $p, p' \in P$ such that $x = d(p)$, $y = d(p')$, and $p \cdot d(p') < 1$ [resp., $x \neq y$ and $p \cdot d(p') \leq 1$]. Note that $W$ is the usual direct revealed preference relation, and $\bar{W}$ is a non-tight direct revealed preference relation satisfying $W \subseteq \bar{W}$. Define a binary relation $S$ [resp., $\bar{S}$] on $d(P)$ by $xSy$ [resp., $x\bar{S}y$] if there exist $z^1, \ldots, z^n$ in $d(P)$ such that $xWz^1W \ldots Wz^nW y$ [resp., $x\bar{W}z^1\bar{W} \ldots Wz^n\bar{W} y$]. Note that $S$ is the usual indirect revealed preference relation, and that $S \subseteq \bar{S}$.

A demand function satisfies the Strong Axiom of Revealed Preference (SARP) if and only if $S$ is acyclic. From the theory of revealed preference, satisfaction of SARP is necessary and sufficient for a single-valued demand function to be consistent with maximization of a locally non-satiated preference preorder.

This note establishes that satisfaction of SARP for continuous demand functions can be tested in principle by a recursive computational algorithm, and a failure can be found in a finite number of steps. Let $P^*$ be any countable dense subset of $P$; e.g., the set of $p \in P$ with rational coordinates. Let $Q^*$ denote the countable set of all finite sequences of points from $P^*$. Define the following:

**Algorithm.** Let $i = 1, 2, \ldots$ index the elements of $Q^*$. For each element $(p^{i1}, \ldots, p^{inp})$ of $Q^*$ in this sequence, terminate the algorithm if $\max_{k=1, \ldots, n} p^{ki} \cdot d(p^{ki+1,i}) < 1$, where $p^{ii} = p^{n+1,i}$; otherwise continue.

The main result is:

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Theorem. Suppose \( d : P \rightarrow R^1_+ \) is continuous. Then, \( d \) fails to satisfy SARP if and only if the Algorithm terminates in a finite number of steps.

The following three lemmas prove the theorem:

Lemma 1. \( xWy \) implies \( xSy \).

Proof. Suppose \( x = d(p) \), \( y = d(p') \), \( p \cdot y \leq 1 \) for some \( p, p' \in P \). If \( p \cdot y < 1 \), then \( xWy \), implying \( xSy \). Alternately, suppose \( p \cdot y = 1 \). Since \( x \neq y \), we can construct a hyperplane with normal \( q \neq 0 \) such that \( q \cdot x > q \cdot y \). For \( \alpha > 0 \) such that \( p + \alpha q \in P \), define \( p^* = (p + \alpha q)/(1 + \alpha(q \cdot (x+y)/2)) \). Then \( \lim_{\alpha \to 0} p^* = p \), implying \( \lim_{\alpha \to 0} d(p^*) = x \). Now, \( p^* \cdot y = (1 + \alpha q \cdot y)/(1 + \alpha(q \cdot (x+y)/2)) < 1 \). Hence, \( x^2 = d(p^*) \) satisfies \( x^2Wy \). Also, the formula for \( p^* \) implies \( p = (1 + \alpha(q \cdot (x+y)/2))p^* - \alpha q \). Hence, \( p \cdot x^2 = 1 + \alpha[(q \cdot (x+y)/2) - q \cdot x^2] \). Since \( \lim_{\alpha \to 0} q \cdot x^2 = q \cdot x > q \cdot y \), one has \( p \cdot x^2 < 1 \) for \( \alpha \) sufficiently small, implying \( xWx^2 \). Hence \( xWx^2Wy \) implies \( xSy \). Q.E.D.

Lemma 2.\(^1\) \( S = S \).

Proof. We only need to show \( \bar{S} \subseteq S \). Suppose \( xSy \), or \( xWz^1W \ldots Wz^nWy \). By Lemma 1, \( xSz^1 \ldots Sz^nSy \). Since \( S \) is transitive by construction, \( xSy \). Q.E.D.

Lemma 3. If \( d : P \rightarrow R^1_+ \) is continuous, then it satisfies SARP if and only if its restriction to \( P^* \) satisfies SARP.

Proof. The 'only if' direction is trivial. To show the 'if' direction, suppose there exist \( x, y \in d(P) \) such that \( xSySx \), so that SARP fails. By Lemma 2, \( xSySx \). Writing out the conditions for these relations, there exist \( p^i \in P \) such that \( p^n = p^1 \) and \( d(p^i)Wd(p^{i+1}) \) for \( i = 1, \ldots, n-1 \) [where \( x = d(p^i) \) and \( y = d(p^k) \) for some \( k \)], or \( p^i \cdot d(p^{i+1}) < 1 \). By continuity of \( d \), there exist \( \hat{p}^i \in P^* \) sufficiently close to \( p^i \) to satisfy \( \hat{p}^i \cdot d((\hat{p}^{i+1}) < 1 \) for \( i = 1, \ldots, n-1 \). Hence, \( d((\hat{p}^i))Sd((\hat{p}^{i-1}))Sd(\hat{p}^1) \), implying that the restriction of \( d \) to \( P^* \) fails to satisfy SARP. Q.E.D.

\(^1\)A result along the lines of Lemma 2 is contained in the unpublished paper by A. Mas-Colell, 'Preferences and income Lipschitzian demand: Continuity and compactness properties', IF-173, Center for Research in Management Science, University of California, Berkeley, CA, October 1972.