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PARETO OPTIMALITY AND COMPETITIVE EQUILIBRIUM IN INFINITE HORIZON ECONOMIES*

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The paper presents a general model of a decentralized economy evolving over an infinite time horizon. Alternative notions of price systems, competitive equilibria, efficiency and optimality are introduced. The main results characterize conditions under which the two fundamental theorems of welfare economics are valid in such a general framework.

1. Introduction

A cornerstone of classical economics is the idea that a competitive equilibrium is optimal in the Paretian sense that no alternative feasible allocation of commodities can improve the lot of one agent without worsening the conditions of some other individual. Equally important is the converse proposition that any given Pareto optimal allocation can be sustained by a competitive equilibrium. A prime achievement of welfare economics has been to establish conditions that are, roughly speaking, 'necessary and sufficient' for the validity of these conclusions in *finite* economies (i.e., economies in which the numbers of commodities and economic agents are finite). On the other hand, it is known that in non-finite economies, these propositions may fail even when the sufficient conditions of the finite case are met. In this paper we shall restrict our attention to a class of infinite horizon economies, typified by von Neumann growth models, in which the production possibilities are not constrained by non-producible

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factors. For these economies, it is verified that the welfare propositions mentioned above hold under conditions closely paralleling those relevant for finite economies. The relation between competitive equilibrium and efficiency or Pareto optimality for static economies has been reviewed in Koopmans (1975). References to the partial extensions of these results to some infinite horizon economies are given in Majumdar–Mitra–McFadden (1976). No attempt will be made, therefore, to review the extensive literature on intertemporal efficiency or optimality. In section 2, a general model of a decentralized economy over time is presented. In section 3, various notions of price systems, competitive equilibria, efficiency and optimality are introduced. The main results in section 4 give necessary and sufficient conditions under which a *strong* competitive equilibrium [see Definition (3.5)] is Pareto optimal, and a Pareto optimal allocation is a *valuation* equilibrium [see Definition (3.6)]. Some other logical connections among the various concepts are examined in section 3, with a number of counter-examples to possible implications.

2. A general decentralized economy

2.1. Commodity space

Consider an economy in which the numbers of commodities and economic agents *within* each period $t=0,1,2\ldots$ is finite. Each economic agent (consumer or firm) is assumed to have a *finite* life.

Let $\mathscr{G}_t$ denote the commodity space in period t; it is a finite-dimensional real linear vector space whose vectors have a component for each commodity existing in the economy in period t. Define the real linear vector space $\mathscr{G}$ consisting of all infinite sequences (or programs) or commodity vectors, $v=(v^0,v^1,\ldots)$ with $v^t\in\mathscr{G}_t$ for $t=0,1,2,\ldots$ Then $\mathscr{G}$ is the commodity space for the infinite horizon economy.

2.2. Firms

The generation of firms initially formed in period t will be numbered $j=1,2,\ldots,J_t$. The typical firm j in generation t will have a finite lifetime (w_{jt} periods), and will have for each period τ ($\tau=0,1,2,\ldots$) an *input–output pair* $(a_{jt}^{\tau},b_{jt}^{\tau+1})$ in $\mathscr{G}_\tau\times\mathscr{G}_{\tau+1}$, where $(a_{jt}^{\tau},b_{jt}^{\tau+1})=(0,0)$ for $\tau<t$, and $\tau\geqq t+(w_{jt}-1)$, and $b_{jt}^0=0$ for $t\geqq 0$.

The typical firm j in generation t will then have, for each period τ, a net output vector $y_{jt}^{\tau}=b_{jt}^{\tau}-a_{jt}^{\tau}$ in $\mathscr{G}_t$, where $y_{jt}^{\tau}=0$ for $\tau<t$ and $\tau>t+(w_{jt}-1)$.

An *input program* for a typical firm is $a_{jt}=(a_{jt}^0,a_{jt}^1,\ldots)$ in $\mathscr{G}$, whose components are zero outside the firm's lifetime, in the sense indicated above. Similarly, an *output program* and a *net output program* for the typical firm are $b_{jt}=(b_{jt}^0, b_{jt}^1,\ldots)$ and $y_{jt}=(y_{jt}^0, y_{jt}^1,\ldots)$, both in $\mathscr{G}$. Let $\mathscr{Y}_{jt}$ denote the finite-dimensional linear subspace of $\mathscr{G}$ spanned by the net output programs y_{jt} in $\mathscr{G}$.

An *aggregate input bundle* in period τ is denoted by

$$a^\tau=\sum_{t=0}^{\tau}\sum_{j=1}^{J_t} a_{jt}^\tau, \qquad \tau\geqq 0. \tag{2.1}$$

An *aggregate output bundle* in period $(\tau+1)$ is

$$b^{\tau+1}=\sum_{t=0}^{\tau}\sum_{j=1}^{J_t} b_{jt}^{\tau+1}, \quad \tau\geqq 0, \quad \text{and} \quad b^0=\sum_{j=1}^{J_0} b_{j0}^0=0. \tag{2.2}$$

An *aggregate net output bundle* in period τ is

$$y^\tau=b^\tau-a^\tau=\sum_{t=0}^{\tau}\sum_{j=1}^{J_t} y_{jt}^\tau, \qquad \tau\geqq 0. \tag{2.3}$$

Aggregate input, output, and net output programs are defined by

$$\begin{aligned} a&=\sum_{t=0}^{\infty}\sum_{j=0}^{J_t} a_{jt}=(a^0,a^1,\ldots) \quad \text{in } \mathscr{G},\\ b&=\sum_{t=0}^{\infty}\sum_{j=1}^{J_t} b_{jt}=(b^0,b^1,\ldots) \quad \text{in } \mathscr{G},\\ y&=\sum_{t=0}^{\infty}\sum_{j=1}^{J_t} y_{jt}=(y^0,y^1,\ldots) \quad \text{in } \mathscr{G}. \end{aligned} \tag{2.4}$$

[Clearly, $y=(b^0-a^0, b^1-a^1,\ldots)$ is in $\mathscr{G}$.]

For each firm j in generation t, a *technology set* $\mathscr{T}_{jt}^\tau$, on $R_\tau^+\times R_{\tau+1}^+$, defines the input–output pairs which the firm can obtain by production in period τ. An input–output pair $(a_{jt}^\tau, b_{jt}^{\tau+1})$ is technologically feasible if it belongs to $\mathscr{T}_{jt}^\tau$.

The following assumptions on the technology sets $\mathscr{T}_{jt}^\tau$ will be used:

Assumption T

(T.1) $\mathscr{T}_{jt}^\tau$ *is independent of the bahavior of any other agent* (i.e., no externalities).

(T.2) $(0,0)\in\mathscr{T}_{jt}^\tau$ (inaction is possible).

(T.3) $(a,b)\in\mathscr{T}^{\tau}_{jt}$ *and* $a'\geqq a$, $0\leqq b'\leqq b$, *implies* $(a',b')\in\mathscr{T}^{\tau}_{jt}$ (free disposal).

(T.4) $(a,b)\in\mathscr{T}^{\tau}_{jt}$, *and* $a=0$ *implies* $b=0$ (impossibility of free production).

(T.5) $\mathscr{T}^{\tau}_{jt}$ *is closed and convex* (continuity and non-increasing returns).

For each firm j in generation t, a *production possibility set* Y_{jt} defines the net output programs y_{jt} which the firm can achieve, i.e.,

$$Y_{jt}=\{y_{jt}\,|\,y^{\tau}_{jt}=b^{\tau}_{jt}-a^{\tau}_{jt},\quad (a^{\tau}_{jt},b^{\tau+1}_{jt})\in\mathscr{T}^{\tau}_{jt}\quad \text{for}\quad \tau\geqq 0\}. \tag{2.5}$$

The following consequences of Assumption T for the sets Y_{jt} may, then, be noted:

(Y.1) Y_{jt} is independent of the behavior of any other economic agent.

(Y.2) The null net output program is in Y_{jt}.

(Y.3) Y_{jt} has a convex, free-disposal hull.[1]

(Y.4) Y_{jt} is closed under pointwise convergence.[2]

(Y.5) If $y_{jt}\in Y_{jt}$, and y_{jt} is bounded below, then y_{jt} is bounded above.[3]

Aggregate production possibilities for the economy are defined by

$$Y=\sum_{t=0}^{\infty}\sum_{j=1}^{J_t}Y_{jt}=\left\{y\in\mathscr{G}\,\middle|\,y=\sum_{t=0}^{\infty}\sum_{j=1}^{J_t}y_{jt},\quad y_{jt}\in Y_{jt}\right\}. \tag{2.6}$$

The following consequence of Assumption T for the set Y, may then be proved:

Lemma 2.1 *If* (Y.2)–(Y.5) *hold, then* Y *is closed under pointwise convergence.*

Proof. Suppose a sequence of programs ${}^{(n)}y$ in Y converges pointwise to

[1]The free disposal hull of Y_{jt} is the set $\{y\in\mathscr{Y}_{jt}\,|\,y\leqq y' \text{ for some } y' \text{ in } Y_{jt}\}$.

[2]A sequence of programs ${}^{(n)}y$ converges pointwise to a program y^* if each component of the vector ${}^{(n)}y$ converges to the corresponding component of y^*. Formally, given t and $\varepsilon>0$, there exists $n_0(t,\varepsilon)$ such that for $n>n_0(t)$, $|{}^{(n)}y^t-y^{*t}|<\varepsilon$. When the vectors are confined to a finite-dimensional subspace, pointwise convergence is equivalent to ordinary Euclidean convergence.

[3]Otherwise, there is a sequence ${}^{(n)}y_{jt}\in Y_{jt}$ such that ${}^{(n)}y^{\tau'}_{jt}$ is bounded below, but ${}^{(n)}y^{\tau'+1}_{jt}$ is unbounded above. This means by definition of the Y_{jt}, that for some $\tau\leqq\tau'$, ${}^{(n)}a^{\tau}_{jt}$ is bounded above, ${}^{(n)}b^{\tau+1}_{jt}$ is unbounded above $|{}^{(n)}b^{\tau+1}_{jt}|>1$, and $({}^{(n)}a^{\tau}_{jt},{}^{(n)}b^{\tau+1}_{jt})\in\mathscr{T}_{jt}$. Define ${}^{(n)}\alpha_{jt}={}^{(n)}a^{\tau}_{jt}/|{}^{(n)}b^{\tau+1}_{jt}|$, ${}^{(n)}\beta^{\tau+1}_{jt}={}^{(n)}b^{\tau+1}_{jt}/|{}^{(n)}b^{\tau+1}_{jt}|$. Then, by (T.2) and (T.5), $({}^{(n)}\alpha^{\tau}_{jt},{}^{(n)}\beta^{\tau+1}_{jt})\in\mathscr{T}_{jt}$ and ${}^{(n)}\alpha^{\tau}_{jt}\to 0$, ${}^{(n)}\beta^{\tau+1}_{jt}$ is bounded, and $|{}^{(n)}\beta^{\tau+1}_{jt}|=1$. Hence, there is a convergent subsequence of n (call it n again) such that ${}^{(n)}\alpha^{\tau}_{jt}\to 0$, and ${}^{(n)}\beta^{\tau+1}_{jt}\to\beta$. By (T.5), $(0,\beta)\in\mathscr{T}_{jt}$. But $|\beta|=1$, which violates (T.4).

$\bar{y} \in \mathscr{G}$. There exists a corresponding sequence ${}^{(n)}y_{jt} \in Y_{jt}$, such that ${}^{(n)}y = \sum_{t=0}^{\infty} \sum_{j=1}^{J_t} {}^{(n)}y_{jt}$.

Suppose each ${}^{(n)}y_{jt}$ sequence is bounded. Then, it has a subsequence which converges pointwise to some $\bar{y}_{jt}$ in Y_{jt} by (Y.4). The Cantor diagonal process can then be used to extract a subsequence converging pointwise to $\sum_{t=0}^{\infty} \sum_{j=0}^{J_t} \bar{y}_{jt}$ in Y. But $\sum_{t=0}^{\infty} \sum_{j=1}^{J_t} \bar{y}_{jt} = \bar{y}$ by the supposition that ${}^{(n)}y$ converges pointwise to $\bar{y}$. Hence, $\bar{y}$ is in Y.

Alternatively, suppose for a subsequence (denote it by n again) ${}^{(n)}y_{jt}$ is unbounded. If ${}^{(n)}y_{jt}$ is bounded below, ${}^{(n)}y_{jt}$ is bounded above by (Y.5). So, under the supposition that ${}^{(n)}y_{jt}$ is unbounded, there is a first period τ in which ${}^{(n)}y_{jt}$ is unbounded below. Since τ is the first period this happens, ${}^{(n)}y^{\tau-1}$ is bounded below, and hence ${}^{(n)}y^{\tau}$ is bounded above. Thus ${}^{(n)}y^{\tau} \to -\infty$ as $n \to \infty$ (for some component), contradicting the hypothesis that ${}^{(n)}y$ converges pointwise to $\bar{y}$. Q.E.D.

A *net output plan* for this economy is a complete description of the net output program of each firm, and may be represented formally as a sequence of vectors of these programs:

$$s = (y, y_{10}, \ldots, y_{J_0}, y_{11}, \ldots, y_{J_1}, y_{12}, \ldots),$$

where $y = \sum_{t=0}^{\infty} \sum_{j=1}^{J_t} y_{jt}$ is the aggregate net output program.

A net output plan is *possible* if each y_{jt} is in the corresponding production possibility set Y_{jt}.

A possible net output plan s is said to be *efficient* if no alternative possible net output plan s' yields an aggregate net output program which is at least as large in every component and larger in at least one component. That is, if y and y' are the net output programs associated with s and s', respectively, and $y' \geqq y$, then $y' = y$.

An aggregate net output program will be called *possible* (*efficient*) if it can be associated with some net output plan which is possible (efficient).

2.3. Consumers

The generation of consumer units intially formed in period t will be numbered $k = 1, 2, \ldots, K_t$. The typical consumer unit k in generation t will consume in each period τ of its finite lifetime (α_{kt} periods) a non-negative commodity bundle c_{kt}^{τ} in $\mathscr{G}_{\tau}$. We define a finite-dimensional linear subspace $\mathscr{C}_{kt}$ of $\mathscr{G}$ spanned by the consumption programs $c_{kt} = (c_{kt}^0, c_{kt}^1, \ldots)$ in $\mathscr{G}$, whose components are zero outside the consumer's lifetime [i.e., $c_{kt}^{\tau} = 0$ for $\tau < t$, and $\tau > t + (\alpha_{kt} - 1)$]. In order to avoid complications of questionable economic interest, assume that there is a positive (finite) number α such that $\alpha \geqq \alpha_{kt}$ for all k, t.

The typical consumer unit will have a *desired set* D_{kt}, a subset of $\mathscr{C}_{kt}$, consisting of the non-negative consumption programs on which it can subsist. On the desired set D_{kt}, the consumer unit will have a preference preordering (i.e., a complete, transitive, reflexive binary relation) $\succsim_{kt}$. For y, y' in D_{kt}, we use the notation $y \succ_{kt} y'$ (resp. $y \sim_{kt} y'$) iff $y \succsim_{kt} y'$ and not $y' \succsim_{kt} y$ (resp. $y \succsim_{kt} y'$ and $y' \succsim_{kt} y$). We shall employ several or all of the following conditions on preferences.

Assumption P.[4] The desired set D_{kt} and the preference preordering $\succsim_{kt}$ have some or all of the following properties for $k=1,2,\ldots,K_t$, and $t=0,1,2,\ldots$

(P.1) *D_{kt} and $\succsim_{kt}$ are independent of the consumption programs of other consumer units, and the net output programs of firms (no externalities).*

(P.2) *D_{kt} is convex and monotone above* (i.e., $c_{kt} \in D_{kt}$, $c'_{kt} \in \mathscr{C}_{kt}$, *and* $c'_{kt} \geqq c_{kt}$ *imply* $c'_{kt} \in D_{kt}$), *and the set of non-negative consumption vectors in $\mathscr{C}_{kt}$, which are not in D_{kt} is closed under pointwise convergence.* (This set may be empty.)

(P.3) *At any $c_{kt} \in D_{kt}$, the upper contour set $U_{kt}(c_{kt}) = \{c'_{kt} \in D_{kt} \mid c'_{kt} \succsim_{kt} c_{kt}\}$ is closed under pointwise convergence, relative to D_{kt}* (continuity of preferences).

(P.4) *At any $c_{kt} \in D_{kt}$, the upper contour set $U_{kt}(c_{kt})$ is convex* (i.e., if $c', c'' \in D_{kt}$ *satisfy* $c' \succsim_{kt} c_{kt}$, *and* $c'' \succsim_{kt} c_{kt}$, *then* $\theta c' + (1-\theta)c'' \succsim_{kt} c_{kt}$ *for* $0 < \theta < 1$).

(P.5) *If $c'_{kt} \geqq c_{kt}$ in D_{kt}, then $c'_{kt} \in U(c_{kt})$* (monotonicity of preferences).

(P.6) *At any $c_{kt} \in D_{kt}$, there exists $c'_{kt} \in D_{kt}$, which is strictly preferred, i.e.,* $c_{kt} \notin U_{kt}(c'_{kt})$ (non-saturation).

(P.7) *At any $c_{kt} \in D_{kt}$, there exists a sequence ${}^{(n)}c_{kt} \in D_{kt}$ converging pointwise to c_{kt} such that each ${}^{(n)}c_{kt}$ is strictly preferred to c_{kt}* (local non-saturation).

An *aggregate consumption bundle* in period τ can be defined by c^τ (in $\mathscr{G}_\tau$),

$$c^\tau = \sum_{t=0}^{\tau} \sum_{k=1}^{K_t} c^\tau_{kt} = \sum_{t=\tau-\alpha}^{\tau} \sum_{k=1}^{K_t} c^\tau_{kt}. \tag{2.7}$$

An *aggregate consumption program* is then given by

[4]Several comments may be useful in clarifying these properties. If D_{kt} consists of all consumption programs which are positive in one subset of commodities and non-negative in the remaining commodities over the lifetime of the consumer unit, then (P.2) is satisfied. If preferences are representable by continuous utility functions, then (P.3) holds. If there is some commodity which is essential to subsistence, divisible, and always desired, and D_{kt} is as described above, then preferences satisfy (P.6) and (P.7).

$$c=\sum_{t=0}^{\infty}\sum_{k=1}^{K_t} c_{kt}=(c^0,c^1,\ldots) \quad \text{in } \mathscr{G}. \tag{2.8}$$

A *distribution plan* in this economy is a complete description of the consumption program of each consumer unit, and may be represented formally as a sequence of vectors $d=(c,c_{10},\ldots,c_{k_00},c_{11},\ldots,c_{k_11},c_{12},\ldots)$, where

$$c=\sum_{t=0}^{\infty}\sum_{k=1}^{K_t} c_{kt} \tag{2.9}$$

is the aggregate consumption program.

A distribution plan d is *desirable* if each c_{kt} is in the corresponding desired set D_{kt}. We let D denote the set of desirable distribution plans. A distribution plan d in D is said to be *Pareto preferable* to a plan d' in D, if $c_{kt}\succsim_{kt}c'_{kt}$ for all consumer units, and $c_{kt}\succ_{kt}c'_{kt}$ for at least one consumer unit.

2.4. Feasible allocations and Pareto optimality

The economy is assumed to have a vector of non-produced resources, a non-negative bundle in $\mathscr{G}_t$, which initially becomes available in period t. This bundle is denoted by e^t, and the *resource supply program* is denoted by $e=(e^0,e^1,e^2,\ldots)$. When y is a possible aggregate net output program, $c=y+e$ will be termed a *possible supply program.*

A possible net output plan s, a desirable distribution plan d, and a resource supply program e define a *feasible allocation* $h=(s,d,e)$ if the material balance condition $c=y+e$ is met by the aggregate consumption program and net output program determined in d and s, respectively. We let H denote the set of feasible allocations in this economy.

A feasible allocation h is *Pareto optimal* if there is no Pareto preferable feasible allocation $\bar{h}$. It is *short-run Pareto optimal* if there is no Pareto preferable feasible allocation $\tilde{h}$, such that every consumer unit living on or after some period L gets the same allocation in $\tilde{h}$ and in h.

3. Concepts of price system, equilibrium, non-decomposability, and reachability

3.1. Prices and equilibrium

A *price system* on a subspace $\mathscr{G}_1$ of the commodity space $\mathscr{G}$ is a linear function P which is not identically zero on $\mathscr{G}_1$ and is *non-negative*, i.e., $P(c)\geq 0$ for any non-negative c in $\mathscr{G}_1$. If $\mathscr{G}_1$ is a topological space, and P is continuous on $\mathscr{G}_1$ we call P a *valuation function* [see Debreu (1954)]. When P is representable as an infinite sequence $p=(p^t)$ with $P(c)=\sum_{t=0}^{\infty} p^t c^t$ for all c in $\mathscr{G}_1$, p is termed a *price sequence*. In this case the notation $p \cdot c=\sum_{t=0}^{\infty} p^t c^t$ is adopted.

Consider the subspace $\mathscr{G}_{1f}$ of $\mathscr{G}_1$ consisting of all sequences c in $\mathscr{G}_1$ which have a *finite* number of non-zero components. Then, for any price-system P *on* $\mathscr{G}_1$, there exists a (possibly zero) sequence $p=(p^t)$ which is a unique representation of P on $\mathscr{G}_{1f}$.[5] If $\mathscr{G}_1$ consists *only* of programs with a finite number of non-zero components, any price system P on $\mathscr{G}_1$ will have a unique representation as a *non-zero price sequence* $p=(p^t)$.

We shall, now, consider a number of possible concepts of a *competitive equilibrium* in our infinite horizon economy.

A feasible allocation $\bar{h}$, and a non-zero price sequence $p=(p^t)$ define a *competitive equilibrium* $(\bar{h},p)$ if:

(i) For all $k=1,2,\ldots,K_t$, and for all $t=0,1,\ldots$,

$$\text{‘}p \cdot c_{kt} \leqq p \cdot \bar{c}_{kt}, \quad c_{kt} \in D_{kt}\text{’} \quad \text{implies} \quad \text{‘}\bar{c}_{kt} \succsim_{kt} c_{kt}\text{’}; \tag{3.1}$$

(ii) For all $j=1,2,\ldots,J_t$, and for all $t=0,1,\ldots$,

$$p^{\tau+1}\, \bar{b}_{jt}^{\tau+1} - p^{\tau} \bar{a}_{jt}^{\tau} \geqq p^{\tau+1} b - p^{\tau} a \quad \text{for} \quad (a,b) \in \mathscr{T}_{jt}^{\tau}, \quad \tau \geqq 0. \tag{3.2}$$

Eq. (3.2) is *equivalent*[6] to the condition that for all $j=1,2,\ldots,J_t$, and for all $t=0,1,\ldots$,

$$p \cdot \bar{y}_{jt} \geqq p \cdot y_{jt} \quad \text{for all} \quad y_{jt} \in Y_{jt}. \tag{3.3}$$

[5]To prove this, note that on the subspace $\mathscr{G}_{1H}$ of $\mathscr{G}_1$ consisting of all programs which are zero after any period H, the function P has a unique continuous representation $(P^0_{(H)},\ldots,P^H_{(H)})$ [Dunford (1958, p. 245)]. This is also true for $\mathscr{G}_{1H-1}$, yielding prices $(P^0_{(H-1)},\ldots,P^{(H-1)}_{(H-1)})$. But $\mathscr{G}_{1H-1}$ is itself a subspace of $\mathscr{G}_{1H}$, implying $P^t_{(H-1)}=P^t_{(H)}$ for $t=0,\ldots,H-1$. Induction on H completes the proof.

[6]$y_{jt}=(y^0_{jt},y^1_{jt},\ldots)=(b^0_{jt}-a^0_{jt},b^1_{jt}-a^1_{jt},\ldots)$ with zeros outside the firm jt's lifetime. Hence

$$\begin{aligned}
p \cdot y_{jt} &= \sum_{\tau=0}^{\infty} p^{\tau}(b^{\tau}_{jt}-a^{\tau}_{jt}) = \sum_{\tau=t}^{t+\omega_{jt}} p^{\tau}(b^{\tau}_{jt}-a^{\tau}_{jt}) \\
&= p^t b^t_{jt} + \sum_{\tau=t}^{t+\omega_{jt}-1} (p^{\tau+1} b^{\tau+1}_{jt} - p^{\tau} a^{\tau}_{jt}) = \sum_{\tau=t}^{t+\omega_{jt}} (p^{\tau+1} b^{\tau+1}_{jt} - p^{\tau} a^{\tau}_{jt}) \\
&\leq \sum_{\tau=t}^{t+\omega_{jt}-1} (p^{\tau+1} \bar{b}^{\tau+1}_{jt} - p^{\tau} \bar{a}^{\tau}_{jt}) + p^t \bar{b}^t_{jt} \quad \text{[by (3.2)]} \\
&= \sum_{\tau=t}^{t+\omega_{jt}} p^{\tau} \bar{y}^{\tau}_{jt} = p \cdot \bar{y}_{jt} \quad \text{proving (3.3).}
\end{aligned}$$

Conversely, if $p \cdot y_{jt} \leqq p \cdot \bar{y}_{jt}$, $y_{jt} \in Y_{jt}$ and (3.2) is violated for some τ^* and $(a,b) \in \mathscr{T}_{jt}$, then

A competitive equilibrium $(\bar{h}, p)$ is a *Malinvaud equilibrium* if, for every feasible allocation $\tilde{h}$, and time period L such that all consumer units living on or after L receive the same consumption program under either $\bar{h}$ or $\tilde{h}$, it follows that

$$\sum_{t=0}^{\infty} p^t(\tilde{y}^t - \bar{y}^t) \leqq 0. \tag{3.4}$$

Condition (3.4) for a Malinvaud equilibrium can be interpreted as requiring that among the set of all feasible aggregate net output programs which differ from $\bar{y}$ in at most a finite number of components, present value is maximized at $\bar{y}$. From material balance, feasible allocations $\tilde{h}$ and $\bar{h}$ satisfy

$$\sum_{t=0}^{L} p^t(\tilde{y}^t - \bar{y}^t) = \sum_{t=0}^{L} p^t(\tilde{c}^t - \bar{c}^t). \tag{3.5}$$

Hence, an equivalent interpretation of the Malinvaud equilibrium is that among the set of all feasible aggregate consumption programs which differ from $\bar{c}$ in a finite number of periods, present value is maximized at $\bar{c}$.

A competitive equilibrium $(\bar{h}, p)$ is a *strong competitive equilibrium* if, for every feasible allocation $\tilde{h}$, the following condition holds:

$$\liminf_{l \to \infty} \sum_{t=0}^{l} \sum_{k=1}^{K_t} p \cdot (\tilde{c}_{kt} - \bar{c}_{kt}) \leqq 0. \tag{3.6}$$

Condition (3.6) can be interpreted as requiring that the incremental 'present value' of consumption associated with a shift from $\bar{h}$ to another feasible allocation $\tilde{h}$ (which may differ from $\bar{h}$ in infinitely many periods) not be positive.

Note that it is not necessary in this definition that 'present value' be a well-defined number. A valuation equilibrium imposes the same economic conditions, and requires in (3.7) that 'present value' be well-defined. Thus, a strong competitive equilibrium $(\bar{h}, p)$ defines a *valuation equilibrium* if the

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consider $y'_{jt} \in Y_{jt}$ given by

$$(\bar{b}^0_{jt} - \bar{a}^0_{jt}, \bar{b}^1_{jt} - \bar{a}^1_{jt}, \ldots, \bar{b}^{\tau^*-1}_{jt} - a, b - \bar{a}^{\tau^*}_{jt}, \bar{b}^{\tau^*+1}_{jt} - \bar{a}^{\tau^*+1}_{jt}, \ldots).$$

Then

$$\begin{aligned} p \cdot y'_{jt} - p \cdot \bar{y}_{jt} &= [p^{\tau^*-1}(-a) + p^{\tau^*} \cdot b] - [p^{\tau^*-1}(-\bar{a}^{\tau^*-1}_{jt}) + p^{\tau^*}\bar{b}^{\tau^*}_{jt}] \\ &= (p^{\tau^*}b - p^{\tau^*-1}a) - [p^{\tau^*}\bar{b}^{\tau^*}_{jt} - p^{\tau^*-1}\bar{a}^{\tau^*-1}_{jt}] > 0, \end{aligned}$$

a contradiction.

non-zero price-sequence $p=(p^t)$ is a continuous linear function on a topological subspace of $\mathscr{G}$ which contains the set of all possible non-negative net supply programs $c=y+e$, and the inequality

$$p\cdot y \leqq p\cdot \bar{y} \tag{3.7}$$

holds for this set of net supply programs.

A competitive equilibrium $(\bar{h},p)$ has the *insignificant future property* if

$$\lim_{\tau\to\infty} p^\tau a^\tau = 0 \quad \text{and} \quad \lim_{\tau\to\infty} p^\tau c^\tau = 0. \tag{3.8}$$

Thus, (3.7) imposes the requirement that the values of aggregate consumption and inputs in period τ go to zero as τ goes to infinity.

We shall now discuss how the various concepts of competitive equilibria are inter-related. The interested reader might compare (3.2), (3.4), (3.6), and (3.7) to the definitions of efficiency prices of different 'types' discussed by Peleg and Yaari (1970). Clearly, one set of implications is immediate from the sequence of definitions: valuation equilibrium implies strong competitive equilibrium implies competitive equilibrium, and Malinvaud equilibrium implies competitive equilibrium. Some less apparent implications will now be derived, and in such derivations (which are only sketched) some accounting identities will be useful.

Let A_T be the value of aggregate consumption from period 1 through period T at prices $p=(p^t)$. Recalling that $c^\tau = e^\tau + y^\tau$,

$$\begin{aligned} A_T &\equiv \sum_{\tau=0}^{T} p^\tau c^\tau \\ &\equiv \sum_{\tau=0}^{T} p^\tau (e^\tau + y^\tau) \\ &= \sum_{\tau=0}^{T} p^\tau e^\tau + p^0 b^0 - p^T a^T + \sum_{\tau=0}^{T-1} [p^{\tau+1} b^{\tau+1} - p^\tau a^\tau]. \end{aligned} \tag{3.9}$$

Using (3.9), we can show that *strong competitive equilibrium implies Malinvaud equilibrium*. Suppose there is a feasible allocation $\tilde{h}$ and a period L such that every consumer unit living on or after L receives the same consumption program as in $\bar{h}$. Consider the identity

$$\sum_{t=0}^{L} \sum_{k=1}^{K_t} p\cdot c_{kt} - \sum_{t=0}^{L} p^t \cdot c^t = \sum_{t=0}^{L} \sum_{k=1}^{K_t} \sum_{\tau=L+1}^{\infty} p^\tau \cdot c_{kt}^\tau. \tag{3.10}$$

For $\tilde{h}$ and $\bar{h}$, the right-hand side of (3.10) is the same, implying

$$\sum_{t=0}^{L} p^t(\tilde{c}^t-\bar{c}^t)=\sum_{t=0}^{L}\sum_{k=1}^{K_t} p\cdot(\tilde{c}_{kt}-\bar{c}_{kt})=\liminf_{l\to\infty}\sum_{t=0}^{l}\sum_{k=1}^{K_t} p(\tilde{c}_{kt}-\bar{c}_{kt}). \quad (3.11)$$

Then (3.5) and (3.9) imply that (3.4) holds.

Next, we shall show that *a valuation equilibrium* $(\bar{h},p)$ *has the insignificant future property*. In order to verify (3.8) note first that

$$p\bar{y}=\sum_{\tau=0}^{\infty} p^\tau(c^\tau+e^\tau)\geqq\sum_{\tau=0}^{\infty} p^\tau c^\tau. \quad (3.12)$$

By definition of a valuation equilibrium, $p\bar{y}$ is finite. From (3.12) it follows directly that

$$\lim_{\tau\to\infty} p^\tau c^\tau=0. \quad (3.13)$$

If, now, $\lim_{\tau\to\infty}\sup p^\tau a^\tau\neq 0$, there must be some $\delta>0$ and a subsequence (retain the same notation for the subsequence) such that $p^\tau a^\tau\geqq\delta$ for all τ. Choose some T^* such that $\sum_{\tau=T^*}^{\infty} p^\tau c^\tau<\delta/2$ and define $\tilde{c}=(\tilde{c}^\tau)$ as $\tilde{c}^\tau=c^\tau$ for all $\tau\leqq T^*$; $\tilde{c}^{T^*+1}=b^{T^*+1}+e^{T^*+1}$; and $\tilde{c}^\tau=0$ for all $\tau>T^*+1$. Then

$$p\tilde{c}\geqq\sum_{\tau=0}^{T^*} p^\tau c^\tau+p^{T^*+1}a^{T^*+1}\geqq\sum_{\tau=0}^{\infty} p^\tau c^\tau+\delta/2. \quad (3.14)$$

But (3.14) contradicts (3.7), implying

$$\lim_{\tau\to\infty} p^\tau a^\tau=0. \quad (3.15)$$

The next interesting implication is that *a competitive equilibrium with the insignificant future property* (3.8) *is a strong competitive equilibrium* (3.6).

To see this, use (3.10) to get

$$\sum_{t=0}^{l}\sum_{k=1}^{K_t} p(\tilde{c}_{kt}-\bar{c}_{kt})=\sum_{t=0}^{l+\alpha} p^t(\tilde{c}^t-\bar{c}^t)-\sum_{t=l+1}^{l+\alpha} p^t(\tilde{c}^t-\bar{c}^t)$$
$$+\sum_{t=0}^{l}\sum_{k=1}^{K_t}\sum_{\tau=l+1}^{\infty} p(\tilde{c}_{kt}^\tau-\bar{c}_{kt}^\tau). \quad (3.16)$$

Now use (3.9) to get the following bound on the first term on the right side of (3.16):

$$\sum_{t=0}^{l+\alpha} p^t(\tilde{c}^t - \bar{c}^t) \leqq p^{l+\alpha}\bar{a}^{l+\alpha}. \tag{3.17}$$

The inequality (3.18) is now derived by using (3.17) and dropping a term from the right side of (3.16) that is negative,

$$\begin{aligned}\sum_{t=0}^{l}\sum_{k=1}^{K_t} p(\tilde{c}_{kt} - \bar{c}_{kt}) \leqq p^{l+\alpha}\bar{a}^{l+\alpha} + \sum_{t=\alpha+1}^{l+\alpha} p^t\bar{c}^t \\ - \sum_{t=l+1}^{l+\alpha} p^t\tilde{c}^t + \sum_{t=0}^{l}\sum_{k=1}^{K_t}\sum_{\tau=l+1}^{\infty} p^\tau \tilde{c}^\tau_{kt} \\ \leqq p^{l+\alpha}\bar{a}^{l+\alpha} + \sum_{t=l+1}^{l+\alpha} p^t\bar{c}^t.\end{aligned} \tag{3.18}$$

Now use the insignificant future property to get (3.6) from (3.18).

Two examples will now be used to illustrate the distinction between the various notions of competitive equilibria. In both these examples we consider an economy with a single commodity and no net production. There is an 'old' consumer in period 0, disappearing at the end of that period. A single consumer unit is born in period $t \geqq 0$, disappearing at the end of period $t+1$. The utility function of all the consumers is the same, namely the sum of consumptions in two periods (the utility of the old consumer in period 0 is just his consumption in period 0).

Example 3.1. Let $e^t = 2$ for all $t \geqq 0$. We examine the following allocations:

Allocation (A): In each period $t \geqq 0$, the available supply (two units of the commodity) is divided equally between the two consumers.

Allocation (B): The available supply is distributed entirely to the 'old' consumer in each period $t \geqq 0$.

Allocation (C): Assuming that the good can be stored free without any depreciation, the 'old' consumer in period 0 gets 1 unit; all other consumers get 2 units in their 'old' age.

A comparison of the allocation (A) with (B) shows that (A) is *not* Pareto optimal. At price $p = (1, 1, 1, \ldots)$, the allocation (A) is a Malinvaud equilibrium [since the absence of production allows (3.4) to be satisfied automatically]. Thus, *a Malinvaud equilibrium need not be Pareto optimal.* The allocation (A), is *not*, however, a strong competitive equilibrium, as can be concluded from a comparison with (B). At $p = (1, 1, \ldots)$ the allocation (C) is a competitive equilibrium, but *not* a Malinvaud equilibrium, as can be concluded from a comparison of (A) and (C). Finally, the allocation (B) is an example of a strong competitive equilibrium which is *not* a valuation equilibrium.[7]

Example 3.2. Let $e^0=1$, $e^t=1/t$ for $t \geqq 1$. The available supply is allotted to the 'old' consumer in each period. At the price system $p=(1,1,\dots)$, this allocation is a competitive equilibrium. It is also true that $p^t c^t = 1/t$ for $t \geqq 1$, so that the insignificant future property holds. However, $\sum_{t=0}^{T} p^t c^t = 1+\sum_{t=1}^{T} 1/t$ and this goes to infinity with T. Thus, we have an example of a competitive equilibrium with the insignificant future property that is *not* a valuation equilibrium.

For a number of simpler models, including the one we considered in Majumdar–Mitra–McFadden (1976), it has been shown that a competitive equilibrium that is also efficient is necessarily long-run Pareto optimal, or that short-run Pareto optimality together with (long-run) efficiency implies long-run Pareto optimality [see Cass and Yaari (1967, p. 249) and Bose (1974) in this connection]. Indeed for 'interior' programs in models in which the technology has appropriate differentiability properties, some remarkably strong implications can be shown to hold, as our earlier exercise seems to indicate. For treating other aspects of intertemporal welfare economics, it might be useful to start with such a simple model. The next example shows that in general efficiency and short-run Pareto optimality need not imply Pareto optimality.

Example 3.3. Again we consider an economy in which a single consumer is born in each period, living for two periods. Similarly, in each time period a 'generation of firms' (consisting of a single firm) is born, living for two periods. The technology set $\mathscr{T}$ for *all* firms is the same and is given by

$$\mathscr{T}=\{(a,b)\geqq 0: Az \leqq a, Bz \geqq b \quad \text{for some} \quad z \geqq 0\}, \tag{3.19}$$

where

$$A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad B=\begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ 0 & 1\end{pmatrix}.$$

Clearly the second activity has the second commodity as a non-depreciating, durable capital good which generates a net production of the first commodity.

The preferences for consumers are given by utility functions (for $t \geqq 0$)

[7]Let $\bar{h}$ denote allocation B and let $\tilde{h}$ denote an alternative feasible allocation in which a proportion θ_t of the aggregate endowment in period t is given to the tth consumer unit. Then $\sum_{t=0}^{H} p\cdot(\tilde{c}_{1t}-\bar{c}_{1t})=-\theta_H$ and (3.6) is satisfied. Since $\sum_{t=0}^{H} p^t \bar{c}^t = H$, $p\cdot\bar{c}$ does not exist, and p is not a valuation function.

$$U_{1t}(c_{1t}^t, c_{1t}^{t+1}) = {}^1c^t/2^t + {}^2c^t + {}^1c^{t+1}/2^{t+1} + {}^2c^{t+1}, \tag{3.20}$$

where ${}^ic$ denotes consumption of commodity i $(=1,2)$.

The resource supply is given by $e^0 = (0,1)$, $e^t = (0,0)$ for $t \geqq 1$. Consider the allocation given by $\bar{a}^t = (0,1)$, $t \geqq 0$; $\bar{c}^0 = (0,0)$, $\bar{c}^t = (\frac{1}{4}, 0)$ for $t \geqq 0$; $\bar{b}^{t+1} = (\frac{1}{4}, 1)$, $t \geqq 0$; $\bar{z}^t = \bar{a}^t$, $t \geqq 0$; $\bar{c}_{1t}^t = (0,0)$, $t \geqq 0$; $\bar{c}_{1t}^{t+1} = (\frac{1}{4}, 0)$, $t \geqq 0$.

One can verify that the allocation is *feasible* and *efficient*. Furthermore, it is *short-run Pareto optimal.*

However, it is not long-run Pareto optiaml. Consider the allocation given by

$$\tilde{a}^0 = (0,1), \quad \tilde{a}^t = \left(0, 1 - \sum_{\tau=0}^{t-1} \tfrac{1}{2}/2^\tau\right), \qquad t \geqq 1,$$

$$\tilde{b}^1 = [\tfrac{1}{4}, 1], \quad \tilde{b}^t = \left\{\frac{1}{4}\left(1 - \sum_{\tau=0}^{t-2} \tfrac{1}{2}/2^\tau\right), \left(1 - \sum_{\tau=0}^{t-2} \tfrac{1}{2}/2^\tau\right)\right\}, \qquad t \geqq 2$$

$$\tilde{z}^t = \tilde{a}, \qquad t \geqq 0,$$

$$\tilde{c}^0 = (0,0), \quad \tilde{c}^1 = [\tfrac{1}{4}, \tfrac{1}{2}], \quad \tilde{c}^t = \left\{\frac{1}{4}\left(1 - \sum_{\tau=0}^{t-2} \tfrac{1}{2}/2^\tau, \tfrac{1}{2}/2^{\tau-1}\right)\right\}, \qquad t \geqq 2,$$

$$\tilde{c}_{1t}^t = (0,0), \quad \tilde{c}_{1t}^{t+1} = \tilde{c}^{t+1}, \qquad t \geqq 0.$$

Then, it can be checked that this allocation is feasible, and has

$$U_{1t}(\tilde{c}_{1t}^t, \tilde{c}_{1t}^{t+1}) > U_{1t}(\bar{c}_{1t}^t, \bar{c}_{1t}^{t+1}) \quad \text{for all} \quad t \geqq 0.$$

3.2. *Non-decomposability and reachability*

A feasible allocation $\bar{h} = (\bar{s}, \bar{d}, e)$ is *non-decomposable* if, for each partition of the consumer units into two non-empty subsets I_1 and I_2, there exists a feasible allocation $\tilde{h} = (\bar{s}, \tilde{d}, e)$, with the *same* net supply program, which is Pareto preferable to the allocation $\bar{h}$ for the consumer units in I_1.

This condition implies, in particular, that no consumer is satiated and that between any two groups I_1 and I_2, there exists at least one commodity desired by some unit in I_1 which is held by some unit in I_2. Non-decomposability also excludes the 'extreme' distributions in which some unit is at subsistence or some other unit has the 'largest' feasible consumption program. A sufficient condition for non-decomposability is that there exists a commodity which is essential and always desirable to every unit, and that at least one unit in each cohort survives for more than one period.

Consider a possible aggregate net output program $\bar{y}$ and a resource supply program e, such that $\bar{c} = \bar{y} + e$ is non-negative. Then, $(\bar{y}, e)$ is *reachable* if for any positive scalar μ there exists a possible net output program $\tilde{y}$ such that $\tilde{c} = \tilde{y}+e$ is non-negative, and there is $L(\mu)$, such that $\tilde{c}^t = \mu \bar{c}^t$ for $t > L(\mu)$.

Reachability implies, roughly, that starting from an arbitrarily small proportion of the original resource program, one can, by pure accumulation, reach, after a sufficiently long term, the path of an originally possible net supply program. In an economy in which all commodities can be produced, resources do not limit the long-run growth rate, and consumption requires the diversion of productive commodities, feasible allocations will generally be reachable.[8]

4. Equilibrium, Pareto optimality, and efficiency

4.1. Equilibrium is optimal

We shall now prove that a strong competitive equilibrium is Pareto optimal.

Theorem 4.1. Under (T.1), (P.1), and (P.7), a strong competitive equilibrium $(\bar{h}, p)$ *is Pareto optimal.*

Proof. Suppose not. Then, there is a feasible allocation $\tilde{h}$, which is Pareto preferable to $\bar{h}$. Then $\tilde{c}_{kt} \succsim_{kt} \bar{c}_{kt}$ for each consumer unit. We claim that this implies that $p \cdot \tilde{c}_{kt} \geqq p \cdot \bar{c}_{kt}$ for all k, t. If not, then $p \cdot \tilde{c}_{kt} < p \cdot \bar{c}_{kt}$ for some k, t. By (P.7), there exists a c'_{kt} in a sequence converging to $\tilde{c}_{kt}$, such that $p \cdot c'_{kt} \leqq p \cdot \bar{c}_{kt}$ and $c'_{kt} \succ_{kt} \tilde{c}_{kt}$. Since $\tilde{c}_{kt} \succsim_{kt} \bar{c}_{kt}$, so $c'_{kt} \succ_{kt} \bar{c}_{kt}$, contradicting (3.1), and establishing our claim.

We also know that $\tilde{c}_{kt} \succ_{kt} \bar{c}_{kt}$ for some k, t. We claim this implies that $p \cdot \tilde{c}_{kt} > p \cdot \bar{c}_{kt}$ for this k, t. If not, then $p \cdot \tilde{c}_{kt} \leqq p \cdot \bar{c}_{kt}$, implying $\bar{c}_{kt} \succsim_{kt} \tilde{c}_{kt}$ by (3.1), a contradiction, which again establishes our claim.

Hence, from the above two results, we know that there is $\theta > 0$ such that

$$\sum_{t=0}^{l} \sum_{k=1}^{K_t} p \cdot (\tilde{c}_{kt} - \bar{c}_{kt}) \geqq \theta > 0 \quad \text{for all } l \text{ sufficiently large,}$$

contradicting (3.5) for a strong competitive equilibrium. Hence $(\bar{h}, p)$ is Pareto optimal.

4.2. Reachable optimal allocations are equilibria

A partial converse of Theorem 4.1 can be proved when a Pareto-optimal allocation is reachable. The resulting price system will define a valuation equilibrium, implying the existence of a finite present-value associated with each aggregate consumption program.

[8]Several examples of economies with this property are given in McFadden (1967).

Theorem 4.2. Under (T.1)–(T.5), and (P.1)–(P.6), if a feasible allocation $\bar{h}$ is efficient,[9] *reachable, non-decomposable, and Pareto optimal, then there exists a non-zero price sequence p such that $(\bar{h}, p)$ is a valuation equilibrium.*

Proof. Let G in $\mathscr{G}$ denote the free disposal hull of the set of possible aggregate net supply programs, i.e., G is the set of all $c \leqq y+e$, where y is a possible aggregate net output program. Let G_0 denote the subset of non-negative net supply programs in G and let $\mathscr{X}$ denote the real linear vector space spanned by G_0. Define $F = G \cap \mathscr{X}$. All programs in G_0 are contained in F, and all efficient programs in F are possible aggregate net supply programs. We call F the set of *admissible* aggregate net supply programs.

The set G is convex, and contains its free-disposal hull, and G_0 is pointwise closed and bounded by Assumption T (and hence, Lemma 1). Hence G and $\mathscr{X}$ satisfy Assumption (A) of the appendix. Hence, by Lemmas A.1–A.6 of the appendix, there exists a norm δ on $\mathscr{X}$, such that $\mathscr{X}$ is a Banach space, and $G_0 = \{c \in \mathscr{X} \mid c \geqq 0 \text{ and } \delta(c) \leqq 1\}$.

Define, for each consumer unit k in cohort t, the finite-dimensional subspace $\mathscr{X}_{kt} = \mathscr{C}_{kt} \cap \mathscr{X}$. Since $\bar{h} = (\bar{s}, \bar{d}, e)$ is Pareto optimal, and non-decomposable, there must exist for any consumer unit kt an alternative feasible allocation which is Pareto preferred by the consumer units other than kt, and hence is strictly inferior for kt. Hence by (P.3), the zero bundle is not in $U_k(\bar{c}_{kt})$. This, along with (P.2)–(P.6), imply that $\bar{a}$ is 'non-extreme', 'non-decomposable', and that preferences satisfy, at $\bar{a}$, Assumption (B) of the appendix.

Since $\bar{a}$ is reachable, given any $\mu > 0$, there is a possible net output plan $\tilde{s}$ such that $\tilde{c} = \tilde{y} + e$ is non-negative, and $\tilde{c}$ and $\mu\bar{c}$ are identical after a finite number of periods, $L(\mu)$. Hence, one can take $\tilde{c}_{kt} = \mu\bar{c}_{kt}$ for $t \geqq L(\mu)$, and so,

$$1 \geqq \delta(\tilde{c}) \geqq \delta\left(\sum_{t=L(\mu)}^{\infty} \sum_{k=1}^{K_t} \mu\bar{c}_{kt} \right) = \mu\delta\left(\sum_{t=L(\mu)}^{\infty} \sum_{k=1}^{K_t} \bar{c}_{kt} \right).$$

Thus, letting $\mu \to \infty$,

$$\lim_{t_0 \to \infty} \delta\left(\sum_{t=t_0}^{\infty} \sum_{k=1}^{K_t} \bar{c}_{kt} \right) = 0.$$

The hypotheses of Theorem A.8 of the appendix are then met; so, there exists a non-negative continuous linear functional $\boldsymbol{P}$ on $\mathscr{X}$, such that:

(i) $\boldsymbol{P}(c) \leqq \boldsymbol{P}(\bar{c}) = 1$ for all $c \in F$, and
(ii) for each kt, $c \in D_{kt}$, and $\boldsymbol{P}(c) \leqq \boldsymbol{P}(\bar{c}_{kt})$ implies $\bar{c}_{kt} \succsim_{kt} c$.

[9]If (P.7) holds, then Pareto Optimality implies efficiency. Hence, if (P.7) is assumed, we do not require the assumption that the allocation $\bar{h}$ is efficient.

P has a unique representation as a price sequence $p=(p^0,p^1,\dots)$ on the subspace $\mathscr{X}_f$ of $\mathscr{X}$ consisting of programs with a finite number of non-zero components. Since D_{kt} is finite-dimensional, (P.6) implies that p is a non-zero price sequence, satisfying (3.1).

For any y_{jt} in Y_{jt},

$$P(\bar{y}+(y_{jt}-\bar{y}_{jt}))\leqq P(\bar{y}),$$

or

$$P(y_{jt}-\bar{y}_{jt})=p\cdot(y_{jt}-\bar{y}_{jt})\leqq 0,$$

so (3.3), and, hence, (3.2) are satisfied.

Since $\lim_{t_0\to\infty}\delta(\sum_{t=t_0}^{\infty}\sum_{k=1}^{K_t}\bar{c}_{kt})=0$, continuity of the linear functional P implies that

$$P(\bar{c})=\lim_{t_0\to\infty}\sum_{t=0}^{t_0}\sum_{k=1}^{K_t}p\cdot\bar{c}_{kt}.^{10}$$

For $c\in G_0$, non-negativity implies

$$p\cdot c=\lim_{t_0\to\infty}\sum_{t=0}^{t_0}p^t c^t\leqq p(\bar{c}),^{11}$$

and p is bounded on G_0. Since G_0 spans $\mathscr{X}$, p is a continuous linear

[10]
$$P(\bar{c})=P\left(\sum_{t=0}^{\infty}\sum_{k=1}^{K_t}\bar{c}_{kt}\right)=P\left(\sum_{t=0}^{t_0}\sum_{k=1}^{K_t}\bar{c}_{kt}+\sum_{t=t_0+1}^{\infty}\sum_{k=1}^{K_t}\bar{c}_{kt}\right)$$
$$=P\left(\sum_{t=0}^{t_0}\sum_{k=1}^{K_t}\bar{c}_{kt}\right)+P\left(\sum_{t=t_0+1}^{\infty}\sum_{k=1}^{K_t}\bar{c}_{kt}\right)=\sum_{t=0}^{t_0}\sum_{k=0}^{K_t}p\cdot\bar{c}_{kt}+P\left(\sum_{t=t_0+1}^{\infty}\sum_{k=1}^{K_t}\bar{c}_{kt}\right).$$

Taking limits on both sides as $t_0\to\infty$,

$$P(\bar{c})=\lim_{t_0\to\infty}\sum_{t=0}^{t_0}\sum_{k=1}^{K_t}p\cdot\bar{c}_{kt}+\lim_{t_0\to\infty}P\left(\sum_{t=t_0+1}^{\infty}\sum_{k=1}^{K_t}\bar{c}_{kt}\right)$$
$$=\lim_{t_0\to\infty}\sum_{t=0}^{t_0}\sum_{k=1}^{K_t}p\cdot\bar{c}_{kt}\left[\because\lim_{t_0\to\infty}\delta\left(\sum_{t=t_0+1}^{\infty}\sum_{k=1}^{K_t}\bar{c}_{kt}\right)=0\right].$$

[11]Since $c\in G_0$, $P(\bar{c})\geqq P(c)$, write $c=(c^0,c^1,\dots)=(c^0,c^1,\dots,c^{t_0},0,0,\dots)+(0,0,\dots,c^{t_0+1},c^{t_0+2},\dots)$ $=c'+c''$. Then

$$P(c)=P(c')+P(c'')\geqq P(c')=\sum_{t=0}^{t_0}p^tc^t,\quad\therefore\sum_{t=0}^{t_0}p^tc^t\leqq P(\bar{c})\quad\text{for each } t_0.$$

Non-negativity implies that $\lim_{t_0\to\infty}\sum_{t=0}^{t_0}p^tc^t\leqq P(\bar{c})$ exists. But $\lim_{t_0\to\infty}\sum_{t=0}^{t_0}p^tc^t=p\cdot c$. Hence $p\cdot c=\lim_{t_0\to\infty}\sum_{t=0}^{t_0}p^tc^t\leqq P(\bar{c})$.

functional on $\mathscr{X}$,[12] and $p\cdot c \leqq p\cdot \bar{c}=1$ for $c\in G_0$. Hence, p is a valuation function on $\mathscr{X}$.

Finally, note that for any feasible allocation, $\tilde{h}$, one has $\sum_{t=0}^{t_0}\sum_{k=1}^{K_t}\tilde{c}_{kt}$ in G_0, implying $\lim_{t_0\to\infty}\sum_{t=0}^{t_0}\sum_{k=1}^{K_t}p\cdot\tilde{c}_{kt}=p\cdot\tilde{c}\leqq 1$. Hence, $(\bar{h},p)$ satisfies (3.5) for a strong competitive equilibrium, and is, therefore, a valuation equilibrium.

Appendix

The set of admissible solutions in many problems in economics and control theory can be characterized as a subset of the non-negative orthant of a real linear vector space.[13] When the admissible set is convex and 'pointwise' bounded, and all solutions smaller than a given admissible solution are also admissible, we establish the existence of a norm on the space spanned by the admissible set such that the admissible set coincides with the non-negative programs in the unit sphere. This result is closely related to the theorem of Kolmogoroff (1934) that a topological linear space is homeomorphic to a normed linear space if and only if there exists a bounded convex neighborhood of the origin. The norm topology introduced here is homeomorphic to the core topology of a linear space introduced by Klee (1951).

For a countable non-empty set T, consider the linear space $\mathscr{Y}$ of all real-valued functions on T, and let $\Omega=\{y\in\mathscr{Y}\,|\,y(t)\geqq 0 \text{ for all } t\in T\}$ denote the non-negative orthant of $\mathscr{Y}$. A partial ordering $\leqq$ on $\mathscr{Y}$ is defined by $y\leqq y'$ if and only if $y'-y\in\Omega$ for $y,y'\in\mathscr{Y}$.

A sequence $\{y_n\}$ in $\mathscr{Y}$ converges *pointwise* to y_0 in $\mathscr{Y}$ [notation: $y_n\to_t y_0$] if, given $\varepsilon>0$ and $t\in T$, there exists $n(\varepsilon,t)$ such that $|y_n(t)-y_0(t)|<\varepsilon$ for $n>n(\varepsilon,t)$.[14] A set y in $\mathscr{Y}$ is *pointwise-closed* (p-closed) if $\{y_n\}\subseteq Y$, $y_n\to_t y_0$ imply $y_0\in Y$. The set Y is *pointwise bounded* (p-bounded) if $\{y(t)|y\in Y\}$ is a

[12] p is bounded on G_0, hence continuous on G_0. Since G_0 spans $\mathscr{X}$, given $c'\in\mathscr{X}$, write $c'=\sum_{i=0}^{n}\alpha_i c_i$ where $c_i\in G_0$. Then

$$P(c')=\sum_{i=0}^{n}\alpha_i P(c_i)=\sum_{i=0}^{n}\alpha_i(p\cdot c_i)=\sum_{i=0}^{n}p\cdot(\alpha_i c_i)=p\cdot c'.$$

Thus p is defined on $\mathscr{X}$, and continuous on G_0. Hence p is bounded on $\mathscr{X}$. Hence p is continuous on $\mathscr{X}$.

[13] A typical example for economics is the problem of optimizing a social objective function over the set of all programs of non-negative consumption of commodities which are feasible in a growing economy over an infinite horizon.

[14] The space $\mathscr{Y}$ is the Cartesian product over T of the space of real numbers, and pointwise convergence is simply convergence in the product topology of $\mathscr{Y}$ when the real line is given its natural topology. Pointwise closure (resp. boundedness) corresponds to closure (resp. boundedness) in the product topology. If sequences are replaced by generalized sequences, all the proofs of this paper hold when T is an arbitrary non-empty set, not necessarily countable.

bounded set for each $t \in \boldsymbol{T}$. The set $\boldsymbol{Y}$ is *monotone below* relative to a set $\boldsymbol{Y}'$ containing $\boldsymbol{Y}$ if $y \in \boldsymbol{Y}$, $y' \in \boldsymbol{Y}'$, $y' \leqq y$ imply $y' \in \boldsymbol{Y}$.

The (*p-closed*) *convex hull* of a set $\boldsymbol{Y}$ is the intersection of all (p-closed) convex sets in $\mathscr{Y}$ containing $\boldsymbol{Y}$.

Suppose a non-empty set $\boldsymbol{Y}_0$ is given which is contained in the non-negative orthant $\boldsymbol{\Omega}$. Let $\boldsymbol{Y}_1$ denote the convex hull of $\boldsymbol{Y}_0$, and let $\mathscr{X}$ denote the real linear space spanned by $\boldsymbol{Y}_1$. Let $\boldsymbol{Y}_2$ denote the intersection of all sets, monotone below relative to $\mathscr{X}$, containing $\boldsymbol{Y}_1$.

Lemma 1. The origin of $\mathscr{X}$ is an internal point of $\boldsymbol{Y}_2$ in $\mathscr{X}$.

Proof. Since $\boldsymbol{Y}_2$ is convex, any point $\hat{y} \in \mathscr{X}$, $\hat{y} \neq \boldsymbol{0}$, can be written as

$$\hat{y} \leqq \alpha_1 y_1 + \alpha_2 y_2, \qquad y_j \in \boldsymbol{Y}_1, \qquad \alpha_j \geqq 0.$$

Then, for $0 \leqq \theta \leqq 1/(\alpha_1 + \alpha_2)$,

$$\theta \hat{y} \leqq \theta \alpha_1 y_1 + \theta \alpha_2 y_2 \leqq \frac{\alpha_1}{\alpha_1 + \alpha_2} y_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} y_2 \in \boldsymbol{Y}_2,$$

and $\theta \hat{y} \in \boldsymbol{Y}_2$ by monotonicity. Q.E.D.

Define the support function of $\boldsymbol{Y}_2, \phi(y) = \inf\{\mu \mid \mu$ a positive scalar, $(1/\mu) y \in \boldsymbol{Y}_2\}$. The function ϕ exists and is non-negative, positive linear homogeneous, and convex [cf. Dunford and Schwartz (1958)]. Define $\delta(y) = \phi(y) + \phi(-y)$. Then, $\delta(y)$ satisfies

(i) $\quad \delta(y) \geqq 0,$

(ii) $\quad \delta(y + y') \leqq \delta(y) + \delta(y'),$

(iii) $\quad \delta(\alpha y) = |\alpha| \delta(y),$

for y, $y' \in \mathscr{X}$ and any scalar α, and is a *pseudo-norm* on $\mathscr{X}$ [cf. Kelley (1955, p. 18)].[15] Note that $y \in \boldsymbol{Y}_0$ implies $\delta(y) \leqq 1$, and that $\delta(y) < 1$ implies $y \in \boldsymbol{Y}_2$.

Lemma 2. If $\boldsymbol{Y}_0$ is p-bounded, then $\delta(y)$ is a norm on $\mathscr{X}$; i.e., $y \neq \boldsymbol{0}$ implies $\delta(y) > 0$.

Proof. Clearly, $\boldsymbol{Y}_0$ p-bounded implies $\boldsymbol{Y}_1$ p-bounded. If $\hat{y} \neq \boldsymbol{0}$, then $\hat{y}(t_0) \neq 0$ for some $t_0 \in \boldsymbol{T}$. We can assume (by changing the sign of $\hat{y}$ if necessary) that $\hat{y}(t_0) > 0$. By p-boundedness, there then exists a positive scalar μ such that $\hat{y}(t_0) > \mu \sup\{y(t_0) \mid y \in \boldsymbol{Y}_1\}$, implying $(1/\mu)\hat{y} \notin \boldsymbol{Y}_1$, and $\delta(\hat{y}) \geqq \phi(\hat{y}) \geqq \mu > 0$. Q.E.D.

[15] Condition (ii) follows from the convexity of ϕ; condition (iii) follows from the homogeneity of ϕ and the sign-symmetry of δ.

Corollary. *The norm δ satisfies the bound $|\hat{y}(t)| \leqq \delta(\hat{y})\sigma(t)$ for any $\hat{y} \in \mathscr{X}$, $t \in \boldsymbol{T}$, where $\sigma(t) = \sup\{y(t) | y \in Y_1\}$.*

A sequence $\{y_n\}$ in $\mathscr{X}$ is a δ-Cauchy sequence if $\lim_{m,n\to\infty} \delta(y_n - y_m) = 0$. By the bound in the Corollary to Lemma 2, if $\{y_n\}$ is a δ-Cauchy sequence in $\mathscr{X}$, then the sequence $\{y_n(t)\}$ is a Cauchy sequence in the real line for each $t \in \boldsymbol{T}$, and there exists a pointwise limit $y_0 \in \mathscr{Y}$ of $\{y_n\}$; i.e., $y_n \to_t y_0$.

Let $\tilde{\mathscr{X}}$ denote the space of all pointwise limits of δ-Cauchy sequences in $\mathscr{X}$, and define on $\tilde{\mathscr{X}}$ the function

$$\begin{aligned}\tilde{\delta}(y) &= \delta(y) \quad \text{if} \quad y \in \mathscr{X},\\ &= \inf\left\{\mu \,\middle|\, \mu = \lim_{n\to\infty} \delta(y_n), \quad \{y_n\} \text{ a } \delta\text{-Cauchy sequence in } \mathscr{X} \text{ with } \quad y^n \to_t y\right\}\\ &\quad \text{if} \quad y \in \tilde{\mathscr{X}}, \quad y \notin \mathscr{X}.\end{aligned}$$

Lemma 3. *If Y_0 is p-bounded, then $\tilde{\delta}$ is a norm on $\tilde{\mathscr{X}}$, and $\tilde{\mathscr{X}}$ is a complete vector space (hence, a Banach space).*

Proof. A direct verification can be made following Köthe (1969, p. 126, §14.3.1).

Corollary. *$\mathscr{X}$ is a dense subspace of $\tilde{\mathscr{X}}$ in the $\tilde{\delta}$ norm.*

The results in the lemmas and corollaries above remain valid if Y_1 is redefined as the p-closed convex hull of Y_0.

Lemma 4. *If Y_1 is p-closed, p-bounded, and monotone below relative to $\boldsymbol{\Omega}$, then $\mathscr{X}$ with the δ-norm is complete, i.e., $\mathscr{X} = \tilde{\mathscr{X}}$.*

Proof. Suppose $\hat{y} \in \tilde{\mathscr{X}}$. Then, there exists a δ-Cauchy sequence $\{y_n\}$ in $\mathscr{X}$ such that $y_n \to_t \hat{y}$. Since $\delta(y_n)$ is bounded, we can assume (by re-scaling) that $\delta(y_n) < 1$. Then, $\phi(y_n) < 1$ and $\phi(-y_n) < 1$, implying $y_n \in Y_2$ and $-y_n \in Y_2$. Then, by definition, there exist $y_n', y_n'' \in Y_1$ such that $y_n \leqq y_n'$ and $-y_n \leqq y_n''$. Since Y_1 is p-bounded, there exists a subsequence (retain notation) such that $y_n' \to_t y_0'$ and $y_n'' \to_t y_0''$ with $y_0', y_0'' \in Y_1$ by p-closure. Then, $\hat{y} \leqq y_0'$ and $-\hat{y} \leqq y_0''$ by the p-closure of the partial ordering $\leqq$. Define $\hat{y}'$ and $\hat{y}''$ by $\hat{y}'(t) = \max\{0, \hat{y}(t)\}$ and $\hat{y}''(t) = \min\{0, \hat{y}(t)\}$. Then, $0 \leqq \hat{y}' \leqq y_0'$ and $0 \leqq -\hat{y}'' \leqq y_0''$, implying by the monotonicity in $\boldsymbol{\Omega}$ of Y_1 that $\hat{y}', -\hat{y}'' \in Y_1$. Hence, $\hat{y} = \hat{y}' + \hat{y}'' \in \mathscr{X}$. Q.E.D.

Corollary. $Y_1 = \{y \in \mathscr{X} | \boldsymbol{0} \leqq y \quad \text{and} \quad \delta(y) \leqq 1\}$.

Let $\mathscr{X}^*$ denote the space of all linear functional on $\mathscr{X}$ which are continuous in the δ-norm. Similarly, let $\tilde{\mathscr{X}}^*$ be the space of all $\tilde{\delta}$-continuous functionals on $\tilde{\mathscr{X}}$.

Lemma 5. If Y_0 is p-bounded, then for each $x \in \mathscr{X}^*$, there exists $\tilde{x} \in \tilde{\mathscr{X}}^*$, such that $x(y) = \tilde{x}(y)$ for all $y \in \mathscr{X}$.

Proof. See Köthe (1969, p. 158).

Let $\Gamma = \mathscr{X} \cap \Omega$ denote the non-negative orthant of $\mathscr{X}$, and let $\Gamma^* = \{x \in \tilde{\mathscr{X}}^* \mid x(y) \geqq 0 \text{ for all } y \in \Gamma\}$ denote the non-negative orthant of $\tilde{\mathscr{X}}^*$.

Lemma 6. If Y_0 is p-bounded, and a convex subset Y_3 of Γ has $\delta(y) \geqq 1$ for all $y \in Y_3$, then there exists $P \in \Gamma^*$ such that $P(y) \leqq 1$ for all $y \in Y_2$ and $P(y) \geqq 1$ for all $y \in Y_3$.

Proof. The sets Y_3 and $Y_4 = \{y \in \mathscr{X} \mid y \leqq y', \delta(y') < 1\}$ are disjoint since $y \in Y_4 \cap \Gamma$ implies $\delta(y) < 1$. Y_4 has a non-empty interior. Hence, there exists a non-zero functional $P \in \mathscr{X}^*$ and some scalar α such that $P(y) \leqq \alpha$ for $y \in Y_4$, and $P(y) \geqq \alpha$ for $y \in Y_3$. Y_4 contains $-\Gamma$, implying $-\lambda P(y) \leqq \alpha$ for $y \in \Gamma$, all positive scalars λ, implying in turn that $P \in \Gamma^*$. By δ-continuity, $P(y) \leqq \alpha$ for y satisfying $\delta(y) \leqq 1$, and hence $P(y) \leqq \alpha$ for $y \in Y_2$. Now, $Y_1 \subseteq Y_2$, $\mathscr{X}$ spanned by Y_1 and $P \neq 0$, imply $P(\hat{y}) > 0$ for some $\hat{y} \in Y_1$. Hence, $\alpha > 0$. Normalizing $\alpha = 1$ completes the proof. Q.E.D.

Consider a given subset Y of $\mathscr{Y}$ and a non-empty, finite or infinite, set of integers $K = \{k \mid k = 0, 1, 2, \ldots\}$. Suppose a linear subspace $\mathscr{Y}_k \subseteq \mathscr{Y}$ is defined for each $k \in K$, and define $Y^k = \{y \in \mathscr{Y}_k \mid y \in Y\}$. Let $\mathscr{S}$ denote the linear space of all vectors $s = (y^0, y^1 \ldots.)$ with $y^k \in \mathscr{Y}_k$, $k \in K$, and define $S_0 = \{s \in \mathscr{S} \mid y^k \in Y^k,\ k \in K;\ y = \sum_{k \in K} y^k \in Y\}$.

We make the following assumption:

Assumption A

Y is convex and monotone below relative to $\mathscr{Y}$, and the set $Y_0 = Y \cap \Omega$ of non-negative points in Y is non-empty, p-bounded, and p-closed.

Let $\mathscr{X}$ denote the real linear space spanned by Y_0, and let $\Gamma = \Omega \cap \mathscr{X}$ denote its non-negative orthant. Define $\mathscr{X}_k = \mathscr{X} \cap \mathscr{Y}_k$ and $\Gamma_k = \Omega \cap \mathscr{X}_k$ for K. Since $Y_0^k = Y^k \cap \Omega \subseteq Y_0$, we have $Y_0^k \subseteq \Gamma_k$. Let $Y_5 = Y \cap \mathscr{X}$. Then, Lemmas 1 – 6 apply to $\mathscr{X}$ and Y_5.[16] We shall again denote by Y_2 the intersection of all sets in $\mathscr{X}$, monotone below with respect to $\mathscr{X}$, containing Y_0.

[16] In the statement of Lemma 6, replace Y_2 by Y_5, and in its proof, replace Y_4 by Y_5.

Suppose, for each $k \in \boldsymbol{K}$, a partial preordering $\succsim_k$ (reflexive, transitive relation) of a subset $\boldsymbol{D}_k$ of $\boldsymbol{\Gamma}_k$ is given. Define $y \succ_k y'$ (resp. $y \sim_k y'$) if and only if $y \succsim_k y'$ and not $y' \succsim_k y$ (resp. both $y \succsim_k y'$ and $y' \succsim_k y$); $y, y' \in \boldsymbol{D}_k$. Let $\boldsymbol{D}$ denote the subset of points, $s=(y^0, y^1, \ldots) \in \mathscr{S}$ such that $y^k \in \boldsymbol{D}_k$ for $k \in \boldsymbol{K}$. For any non-empty subset $\boldsymbol{K}_1$ of $\boldsymbol{K}$, define a partial preordering $\succsim_{K_1}$ on D by $s \succsim_{K_1} s'$ if and only if $y \succsim_k y'$ for every $k \in \boldsymbol{K}_1$; $s, s' \in D$. Define $s \succ_{K_1} s'$ (resp. $s \sim_{K_1} s'$) if and only if $s \succsim_{K_1} s'$ and not $s' \succsim_{K_1} s$ (resp. both $s \succsim_{K_1} s'$ and $s' \succsim_{K_1} s$), $s, s' \in \boldsymbol{D}$. Define $\boldsymbol{S}_1 = \boldsymbol{S}_0 \cap \boldsymbol{D}$.

A point $\bar{s} \in \boldsymbol{S}_1$ is *maximal* in $\boldsymbol{S}_1$ relative to a non-empty subset $\boldsymbol{K}_1$ of $\boldsymbol{K}$ if any $s' \in \boldsymbol{S}_1$ with $s' \succsim_{K_1} \bar{s}$ has $s' \sim_{K_1} \bar{s}$. A point $\bar{s} \in \boldsymbol{S}_1$ is *non-decomposable* if, for every partition of $\boldsymbol{K}$ into two proper subsets $\boldsymbol{K}_1, \boldsymbol{K}_2$, there exists $s' \in \boldsymbol{S}_1$ with $y(s') = y(\bar{s})$ such that $s' \succ_{K_1} \bar{s}$. For $\bar{s} \in \boldsymbol{S}_1$, define the set $\boldsymbol{U}_k(\bar{y}^k) = \{y \in D_k \mid y \succsim_k \bar{y}^k\}$. The point $\bar{s} \in \boldsymbol{S}_1$ is *non-extreme* if the origin is not contained in the p-closure of $\boldsymbol{U}_k(\bar{y}^k)$, $k \in \boldsymbol{K}$.

We shall consider the following assumption on the relations $\succsim_k$ for a given non-extreme $\bar{s} \in \boldsymbol{S}_1$:

Assumption B

$\boldsymbol{U}_k(\bar{y}^k)$ is convex, and $y \in \boldsymbol{U}_k(\bar{y}^k)$, $y \leqq y' \in \boldsymbol{\Gamma}_k$ implies $y' \in \boldsymbol{U}_k(\bar{y}^k)$. If $y' \succ_k \bar{y}$, $y'' \succsim_k \bar{y}^k$, then $\theta y' + (1-\theta) y'' \succ_k \bar{y}^k$ for $0 < \theta < 1$, and $\theta' y' \succ_k \bar{y}^k$ for some $\theta' < 1$. At any $\tilde{s} \in \boldsymbol{S}_1$, there exists $k' \in \boldsymbol{K}$ and $y \in \boldsymbol{\Gamma}_k$, such that $\tilde{y}^{k'} + \hat{y} \succ_{k'} \tilde{y}^{k'}$.

Define the set

$$\boldsymbol{U}(\bar{s}) = \{y(s) \in \boldsymbol{\Gamma} \mid s \succ_K \bar{s}\} \quad \text{for} \quad \bar{s} \in \boldsymbol{S}_1.$$

If $\bar{s}$ is maximal in $\boldsymbol{S}_1$ relative to $\boldsymbol{K}$, then $\boldsymbol{U}(\bar{s})$ and $\boldsymbol{Y}_0$ are disjoint. If, further, (B) holds, then $\boldsymbol{U}(\bar{s})$ is convex, and Lemma 6 establishes the existence of a non-zero continuous linear functional (in the δ-norm) separating $\boldsymbol{U}(\bar{s})$ and $\boldsymbol{Y}_5$. A stronger separation theorem will now be established.

Theorem 8. *If (A) and (B) hold at a point $\bar{s} \in \boldsymbol{S}_1$ which is maximal relative to $\boldsymbol{K}$, non-decomposable, and non-negative, and if* $\lim_{k_0 \to \infty} \delta(\sum_{k \in K} \bar{y}^k) = 0$, *then there exists $\boldsymbol{P} \in \boldsymbol{\Gamma}^*$ such that $\boldsymbol{P}(y) \leqq 1$ for all $y \in \boldsymbol{Y}_5$, $\boldsymbol{P}(y) > 1$ for all $y \in \boldsymbol{U}(\bar{s})$, and, for each $k \in \boldsymbol{K}$, $\boldsymbol{P}(y^k) \leqq \boldsymbol{P}(\bar{y}^k)$ and $y^k \succsim_k \bar{y}^k$ imply $y^k \sim_k \bar{y}^k$.*

Proof. For the set $\boldsymbol{Y}_5$, define the support function

$$\phi_5(y) = \inf\{\mu \mid \mu \text{ a positive scalar}, \quad (1/\mu) y \in \boldsymbol{Y}_5\}.$$

Let $\phi_2(y)$ denote the support function of $\boldsymbol{Y}_2$. Since $\boldsymbol{Y}_2 \subseteq \boldsymbol{Y}_5$, it follows that $\phi_5(y) \leqq \phi_2(y)$, and since $\boldsymbol{Y}_2 \cap \boldsymbol{\Gamma} = \boldsymbol{Y}_5 \cap \boldsymbol{\Gamma}$, $\phi_5(y) = \phi_2(y)$ for $y \in \boldsymbol{\Gamma}$.

Consider the function

$$f_k(y)=\inf\{\alpha>0\,|\,(1/\alpha)y\notin U_k(\bar{y}^k)\},\quad k\in K,\quad y\in\Gamma.$$

We first show that $f_k(y)$ is well-defined and bounded above on sets bounded in the δ-norm. Suppose one could find a sequence $\{y_n\}$ in Γ with $\delta(y_n)\leqq 1$ such that $(1/n)y_n\in U_k(\bar{y}^k)$. Then, $(1/n)y_n\to_t 0$, contradicting the hypothesis that $\bar{s}$ is non-extreme. Hence, $f_k(y)$ is defined and bounded on $y\in\Gamma$ with $\delta(y)\leqq 1$. By Assumption (B), $f_k(y)$ is concave and positive linear homogeneous on Γ and satisfies $f_k(y)\geqq 1$ for $y\in U(\bar{y}^k)$ and $f_k(\bar{y}^k)=1$.

Define

$$\mu_k=-1+\sup\{f_k(y+\bar{y}^k)\,|\,y\in\Gamma,\quad \delta(y)\leqq 1\}.$$

Since $\bar{y}^k\in Y_0$ has $f_k(2\bar{y}^k)=2$, we have $\mu_k\geqq 1$. Since $f_k(y)$ is bounded on Y_0, μ_k is finite.

Let $v=(v_0,v_1,\dots)$ denote a sequence of real numbers with a component for each $k\in K$, and define a real Banach space

$$V=\left\{v\,\middle|\,\sum_{k\in K}|v_k|/\mu_k=\zeta(v)<+\infty\right\}.$$

Then, V is homeomorphic to l_1, and the space of continuous linear functionals on V is

$$V^*=\left\{z=(z_0,z_1,\dots)\,\middle|\,\sup_{k\in K}(\mu_k|z_k|)<+\infty\right\}.$$

Let R denote the real line, and form the linear space $W=V\times R\times\mathscr{X}$ with norm

$$\|w\|=\|(v,r,y)\|=\sum_{k\in K}|v_k|/\mu_k+|r|+\delta(y).$$

Define the set

$$E=\Bigg\{(v,r,y)\in W\,\Big|\,v_k\leqq f_k(y_1^k)-f_k(\bar{y}^k),\quad r\leqq 1-\phi_5(y_2),$$
$$y\geqq\sum_{k\in K}y_1^k-y_2;\quad y_1^k\in\Gamma;\quad y_2,\quad \sum_{k\in K}y_1^k\in\mathscr{X}\Bigg\}.$$

Since each function f_k is concave and ϕ_5 is convex, the set E is convex. The point $(0,0,0)$ is in E (i.e., take $y_1^k=\bar{y}_1^k$, $y_2=\bar{y}$). We will now show that $(0,0,0)$

is a non-interior point of $\boldsymbol{E}$. Suppose there exists $(\boldsymbol{v},r,\boldsymbol{y})\in\boldsymbol{E}$ with $\boldsymbol{v}\geqq\boldsymbol{0}$, $r>0$, and $\boldsymbol{y}\geqq\boldsymbol{0}$. Then, there exists $\tilde{s}\in\boldsymbol{S}_1$ such that $f_k(\tilde{y}^k)\geqq f_k(\bar{y}^k)$ for all k, implying $\tilde{y}^k\sim_k\bar{y}^k$ by the maximality of $\bar{s}$, and such that $\phi_5(y(\tilde{s}))=1-\varepsilon<1$. By Assumption (B), there exists $k'\in\boldsymbol{K}$ and $\boldsymbol{y}\in\boldsymbol{\Gamma}_k$, such that $\theta\hat{y}+\tilde{y}^{k'}\succsim_{k'}\theta\tilde{y}+(1-\theta)\tilde{y}^{k'}\succ_{k'}\tilde{y}^{k'}$ for $0<\theta<1$. Choose $\theta=\varepsilon/(1+\delta(\hat{y}))$. Then, $\phi_5(\theta\hat{y})<\varepsilon$, and s' defined with $y'^k=\tilde{y}^k$, $k\neq k'$, and $y'^{k'}=\tilde{y}^{k'}+\theta\hat{y}$ has $s'\succ_k\bar{s}$, contradicting the maximality of $\bar{s}$. Hence, $(\boldsymbol{v},r,\boldsymbol{y})\in\boldsymbol{E}$ and $r>0$, $\boldsymbol{y}\geqq\boldsymbol{0}$ implies that some component of $\boldsymbol{v}$ is negative, and $(\boldsymbol{0},0,\boldsymbol{0})$ is non-interior to $\boldsymbol{E}$.

We next show that $\hat{w}=(\boldsymbol{0},-3,\boldsymbol{0})$ is an interior point of $\boldsymbol{E}$. Consider any point $w=(\boldsymbol{v},r,\boldsymbol{y})$ satisfying $\|w-\hat{w}\|\leqq\frac{1}{2}$, and define $\theta_k=(|v_k|+2^{-(k+2)})/\mu_k$. Then $\sum_{k\in K}\theta_k\leqq\zeta(\boldsymbol{v})+\frac{1}{2}\leqq1$. Let $\tilde{y}^k$ denote a point in Y_0^k which satisfies $f_k(\tilde{y}^k+\bar{y}^k)-1\geqq\max\{\mu_k-2^{-(2+k)},f_k(\bar{y}^k)\}$, and define $y^{*k}=\theta_k\tilde{y}^k+\bar{y}^k$. By Lemma 5, $\mathscr{X}$ is complete and the unit sphere is closed in δ-norm, implying that $y^*=\sum_{k\in K}y^{*k}$ has $\phi_5(y^*)\leqq2$. Define

$$v_k^*=f_k(y^{*k})-f_k(\bar{y}^k)\geqq\theta_k(\mu_k-2^{-(k+2)}),$$

by concavity. Then, the point $(\boldsymbol{v}^*,1-\phi_5(y^*-y),y)$ is in $\boldsymbol{E}$. But

$$1-\phi_5(y^*-y)\geqq1-\phi_5(y^*)-\delta(y)\geqq-2,$$

and

$$v_k^*\geqq\theta_k(\mu_k-2^{-(k+2)})\geqq v_k,$$

imply

$$(\boldsymbol{v}^*,1-\phi_5(y^*-y),y)\geqq(\boldsymbol{v},r,y)=w,$$

and $w\in\boldsymbol{E}$.

Since $(\boldsymbol{0},0,\boldsymbol{0})$ is a boundary point of a convex set with a non-empty interior, there exists a non-zero linear functional $(z,b,-\boldsymbol{P})\in V^*\times R\times\mathscr{X}^*$ such that $z(\boldsymbol{v})+br-\boldsymbol{P}(y)\leqq0$ for all $(\boldsymbol{v},r,y)\in\boldsymbol{E}$ [cf. Dunford and Schwartz (1957, pp. 447–449)]. From the construction of $\boldsymbol{E}$, $z_k\geqq0$ for $k\in\boldsymbol{K}$, $b\geqq0$, and $\boldsymbol{P}\in\Gamma^*$.

Take $y_1^k=\bar{y}^k$, $k\in\boldsymbol{K}$ and $y_2\in\mathscr{X}$. Then one has $b(1-\phi_5(y_2))-\boldsymbol{P}(\bar{y}-y_2)\leqq0$. Suppose $b=0$. Then $\boldsymbol{P}(y_2)\leqq\boldsymbol{P}(\bar{y})$ for all $y_2\in\mathscr{X}$ implies $\boldsymbol{P}=\boldsymbol{0}$. But then, taking $y_1^k-(1+2^{-k})\bar{y}^k$ and $y_2=\bar{y}$, one has $f_k(y_1^k)-f_k(\bar{y}^k)=2^{-k}=v_k$, implying $\boldsymbol{v}\in V$ and $z(\boldsymbol{v})=\sum_{k\in K}2^{-k}z_k\leqq0$. But this contradicts the previous conclusion that $(z,b,-\boldsymbol{P})$ is non-zero and $z_k\geqq0$, and hence the supposition that $b=0$ is false. Normalize $b=1$. Setting $y_1^k=\bar{y}^k$ and $y_2=\boldsymbol{0}$ and $2\bar{y}$ establishes that $b=\boldsymbol{P}(\bar{y})=1$.

Next consider $y_1^{k'}$ equal to $\boldsymbol{0}$ or $2\bar{y}^{k'}$ for some $k'\in\boldsymbol{K}$, $y_1^k=\bar{y}^k$ for $k\in\boldsymbol{K}$, $k\neq k'$, and $y_2=\bar{y}$. Then, $-z_{k'}+\boldsymbol{P}(\bar{y}^{k'})\leqq0$ and $+z_{k'}-\boldsymbol{P}(\bar{y}^{k'})\leqq0$, implying $z_{k'}=\boldsymbol{P}(\bar{y}^{k'})$. Now,

$$\lim_{k_0\to\infty} \delta\left(\sum_{\substack{k\in K\\ k\geqq k_0}} \bar{y}^k\right)=0$$

implies

$$P(\bar{y})=\lim_{k_0\to\infty} P\left(\sum_{\substack{k\in K\\ k\leqq k_0}} \bar{y}^k\right)=\lim_{k_0\to\infty} \sum_{\substack{k\in K\\ k\leqq k_0}} z_k=1,$$

by the continuity of the linear functional. Hence, at least one z_k' is positive.

Suppose not all z_k are positive. Partition K into sets K_1, K_2 such that $z_k>0$ for $k\in K_1$; $z_k=0$ for $k\in K_2$. By non-decomposability there exists $\tilde{s}\in S_1$ with $\tilde{s}\succ_{K_1}\bar{s}$. Hence, for some $k'\in K_1$, $\tilde{y}^{k'}\succ_{k'}\bar{y}^{k'}$. By Assumption (B), $\theta'\tilde{y}^{k'}\succ_{k'}\bar{y}^{k'}$ for some $\theta'<1$, implying $f_{k'}(\tilde{y}^{k'})>1$. Define $v_k=2^{-k}[f_k(y^k)-f_k(\bar{y}^k)]$. Then, $(v,0,0)\in E$ implies $\sum_{k\in K_1} z_k v_k\leqq 0$, contradicting the previous results that $z_k>0$, $v_k\geqq 0$ for $k\in K_1$ and $v_{k'}>0$. Hence, $z_k>0$ for $k\in K$.

Finally, consider $y_2=\bar{y}$, $y_1^k=\bar{y}^k$ for $k\in K$, $k\neq k'$. Then

$$z_k[f_{k'}(y_1^k)-f_{k'}(\bar{y}^k)]-P(y_1^{k'}-\bar{y}^{k'})\leqq 0,$$

or

$$z_k f_k(y^k)\leqq P(y^k) \quad \text{and} \quad z_k f_k(\bar{y}^k)=P(\bar{y}^k) \quad \text{for} \quad k\in K, \quad y^k\in \Gamma_k.$$

Hence, $P(y^k)\leqq P(\bar{y}^k)$ implies $f_k(y^k)>f_k(\bar{y}^k)$. But $y^k\succ_k\bar{y}^k$ would imply $f_k(y^k)>f_k(\bar{y}^k)$ by Assumption (B). Hence, $P(y^k)\leqq P(\bar{y}^k)$ implies $\bar{y}^k\succsim_k y^k$.

Take $\tilde{y}\in U(\bar{s})$, and let $\tilde{s}$ be an associated point in S_1. Then, $f_k(\tilde{y}^k)\geqq f_k(\bar{y}^k)$, implying $P(\tilde{y}^k)\geqq P(\bar{y}^k)$ for $k\in K$. Further $f_{k'}(\tilde{y}^{k'})>f_{k'}(\bar{y}^{k'})$ for some $k'\in K$, implying $P(y^{k'})>P(\bar{y}^{k'})$. Now, $P(\tilde{y})\geqq\lim_{k_0\to\infty} P(\sum_{k\in K, k=k_0}\tilde{y}^k)$ by the non-negativity of the $\tilde{y}^k$, and $P(\bar{y})=\lim_{k_0\to\infty} P(\sum_{k\in K, k\leqq k_0}\bar{y}^k)$ was previously established. Hence, $P(\tilde{y})>P(\bar{y})$ for all $\tilde{y}\in U(\bar{s})$.

The condition $\phi_5(y_2)\geqq P(y_2)$ is obtained by setting $y_k^k=\bar{y}^k$ for $k\in K$. Then $\phi_5(y_2)\leqq 1$ for $y_2\in Y_5$ and $1=P(\bar{y})$ implies $P(y)\leqq P(\bar{y})$ for $y\in Y_5$. This completes the proof of the theorem. Q.E.D.

References

Bose, A., 1974, Pareto optimality and efficient capital accumulation, Discussion Paper no. 74-4 (Department of Economics, University of Rochester, Rochester, NY).

Cass, D. and M. Yaari, 1967, Individual saving, aggregate capital accumulation and efficient growth, in: K. Shell, ed., Essays on the theory of optimal economic growth (M.I.T. Press, Cambridge, MA).

Debreu, G., 1954, Valuation equilibrium and pareto optimum, Proceedings of the National Academy of Sciences USA 40, 588–592.

Dunford, N. and J. Schwartz, 1957, Linear operators, Part I (Interscience, New York).

Kelly, J., 1955, General topology (Van Nostrand, Princeton, NJ).

Klee, V.L., 1951, Convex sets in linear spaces, Duke Math Journal, 443–466, 875–883.