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On the Controllability of Decentralized Macroeconomic Systems: The Assignment Problem

ON THE CONTROLLABILITY OF DECENTRALIZED MACROECONOMIC SYSTEMS: THE ASSIGNMENT PROBLEM*

Daniel McFadden

1. Introduction

In classical control theory, the operation of a system is controlled by a single supervisor with complete information on the system's state. By contrast, control responsibility in economic systems is frequently spread among a number of institutions, each operating with limited information on the state of the system and the behavior of other institutions. In this environment, each controlling agency must form a decentralized behavioral strategy based on the partial information it receives.

Consider an economy with K controlling institutions, indexed $k=1,\ldots,K$, with Agency k operating an abstract control vector $\underset{\sim}{u}_k$. Let $\underset{\sim}{x}(t)$ describe the state of the system at time t, and let $\underset{\sim}{z}_k(t)$ denote the signal received by Agency k containing information on $\underset{\sim}{x}(t)$. Generally, the description $\underset{\sim}{x}(t)$ of the state of the system can include inter-agency messages and historical data.

As noted in Figure 1, the economy has an information transformer $\underset{\sim}{P}_k: \underset{\sim}{x}(t) \to \underset{\sim}{z}_k(t)$ determining the signal transmitted to Agency k. On the basis of the information contained in this signal, the agency's strategy determines a control $\underset{\sim}{s}_k: \underset{\sim}{z}_k(t) \to \underset{\sim}{u}_k(t)$. The state $\underset{\sim}{x}(t)$ and controls $\underset{\sim}{u}_1(t),\ldots,\underset{\sim}{u}_K(t)$ then lead, through a state transformer $\underset{\sim}{A}$, to a new state $\underset{\sim}{x}(t+1)$. We pose four questions on the control of this system.

1. For a particular structure of institutional authority and a particular information transformer, do there exist feasible decentralized strategies $(\underset{\sim}{s}_1,\ldots,\underset{\sim}{s}_K)$ which steer the system to a target state; i.e., is the system controllable?

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* Technical Report No. 6. Project for the Estimation and Optimization of Economic Growth.

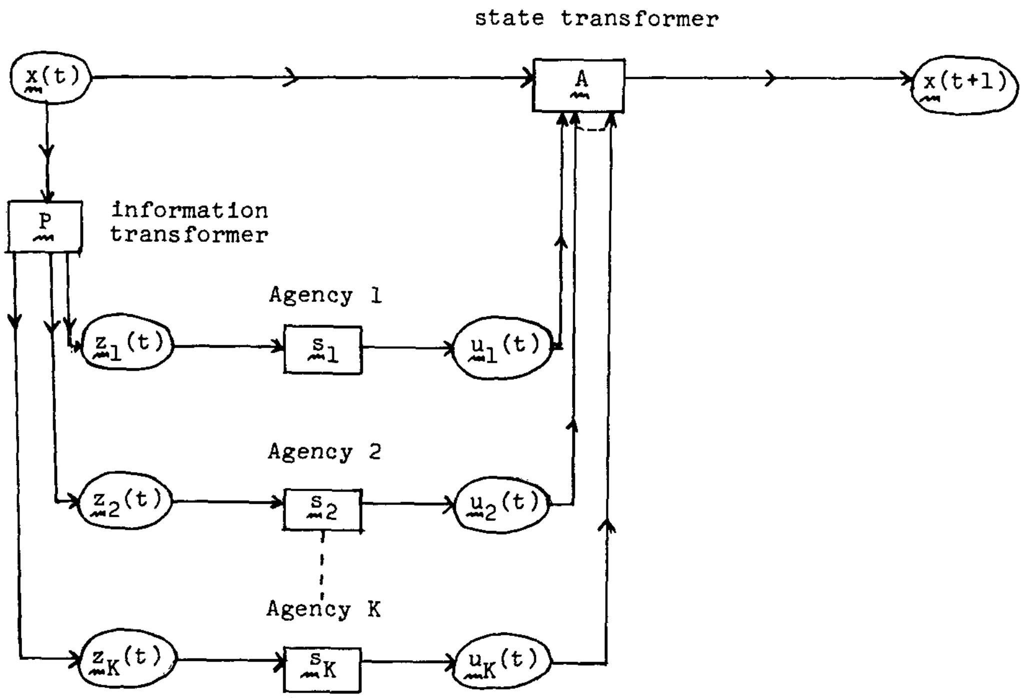

Figure 1: An abstract decentralized macroeconomic system

2. Given a performance function assigning losses to deviations from a target state, does there exist within a specified class of feasible strategies an optimal decentralized strategy?

3. When alternative designs for the economic system are available yielding different patterns of institutional decentralization and/or information transformers, can designs be found which make the system controllable? (This is termed the <u>assignment problem</u>.)

4. Weighing performance and operating costs under alternative economic designs, is there an optimal design?

In this lecture, I will answer some of these questions for a simple economic model, and will pose a number of unsolved problems.

2. A Simple Macroeconomic System

A highly simplified version of a common economic model will be used to illustrate the problems confronted in the control of a macroeconomic system. The United States has the short run economic objectives of full employment without inflation (internal balance) and balance of international payments (external balance), which must be achieved through two principal policy instruments, changes in the rate of interest and in the government budget deficit. The country has, correspondingly, two government institutions, a central bank (the Federal Reserve) which controls changes in the interest rate, and a Congress which controls changes in the government deficit. While it is politically impossible to combine these institutions into a single controlling agency, it is possible to establish general directives for them to follow.* Can this system be controlled? To answer this question, we will first give a simple algebraic statement of the system. Define the following variables:

- $Y(t)$ domestic U. S. production (= income of consumers)
- $X(t)$ U. S. aggregate expenditure
- $C(t)$ U. S. Aggregate consumption
- $S(t)$ U. S. aggregate saving
- $I(t)$ U. S. domestic investment
- $M(t)$ U. S. imports of foreign goods and services
- $K(t)$ Net capital outflows from the U. S.
- $T(t)$ Taxes net of transfers
- $G(t)$ U. S. government expenditures for goods and services
- $B(t)$ U. S. net surplus in international balance of payments

All the variables above are annual rates measured in period t, and are measured in billions of dollars, deflated to a uniform price level. Define the additional variables

- E U. S. exports of goods and services (annual rate in dollars), assumed constant
- Y_F the full-employment, no inflation level of domestic production, assumed constant

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* This problem has been analyzed by Robert MUNDELL [9] and Harold VOTEY [13], and the algebraic model presented here is a simplified version of the one they analyze.

$r(t)$ U. S. domestic interest rate

r_f foreign interest rate, assumed constant

Time arguments will be suppressed where there is no ambiguity.

Four accounting identities link these variables:

(1) $$Y = C + S + T$$

(2) $$X = C + I + K + G$$

(3) $$B = E - M - K$$

(4) $$B = Y - X$$

Further, the following functional relations are found empirically to be roughly valid:*

(5) $$S = \alpha_1 Y - \alpha_0 \approx .25Y - 40$$

(6) $$M = \beta_1 Y - \beta_0 \approx .13Y - 45$$

(7) $$I = -\gamma_1 r + \gamma_0 \approx -375r + 113$$

(8) $$K = -\delta_1 r + \delta_0 \approx -76r + 13.7$$

This model yields the very simple dynamic system

(9) $$\Delta \underset{\sim}{x} = \underset{\sim}{A}\underset{\sim}{u}$$

$$\underset{\sim}{x} = \begin{bmatrix} B \\ \\ Y \end{bmatrix}, \quad \underset{\sim}{u} = \begin{bmatrix} \Delta r \\ \\ \Delta D \end{bmatrix}, \quad \underset{\sim}{A} = \begin{bmatrix} \delta_1 + \mu\beta_1\gamma_1 & -\mu\beta_1 \\ \\ -\mu\gamma_1 & \mu \end{bmatrix} = \begin{bmatrix} 204.2 & -.342 \\ \\ -986 & 2.632 \end{bmatrix}$$

where $\mu = (\alpha_1 + \beta_1)^{-1}$, $D = G - T$ (the net government deficit), and $\Delta \underset{\sim}{x} = \underset{\sim}{x}(t+1) - \underset{\sim}{x}(t)$, $\Delta r = r(t+1) - r(t)$, etc. From an initial state $\underset{\sim}{x}(0)$, the central planner wishes to steer this economy to the target

$$\underset{\sim}{x}^* = \begin{bmatrix} 0 \\ \\ Y_F \end{bmatrix}$$

* Variables are measured in billions of U. S. dollars in 1958 prices (except r, which is a proportion). The numerical functions are adapted from econometric estimates in a more complex model considered by VOTEY [13], modified so that the model approximates the 1963 U. S. national accounts. In this year, the U. S. had $D = \$9$ bil., $r = .08$ (= depreciation + yield on long-term corporate bonds + anticipated rate of inflation), $E = \$32$, and $Y_F = \$579$ bil. The model then yields roughly the values actually observed: $Y = \$551$ bil., $I = \$83$ bil., $C = \$353$ bil., $M = \$26.5$ bil., $G = \$109$ bil., $T = \$100$ bil., and $S = \$98$ bil. (see [11], [12]).

where international payments are in balance and a level Y_F of output is achieved yielding internal balance. In the example, $x'(0) = (-2.0, 551)$ and $x^{*'} = (0, 579)$.

Assume in this example that every admissible control has the linear, time-independent form $u = S(x - x^*)$, where S is a 2×2 matrix. Let $\mathcal{S}$ denote the set of matrices S which yield admissible controls. Does there exist $S \in \mathcal{S}$ such that the dynamic system $\Delta x = AS(x - x^*)$ is stable? In the case that A is non-singular, as in the numerical example, and the set of matrices $\mathcal{S}$ is unrestricted, the system is trivially controllable: $S = -A^{-1}$ is one stable control. With decentralization of control responsibility and limited information on the state of the system, stability is less immediate. Consider in this example the case in which the Central Bank observes the balance of payments, the Congress observes the output level, and no communication between the institutions is possible.* Then, S must be a diagonal matrix for each admissible control. For example, let

$$\mathcal{S} = \left\{ S \middle| S = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}, \quad |s_i| \leq 1 \right\}$$

Direct calculation of the characteristic roots of the two-dimensional system $\Delta x = AS(x - x^*)$ for $S \in \mathcal{S}$ and an arbitrary matrix $A = (a_{ij})$ verifies that this system is controllable from an arbitrary starting point if and only if A is non-singular and has at least one non-zero diagonal element.** In the numerical example, one finds that the system is stable if and only if $s_2 < 0$, $-.00981 < s_1 < 0$, and

$$s_2 > - \frac{1.521 + 155.1\, s_1}{2 + 76\, s_1},$$

* In the U. S. economy, control responsibility for "monetary" variables, such as the balance of payments, the rate of inflation, and the interest rate are the primary responsibility of the Federal Reserve Bank; and control responsibility for "real" variables, such as unemployment, are the primary responsibility of the Congressional-Executive branch of government. Correspondingly, the Federal Reserve serves as the basic data-gathering agency for "monetary" variables and bureaus of the Executive branch serve as the basic data-gathering agencies for "real" variables. While in practice there is considerable exchange of information between these institutions, at least with some time lag, and at least some degree of discretionary coordination on both targets and controls, the example nevertheless captures an important aspect of the problem of macroeconomic control, and probably does less violence to reality than an assumption of monolithic control.

** Choosing s_1, s_2 such that $a_{11}s_1 + a_{22}s_2 < 0$ and $s_1 s_2 \det(A) > 0$, one guarantees that the characteristic roots of AS have negative real parts. Then, s_1, s_2 can be taken sufficiently small in magnitude so that the characteristic roots of $I + AS$ are less than one in modulus.

and that the characteristic roots of AS are real for all $s_1 < 0$, $s_2 < 0$. We see that in the two-dimensional case, decentralized controllability is obtained under quite general conditions. We shall now establish an analogous result for a more general class of problems, and also explore some of the other questions that we posed initially.

3. Decentralized Controllability

Employing a basic theorem on the stabilization of matrices due to FISHER and FULLER [2], we can establish a simple sufficient condition for the controllability of a decentralized linear, time-independent macroeconomic system of the type described in the above example. Consider an economic system whose states are described by real N-dimensional vectors, $x \in R^N$. There are K controlling agencies, indexed $k=1,\dots,K$. Agency k receives a signal $z_k \in R^N$ and operates a control $u_k \in R^{J_k}$, using a linear strategy $u_k = S_k(z_k - z_k^*)$, where z_k^* is the signal generated when the system is in its target state, and S_k is a $J_k \times N$ matrix. The information transformer is assumed to be an orthogonal projection onto a coordinate subspace: $z_k = P_k x$, where P_k is an $N \times N$ diagonal matrix with "zero" and "one" diagonal elements.* The state transformer is assumed to have the linear form $\Delta x = Au$, where A is a $N \times J$ matrix, $J = \Sigma_{k=1}^K J_k$, and $u' = (u_1', \dots, u_K')$. The dynamic system is then

$$\Delta x = ASP(x - x^*), \tag{10}$$

where

$$S = \begin{bmatrix} S_1 & & & 0 \\ & S_2 & & \\ 0 & & \ddots & \\ & & & S_K \end{bmatrix}, \qquad P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_K \end{bmatrix},$$

and S is contained in a set of feasible strategies $\mathcal{S}$.

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* Thus, P_k passes a subset of the components of x without alteration. Our results are unchanged if P_k is, more generally, any idempotent matrix. The information transformer in actual macroeconomic control problems is usually considerably more complex than this, containing systematic biases, time lags, aggregates of state variables, and stochastic components. The problems of (centralized) control under some of these conditions have been studied by THEIL [10] and ZELLNER [14]. We shall not take up these problems here.

An obvious necessary condition for the system (10) to be controllable from any initial $\underset{\sim}{x}(0)$ is that each of the matrices $\underset{\sim}{A}$, $\underset{\sim}{S}$, $\underset{\sim}{P}$ be at least of rank N. Then, in particular, the number of control variables J must be at least N; the KN × N matrix $\underset{\sim}{P}$ must be of full rank, requiring that information on all components of the state variable be transmitted to the controlling agencies; and the strategies $\underset{\sim}{S}_k$ must satisfy

$$N \leq \text{rank } (\underset{\sim}{S}) = \sum_{k=1}^{K} \text{rank } (\underset{\sim}{S}_k) \leq \sum_{k=1}^{K} \min (J_k, \text{ rank } \underset{\sim}{P}_k).$$

These weak necessary conditions are not generally sufficient. In the system

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 \\ & \\ 1 & 0 \end{bmatrix}, \quad \underset{\sim}{S} = \begin{bmatrix} s_{11} & 0 & 0 & 0 \\ & & & \\ 0 & 0 & 0 & s_{22} \end{bmatrix}, \quad \underset{\sim}{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

for example, no stable control exists for any s_{11}, s_{22}. We now consider conditions which are sufficient for stability, starting with the case of full decentralization in which there are N controls, each administered by a distinct agency which receives information only on the state variable it is attempting to control. Then, $\underset{\sim}{A}$ is a square matrix, and $\underset{\sim}{L} \equiv \underset{\sim}{S}\underset{\sim}{P} = \text{diag } (s_{11},\ldots,s_{NN})$. The following theorem is due to FISHER and FULLER [2].

THEOREM 1: Consider the dynamic system $\Delta \underset{\sim}{x} = \underset{\sim}{A}\underset{\sim}{u}$, $\underset{\sim}{u} = \underset{\sim}{L}(\underset{\sim}{x} - \underset{\sim}{x}^*)$, where $\underset{\sim}{A}$ and $\underset{\sim}{L}$ are real square matrices of order N, and $\underset{\sim}{L} = \text{diag } (s_{11},\ldots,s_{NN})$, with the s_{nn} unrestricted. A sufficient condition for the existence of a decentralized linear control $\underset{\sim}{u} = \underset{\sim}{L}(\underset{\sim}{x} - \underset{\sim}{x}^*)$ such that this system is stable from any $\underset{\sim}{x}(0)$ is that $\underset{\sim}{A}$ have a nested sequence of non-zero principal minors.*

This theorem was suggested by the economist J. R. HICKS [3] in a literary discussion of the stability of market equilibrium. A formal proof of the theorem we have stated has been given in [2] and extended slightly in [8]. We will not repeat it here, but will illustrate the method of proof for the case N = 2: The matrix $\underset{\sim}{A}\underset{\sim}{L}$ has, in this case, the characteristic polynomial $P(\lambda) = \lambda^2 - [a_{11}s_{11} + a_{22}s_{22}]\lambda + s_{11}s_{22}|A|$. Suppose

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* A principal minor of order n of an N × N matrix $\underset{\sim}{A}$ is a determinant formed by striking out N-n columns and the corresponding symmetric rows of $\underset{\sim}{A}$. A sequence of principal minors of order 1,2,...,N is <u>nested</u> if the determinant of order n-1 in this sequence is itself a principal minor of the determinant of order n in the sequence, n=2,...,N.

$a_{11} \neq 0$ and $|A| \neq 0$, satisfying the hypothesis of the theorem. Define a second polynomial

$$Q(\lambda) = \left(\lambda - s_{11} \frac{a_{11}}{1}\right) \left(\lambda - s_{22} \frac{|A|}{a_{11}}\right) = \lambda^2 - \left[a_{11}s_{11} + \frac{|A|}{a_{11}} s_{22}\right] + s_{11}s_{22}|A|$$

whose roots are given by ratios of successive principal minors of A, multiplied by corresponding diagonal elements of L. Take s_{11}, s_{22} so that $\text{sign}(s_{11}) = -\text{sign}(a_{11})$ and $\text{sign}(s_{22}) = -\text{sign}\left(\frac{|A|}{a_{11}}\right)$, $|s_{11}|$ is small, and $|s_{22}/s_{11}|$ is small. Then, the roots of $Q(\lambda)$ can be made distinct, real, negative, and less than one in modulus. Since the coefficients of $P(\lambda)$ and $Q(\lambda)$ can be made arbitrarily close for $|s_{22}/s_{11}|$ sufficiently small, the roots of $P(\lambda)$ can be made arbitrarily close to the roots of $Q(\lambda)$, so that they will be real, negative, and less than one in modulus. Hence, the roots of $\underset{\sim}{I} + \underset{\sim}{A}\underset{\sim}{L}$ can be less than one in modulus. The N-dimension case is proved by FISHER-FULLER in this same manner.

Consider the set $\{1,\cdots,N\}$ of indices of the components of $\underset{\sim}{x}$, and let $\ell_i = \{1,\cdots,i\}$ and $r_i = \{i+1,\cdots,N\}$ be subsets of indices, $i=1,\cdots,N$. (The subscript i on ℓ_i, r_i will be deleted when there is no ambiguity.) For the case $\underset{\sim}{A}$ square and $\underset{\sim}{L} = \text{diag}(s_{11},\cdots,s_{NN})$, define commensurately with these subsets the partitioned vectors and matrices

$$\underset{\sim}{x} = \begin{bmatrix} \underset{\sim}{x}_{\ell_i} \\ \underset{\sim}{x}_{r_i} \end{bmatrix}, \quad \underset{\sim}{u} = \begin{bmatrix} \underset{\sim}{u}_{\ell_i} \\ \underset{\sim}{u}_{r_i} \end{bmatrix}, \quad \underset{\sim}{A} = \begin{bmatrix} \underset{\sim}{A}_{\ell_i\ell_i} & \underset{\sim}{A}_{\ell_i r_i} \\ \underset{\sim}{A}_{r_i\ell_i} & \underset{\sim}{A}_{r_i r_i} \end{bmatrix}, \quad \underset{\sim}{L} = \begin{bmatrix} \underset{\sim}{L}_{\ell_i\ell_i} & 0 \\ 0 & \underset{\sim}{L}_{r_i r_i} \end{bmatrix}.$$

If the hypotheses of Theorem 1 are satisfied, then the indices $\{1,\cdots,N\}$ can be assigned so that the nested seouence of non-zero principal minors lie in the left-hand corner of $\underset{\sim}{A}$; i.e., $|\underset{\sim}{A}_{\ell_i\ell_i}| \neq 0$ for $i=1,\cdots,N$. Define $\rho_i = |\underset{\sim}{A}_{\ell_i\ell_i}|/|\underset{\sim}{A}_{\ell_{i-1}\ell_{i-1}}|$, $i=1,\cdots,N$; with $|\underset{\sim}{A}_{\ell_0\ell_0}| = 1$ and $\rho_0 = 1$ by convention. The characterization of stable controls is strengthened by the following:

__LEMMA 1.1:__ Suppose a dynamic system satisfies the hypotheses of Theorem 1, and the components of $\underset{\sim}{x}$ are indexed so that $|\underset{\sim}{A}_{\ell_i\ell_i}| \neq 0$ for $i=1,\cdots,N$. Then, there exists $\varepsilon > 0$ such that if $|s_{11}| < \varepsilon$, $|s_{ii}/s_{i-1,i-1}| < \varepsilon$, $i=2,\cdots,N$; and $\text{sign}(s_{ii}) = -\text{sign}(\rho_i)$, $i=1,\cdots,N$, then the characteristic roots λ_i of $\underset{\sim}{I} + \underset{\sim}{A}\underset{\sim}{L}$ are distinct, real, non-negative, and less than one in modulus, satisfying

$$0 < 1 + \frac{3}{2} s_{11}\rho_1 < \lambda_1 < 1 + \frac{1}{2} s_{11}\rho_1 < 1 + \frac{3}{2} s_{22}\rho_2 < \lambda_2 < 1 + \frac{1}{2} s_{22}\rho_2 < \cdots$$

$$\cdots < 1 + \frac{3}{2} s_{NN}\rho_N < \lambda_N < 1 + \frac{1}{2} s_{NN}\rho_N < 1.$$

If, in particular, the inequality

$$\varepsilon \leq \frac{\min_{i=0,\cdots,N-1} \{|\rho_i/\rho_{i+1}|\}}{2^{N+1}\left[1 + 3N + 3N\|A\|^N \max_{i=0,\cdots,N-1} \{|\rho_i/\rho_{i+1}|\}\right]}$$

holds, where $\|A\| = \Sigma_{i,j=1}^{N} |a_{ij}|$, then the conclusion of this corollary holds.

A proof of this corollary is given in the appendix of [8], and is based on the proof of FISHER and FULLER. The bound on ε established above is derived from a bound $\varepsilon \leq \min_{i=0,\cdots,N-1} \{\{|\rho_i/\rho_{i+1}|\}/2^{N+1} (1 + \Sigma_{s-1}^{N-1} m_s)$ given in [8], where

$$m_s = \left(\frac{3}{2}\right)^{N-s} \left\{ \left|\frac{\beta_s}{\alpha_s} \frac{\rho_s}{\rho_{s+1}} - 1 \right| + \left|\frac{\gamma_s}{\alpha_s} \frac{\rho_s}{\rho_{s+1}} - 1 \right| + \cdots\right\}$$

with $\alpha_s = |A_{\ell_s \ell_s}|$ and $\beta_s, \gamma_s, \cdots$ denoting the remaining principal minors of A of order s.

The dynamic process characterized in this corollary has a simple geometric structure. Let $B = AL$ and $y = x - x^*$, and consider the partitioned system

(11) $$\begin{bmatrix} \Delta y_{\ell_i} \\ \Delta y_{i+1} \\ \Delta y_{r_{i+1}} \end{bmatrix} = \left[\begin{array}{c|c|c} B_{\ell_i \ell_i} & B_{\ell_i, i+1} & B_{\ell_i, r_{i+1}} \\ \hline B_{i+1,\ell_i} & b_{i+1,i+1} & B_{i+1, r_{i+1}} \\ \hline B_{r+1,\ell_i} & B_{r_{i+1}, i+1} & B_{r_{i+1} r_{i+1}} \end{array}\right] \begin{bmatrix} y_{\ell_i} \\ y_{i+1} \\ y_{r_{i+1}} \end{bmatrix} .$$

The conditions $|B_{\ell_i \ell_i}| \neq 0$, $i=1,\cdots,N$ imply that this system can be solved recursively for vectors $\hat{y}_{\ell_i} = -B_{\ell_i \ell_i}^{-1} B_{\ell_i r_i} y_{r_i}$ which determine a "partial equilibrium" in the ℓ_i component of the state vector (i.e., $\Delta y_{\ell_i} = 0$) for each possible y_{r_i} $i=1,\cdots,N$, by use of the formulae:

(12) $$\beta_{i+1} \equiv \left[b_{i+1,i+1} - B_{i+1,\ell_i} B_{\ell_i \ell_i}^{-1} B_{\ell_i, i+1}\right]$$

(13) $$\left|B_{\ell_{i+1}\ell_{i+1}}\right| = \beta_{i+1}\left|B_{\ell_i\ell_i}\right| \neq 0$$

(14) $$\hat{y}_{i+1} = \beta_{i+1}^{-1}\left\{B_{i+1,\ell_i}B_{\ell_i\ell_i}^{-1}B_{\ell_i r_{i+1}} - B_{i+1,r_{i+1}}\right\}y_{r_{i+1}}$$

(15) $$B_{\ell_{i+1}\ell_{i+1}}^{-1} = \left[\begin{array}{c|c} I - \beta_{i+1}^{-1}B_{\ell_i\ell_i}^{-1}B_{\ell_i,i+1}B_{i+1,\ell_i}B_{\ell_i\ell_i}^{-1} & \beta_{i+1}^{-1}B_{\ell_i\ell_i}^{-1}B_{\ell_i,i+1} \\ \hline \beta_{i+1}^{-1}B_{i+1,\ell_i}B_{\ell_i\ell_i}^{-1} & \beta_{i+1}^{-1} \end{array}\right]$$

If the condition $y_{\ell_i} = \hat{y}_{\ell_i}$ were imposed on the dynamic process for some i, then one would obtain

$$\Delta y_{i+1} = \beta_{i+1}y_{i+1} + \left[B_{i+1,r_{i+1}} - B_{i+1,\ell_i}B_{\ell_i\ell_i}^{-1}B_{\ell_i r_{i+1}}\right]y_{r_{i+1}}$$

$$= \beta_{i+1}(y_{i+1} - \hat{y}_{i+1}).$$

Since $\left|B_{\ell_i\ell_i}\right| = \left|A_{\ell_i\ell_i}\right| \cdot \prod_{j=1}^{i} s_{jj}$, we have $\beta_i = \rho_i s_{ii}$. Since, by construction, $\rho_i s_{ii} \in (-1, 0)$, it follows that y_{i+1} would converge to $\hat{y}_{i+1}$ if one fixed the value of $y_{r_{i+1}}$ and imposed the condition $y_{\ell_i} = \hat{y}_{\ell_i}$. But given that the s_{ii} decline sharply in magnitude for increasing i under the conditions imposed in the corollary, these conditions are closely approximated: the first component of x adjusts rapidly toward its "partial equilibrium" value while the relative adjustment in the remaining components is small. The second component adjusts, less rapidly, toward its "partial equilibrium" value, while the first component is approximately maintained in partial equilibrium and the components $3,\cdots,N$ change relatively little. The process continues in this essentially recursive manner, converging eventually to the "full equilibrium" $y = 0$. Figure 2 illustrates typical trajectories for the process.

A generalization of Theorem 1 to establish controllability in the large of a nonlinear system $\Delta x = a(u)$, $u = Lh(x)$, L diagonal, $h(x)' = (h_1(x_1),\cdots,h_N(x_N))$, can be proved when the geometric structure of recursive "partial equilibria" of the form noted above continues to hold and mild regularity conditions are met (see [8]). The proof is based directly on the geometric structure. A second extension considers more general

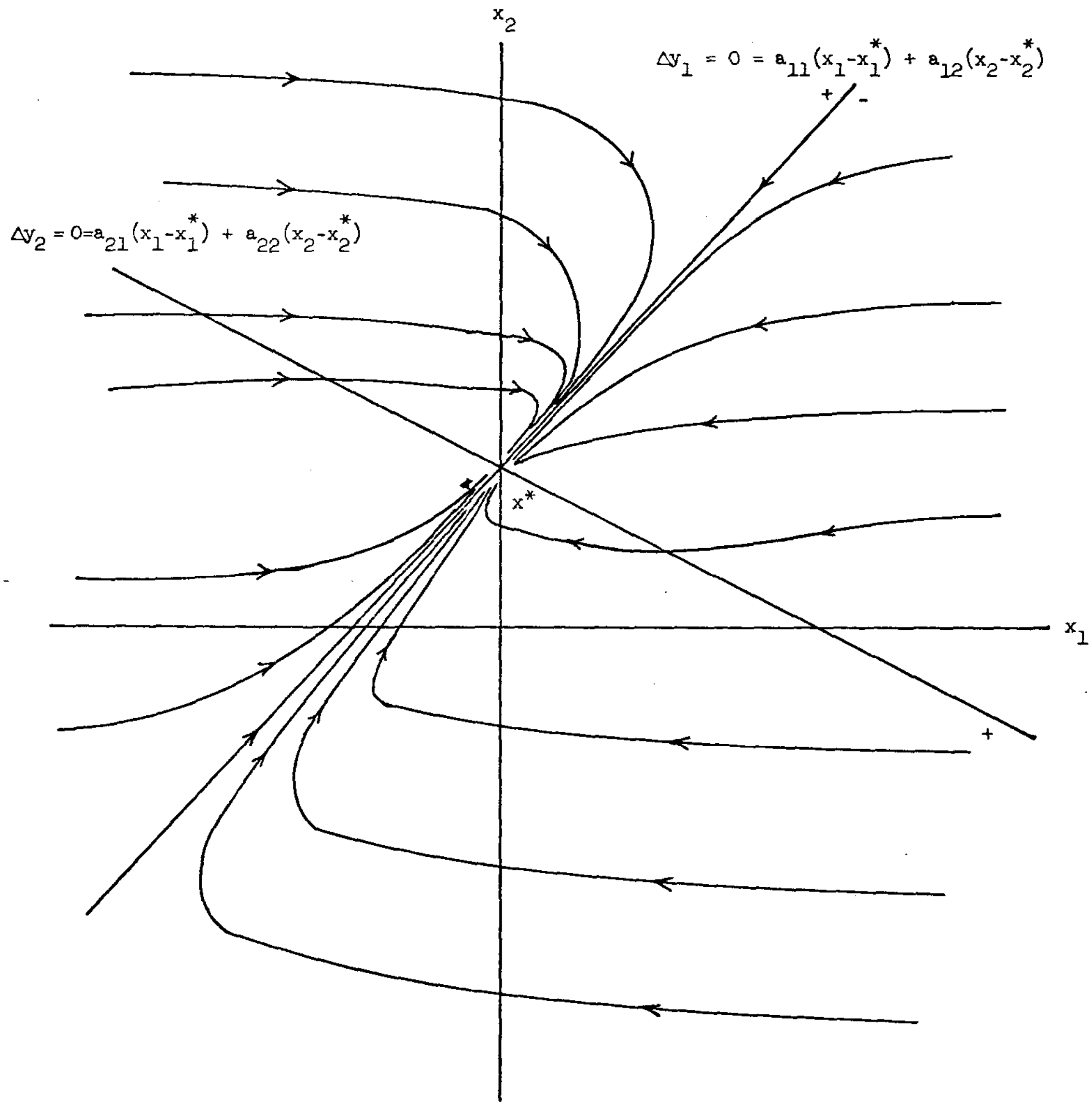

Figure 2

structures of institutional authority and information transfer than the fully decentralized case treated above:

Consider the dynamic system (10), and suppose that the rank conditions necessary for controllability are met. Consider the case in which information on each state variable is transmitted to one and only one agency. Letting P_{ik}' denote the ith row of P_k, $i=1,\cdots,N$, $k=1,\cdots,K$, we note that in this case exactly N of the rows P_{ik}' have non-zero elements. Form the matrix $W = (P_{i_1k_1}, P_{i_2k_2}, \cdots, P_{i_Nk_N})$ whose columns are given, in order, by these rows of P with non-zero elements. Since P is of rank N, W is a permutation matrix, formed by permuting the columns of an identity matrix, and $W' = W^{-1}$. Define $v = W'(x - x^*)$. Then, the system (10) can be written in the equivalent form

$$\Delta v = CLv$$

with $C = W'A$, a $N \times J$ matrix; L = block diagonal $(L_1, \cdots, L_K) = SPW$, a $J \times N$ matrix; and L_k a $J_k \times N_k$ sub-matrix, where N_k equals the number of components of x (possibly zero) on which information is transmitted to agency k. The rank conditions necessary for controllability imply rank $(L_k) = N_k \leq J_k$. When the sub-matrices S_k of S are unrestricted and the inequality $N_k \leq J_k$ holds for all agencies, then L_k is unrestricted and the rank conditions are met. Let H_k denote a non-singular $N_k \times N_k$ matrix, and define $G_k = L_kH_k^{-1}$ and the partitioned matrix

$$(16) \qquad F = C \begin{bmatrix} G_1 & & & 0 \\ & G_2 & & \\ 0 & & \ddots & \\ & & & G_K \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1K} \\ F_{21} & F_{22} & & \\ \vdots & & \ddots & \vdots \\ F_{KL} & & \cdots & F_{KK} \end{bmatrix}$$

where the partitioning is commensurate with the partitioning of L. The following result can now be established.

THEOREM 2: Consider the dynamic system $\Delta v = FHv$ defined above, where $F = CG$ and G and H are block diagonal matrices with blocks G_k and H_k, respectively, and $L = GH$. A sufficient condition for the existence of a decentralized linear control $u = Lv$ such that this system is stable from any $v(0)$ is that there exist a sequence of agencies $k_1, \cdots, k_K$ and a matrix G such that the following nested sequence of principal minors is non-zero:

$$\left|F_{k_1k_1}\right| \neq 0, \quad \begin{vmatrix} F_{k_1k_1} & F_{k_1k_2} \\ F_{k_2k_1} & F_{k_2k_2} \end{vmatrix} \neq 0, \cdots, \begin{vmatrix} F_{k_1k_1} & F_{k_1k_K} \\ \vdots & \\ F_{k_Kk_1} & F_{k_Kk_K} \end{vmatrix} \neq 0 .$$

Proof: Without loss of generality, we can take $(k_1, \cdots, k_K) = (1, \cdots, K)$. Define $\underset{\sim}{E} = \underset{\sim}{F}\underset{\sim}{H}$, let $\underset{\sim}{E}_{ij} = \underset{\sim}{F}_{ij}\, \underset{\sim}{H}_j$ denote a sub-matrix, and define

$$\underset{\sim}{D}_{\ell_i \ell_i} = \begin{bmatrix} \underset{\sim}{D}_{11} & \cdots & \underset{\sim}{D}_{1i} \\ \vdots & \ddots & \vdots \\ \underset{\sim}{D}_{i1} & \cdots & \underset{\sim}{D}_{ii} \end{bmatrix},$$

for $\underset{\sim}{D} = \underset{\sim}{E}, \underset{\sim}{F}$. Define $\underset{\sim}{H}_1 = \sigma_1 \underset{\sim}{F}_{11}^{-1}$ and, recursively,

$$\underset{\sim}{H}_{ii} = -\sigma_i \left[\underset{\sim}{F}_{ii} - \underset{\sim}{E}_{i\ell_{i-1}} \underset{\sim}{E}_{\ell_{i-1}\ell_{i-1}}^{-1} \underset{\sim}{F}_{\ell_i i} \right]^{-1},$$

where the σ_i are scalars in $(0, 2)$ and the inverse exists by the hypothesis that $|\underset{\sim}{F}_{\ell_i \ell_i}| \neq 0$. Then, this system can be written in the partitioned form of (11), replacing y_i by the subvector $\underset{\sim}{v}_i$ of $\underset{\sim}{v}$ commensurate with $\underset{\sim}{L}_i$, and replacing b_{ij} by the submatrix $\underset{\sim}{E}_{ij}$. The formulae (12), (14), and (15) then continue to hold with these substitutions, establishing (1) the existence of recursively defined subvectors $\hat{\underset{\sim}{v}}_{\ell_k} = -\underset{\sim}{E}_{\ell_k \ell_k}^{-1} \underset{\sim}{E}_{\ell_k r_k} \underset{\sim}{v}_{r_k}$ which determine a "partial equilibrium" in the subvectors $(\underset{\sim}{v}_1, \cdots, \underset{\sim}{v}_k)$ for each possible $(\underset{\sim}{v}_{k+1}, \cdots, \underset{\sim}{v}_K)$, (2) $\beta_k = -\sigma_k \underset{\sim}{I}$, and (3) the stability of the "partial" dynamic process $\Delta \underset{\sim}{v}_{k+1} = -\sigma_k (\underset{\sim}{v}_{k+1} - \hat{\underset{\sim}{v}}_{k+1})$ which results when the condition $\underset{\sim}{v}_{\ell_k} = \hat{\underset{\sim}{v}}_{\ell_k}$ is imposed and $\underset{\sim}{v}_{r_{k+1}}$ is fixed. The geometry of this process is then identical to the one considered previously, and the stability proof given in [8] establishes controllability for $|\sigma_1|$ and $|\sigma_k / \sigma_{k-1}|$ sufficiently small.

Q.E.D.

These theorems provide preliminary results on the decentralized controllability of one class of economic systems. It would be of considerable interest to economists to have answers to the following further questions:

1. What conditions are sufficient for controllability when the "target" is a subspace of the state space rather than a point (i.e., some state variables are irrelevant to the planner)?*

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* This case would occur, in particular, when a portion of the description of a state at time t contains inter-agency "messages" on past control activities and anticipated behavior, and the "messages" themselves are not "real" economic variables.

2. Is decentralized control possible when there are more complex signals than in the model above, allowing incomplete decentralization, storage of information with adaptive control procedures, more complex administrative hierarchies, and/or parametric "messages", such as prices?

3. Can an economic system with exogenous changes be steered with decentralized controls? What forecasts are required? What information on forecasts must be included in signals? Can "mean controllability" be maintained when stochastic elements are present? What historical information must be stored?

4. Optimal Decentralized Control

Costs are associated with deviations from the target state in a macroeconomic system, and the objective of the planner will be to choose a control strategy which minimizes these costs. Consider the dynamic system given in (10),

$$\Delta x = Au, \qquad u = Sz, \qquad z = P(x-x^*), \tag{17}$$

with $x \in R^N$, $u \in R^J$, $z \in R^{KN}$ and S chosen from a set $\mathcal{S}$ of feasible decentralized control strategies. Suppose that costs are described by a quadratic loss function

$$c = \sum_{t=0}^{\infty} \delta^t (x(t) - x^*)' Q(x(t)-x^*), \tag{18}$$

where δ is a discount factor applied to future costs, $\delta \in (0, 1)$; Q is a real symmetric positive-definite matrix; and the range of c is the extended half-line $[0, \infty]$. From (17), $x(t) - x^* = (I + ASP)^t(x(0) - x^*)$, and c can be written

$$c = \sum_{t=0}^{\infty} \delta^t \, y'_0 \, (E')^t (E)^t \, y_0, \tag{19}$$

where $y_0 = Q^{\frac{1}{2}}(x(0) - x^*)$ and $E = I + Q^{\frac{1}{2}} ASPQ^{-\frac{1}{2}}$.

Several difficulties may occur in treating the minimization of (19) over S in $\mathcal{S}$ as a conventional finite-dimensional non-linear program. The set of S for which the system (17) is stable from x(0), or more generally, from all starting vectors, is not usually compact or convex. Hence, these properties may fail for a feasible set $\mathcal{S}$ requiring stability, and an optimum may not exist. Further, the objective function is not usually convex in S, and standard programming algorithms are of limited usefulness. It would be desirable to have a good algorithm for solving this program which exploits the structure of (17) and (19).

In the simple macroeconomic system of Section 2, numerical methods have been used to calculate optima for two alternative feasible sets. Suppose that at the initial state x(0), the society considers a $1 decrease in the balance of payments surplus to imply the same economic loss as a $7 decrease in GNP. Roughly, the society considers a one billion dollar change in the balance of payments to be weighed the same as a one percent change in unemployment. If the costs of deviations from the balance of payments and full employment targets are additive, equation (18) then has approximately $Q = \text{diag}\ (100, 1)$.* The system (17) then can be written for this example, for $\delta = .9$, as

$$\Delta y = Q^{\frac{1}{2}} ASPQ^{-\frac{1}{2}} y$$

$$Q^{\frac{1}{2}} ASPQ^{-\frac{1}{2}} = \begin{bmatrix} 204.2s_1 & -3.42s_2 \\ -94.6s_1 & 2.632s_2 \end{bmatrix}$$

$$y = Q^{\frac{1}{2}}(x-x^*) \qquad y(0)' = (-20, -28).$$

When all diagonal matrics SP are feasible, i.e. $\mathcal{S} = \{S | SP = \text{diag}\ (s_1, s_2)\}$, the condition $y(1) = (I + Q^{\frac{1}{2}} ASPQ^{-\frac{1}{2}}) y(0) = 0$ obtains for $s_1 = -.0371$, $s_2 = -1.42$. This control strategy applied in the US economy in 1963 would have produced internal and external balance in the following year, but would have required that the Federal Reserve Bank raise the interest rate by 7.42 percentage points and that the Congress raise the government deficit by 39.76 billion dollars. For this strategy, y_0 is an eigenvector of $Q^{\frac{1}{2}} ASPQ^{-\frac{1}{2}}$ (with the corresponding root -1), and the second characteristic root of this matrix is -11.315. Hence, this strategy is unstable for any starting point not proportional to y_0. In particular, if we make the realistic modification in (17) of allowing random noise in the signals to control agencies, this strategy would be explosive with probability one.

Now suppose that the set of feasible strategies contains the diagonal matrices SP such that the characteristic roots of $I + Q^{\frac{1}{2}} ASPQ^{-\frac{1}{2}}$ are bounded in modulus by, say, 0.9. The optimal strategy is found to be $s_1 = -.00624$ and $s_2 = -.2385$. This strategy requires that the interest rate be raised 0.62 percentage points for each one billion dollar deficit in the balance of payments, and that the government debt be increased 0.24 billion dollars for each one billion dollar deficiency in GNP. The characteristic roots of $I + Q^{\frac{1}{2}} ASPQ^{-\frac{1}{2}}$ for this strategy are 0.832 and -0.9, with y_0 and eigenvector for the positive root.

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* In this example, it would probably be more realistic to assume that $\tilde{Q}$ has a negative off-diagonal term, implying that a balance of payments deficit entails a greater economic loss in an inflationary situation than in a less-than-full-employment economy.

5. The Assignment Problem

Thus far, we have discussed control in economic systems in which the information transformer is fixed. At a more basic level, we may consider the possibility that the planner can choose among alternative designs for the structure of information flows to optimize performance of the system. This problem of determining an efficient economic organization is the fundamental question of economics, and has been studied extensively in an abstract framework by MARSCHAK [5, 6], HURWICZ [4], and MARSCHAK and RADNER [7]. We shall not repeat here a formal statement of this problem, but shall examine a very special case, called the assignment problem, which arises in the context of decentralized macroeconomic control (see MUNDELL [9] and COOPER [1]).

Consider the macroeconomic system of equation (10), and suppose now that the planner can choose the information transformer $\underset{\sim}{P}$ from a set of alternative designs $\mathcal{P}$. We say that the assignment of signals is unrestricted if any list of feasible signals $(\underset{\sim}{z}_1,\cdots,\underset{\sim}{z}_K)$ going to agencies $1,\cdots,K$, respectively, could be re-assigned so that agency k receives a signal $\underset{\sim}{z}_{i_k}$, where $(i_1,\cdots,i_K)$ is a permutation of $(1,\cdots,K)$. Equivalently, if assignment is unrestricted, then $\underset{\sim}{P} \in \mathcal{P}$, $\underset{\sim}{P}' = (\underset{\sim}{P}_1',\cdots,\underset{\sim}{P}_K')$, implies $\underset{\sim}{\tilde{P}} \in \mathcal{P}$, where $\underset{\sim}{\tilde{P}}' = (\underset{\sim}{P}_{i_1}',\underset{\sim}{P}_{i_2}',\cdots,\underset{\sim}{P}_{i_K}')$ and $(i_1',\cdots,i_K')$ is a permutation of $(1,\cdots,K)$.

THEOREM 4: Consider the macroeconomic system of equation (10). Suppose that the assignment of signals is unrestricted, and that $\underset{\sim}{\tilde{S}} \in \mathcal{S}$ and $\underset{\sim}{S}$ any $J \times KN$ matrix with $|s_{ij}| \leq |\tilde{s}_{ij}|$ $i=1,\cdots,J;\ j=1,\cdots,KN$, implies $\underset{\sim}{S} \in \mathcal{S}$. If there exist $\underset{\sim}{\hat{S}} \in \mathcal{S}$ and $\underset{\sim}{\hat{P}} \in \mathcal{P}$ such that $\underset{\sim}{A}\underset{\sim}{\hat{S}}\underset{\sim}{\hat{P}}$ is of rank N, then there exist $\underset{\sim}{S} \in \mathcal{S}$ and $\underset{\sim}{P} \in \mathcal{P}$ such that the system is controllable from any $\underset{\sim}{x}(0)$.

Proof: The following result will be employed in the proof: If $\underset{\sim}{A}, \underset{\sim}{B}$ are $N \times J$ matrices with columns $\underset{\sim}{a}_j, \underset{\sim}{b}_j$ respectively, and if $\underset{\sim}{A}\underset{\sim}{B}'$ is of rank N, then there exist columns $(j_1,\cdots,j_N)$ such that $\Sigma_{i=1}^{N} \underset{\sim}{a}_{j_i}\underset{\sim}{b}_{j_i}'$ is of rank N. Suppose, to the contrary, that more than N columns were needed to obtain a matrix of full rank. Then one would have, for some i',

$$\text{rank} \left(\sum_{i=1}^{i'-1} \underset{\sim}{a}_{j_i}\underset{\sim}{b}_{j_i}' \right) = \text{rank} \left(\sum_{i=1}^{i'} \underset{\sim}{a}_{j_i}\underset{\sim}{b}_{j_i}' \right) ,$$

implying that each column of $a_{j_{i'}} b'_{j_{i'}}$ can be written as a linear combination of the columns of $\Sigma_{i=1}^{i'-1} a_{j_i} b'_{j_i}$. Then, $a_{j_{i'}} b'_{j_i}$ could be deleted from the sum without reducing the subspace spanned by $\Sigma_{i=1}^{i'-1} a_{j_i} b'_{j_i}$. This yields a contradiction of the supposition and proves the result.

Applying this result to the matrices A and $(\hat{S}\hat{P})'$, one obtains a partition $A = (A^{(1)}, A^{(2)})$ and $(\hat{S}\hat{P})' = [(\hat{S}^{(1)}\hat{P})', (\hat{S}^{(2)}\hat{P})']$, where $A^{(1)} = (a_{j_1},\cdots,a_{j_N})$ is $N \times N$ of full rank and $(\hat{S}\hat{P})$ is partitioned commensurately. Applying the result again to $(\hat{S}^{(1)}\hat{P})'$, one obtains a partition $\hat{S}^{(1)} = (\hat{S}^{(11)}, \hat{S}^{(12)})$, $\hat{P}' = (\hat{P}^{(1)'}, \hat{P}^{(2)'})$, where $\hat{S}^{(11)}$ and $\hat{P}^{(1)}$ are $N \times N$ and $\hat{S}^{(11)}\hat{P}^{(1)}$ is of full rank. From the assumption on $\mathcal{S}$, a feasible control can be formed from $\hat{S}$ by replacing $\hat{S}^{(2)}$ and $\hat{S}^{(12)}$ by zero, yielding $\Delta x = A^{(1)}\hat{S}^{(11)}\hat{P}^{(1)}x$. The remainder of the proof will utilize the following result: If an $N \times N$ matrix A is non-singular, then there exists a permutation matrix W such that AW has non-zero principal minors and diagonal elements. The result is established by induction. Note that AW is a permutation of the columns of A, and suppose that columns $i+1,\cdots,N$ have been chosen so that the left-hand principal minors of order $i+1,\cdots,N$ are non-zero. The $i+1$ order principal minor can be expanded in terms of the elements of its last row. Since the $i+1$ order minor is non-zero, at least one element of this row and its corresponding i order minor must be non-zero. Choose the column containing this non-zero element to be the ith column of the permuted matrix. The corresponding minor then becomes the non-zero i order principal minor.

We now complete the proof of the theorem. By the result above, there exists a permutation matrix W such that $\hat{S}^{(11)}W$ has non-zero diagonal elements. Then, by the assumption on $\mathcal{S}$, a feasible control strategy can be obtained from $\hat{S}W$ by reducing the magnitude of off-diagonal elements to zero. Denote the resulting matrix by L. The dynamic system can now be written $\Delta v = CLv$, where $v = W'\hat{P}^{(1)}x$ and $C = W'\hat{P}^{(1)}A$. Again applying the result above, there exists a permutation matrix $\hat{W}$ such that $\hat{W}A$ has non-zero principal minors. By construction and the properties of $\hat{P}^{(1)}$, this matrix is a permutation matrix, and by the unrestricted assignment assumption, it can be replaced by any other $N \times N$ permutation matrix. Replacing $\hat{P}^{(1)}$ by $W\hat{W}$, the system satisfies the hypotheses of Theorem 1, and controllability follows.

Q.E.D.

This theorem establishes, for one simple case, a condition under which an information structure can be selected to make a system controllable. The question of controllability can be posed for more general economic systems and more complex sets of feasible information transformers, as at the end of Section 3. It would also be of interest to explore the conditions under which reorganization of institutions can lead to controllability. (One might view alternative organizations as the choice of different sets of feasible strategies $\mathscr{S}$ from a specified class.)

A second class of questions can be posed on optimal control when the information transformer is a decision variable and there is an economic cost associated with each information structure. What degree of decentralization is optimal? What criterion must a "marginal" signal meet to be included in the optimal information transformer? Answers to these questions in concrete macroeconomic policy models, particularly under the actual conditions of signal "noise", exogenous trends, and non-linearity of the system, would provide a useful complement to the abstract treatments of these problems cited at the beginning of this section.

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