Journal of Mathematical Economics 2 (1975) 225–242. © North-Holland Publishing Company

TCHEBYSCHEFF BOUNDS FOR THE SPACE OF AGENT CHARACTERISTICS*

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1. Introduction

The concept of a measure space of agents has been used to deduce implications for the structure of demand and equilibrium in an economy. Conversely, properties of demand and equilibrium contain implications for the distribution of agents' characteristics. We are concerned with these converse implications. A precedent for our interest in these problems is provided by the classical economic theory of revealed preference for a single consumer. Jointly with Professor M.K. Richter, the author has previously generalized classical revealed preference axioms to the case of market data generated by a population of agents [McFadden and Richter (1971), McFadden (1973a, 1973b)]. This analysis was concerned with the *existence* of a measure space of agent characteristics consistent with the market data. Recently, Professor R.E. Hall has explored in the context of an empirical application the problem of bounding the consistent measures on the space of agent characteristics [Hall (1973)]. His ingenious application of the results of Krein (1959) and others on Tchebyscheff systems has suggested that analogous results might be obtained in a more abstract context. We take this approach, starting in section 2 with a restatement of the classical moment problem of Tchebyscheff and Markov. Section 3 lists some basic properties of the space of agent characteristics. Section 4 formulates the questions of existence and limiting values. Section 5 gives conditions for existence. Section 6 gives limiting values for an integral. Section 7 gives limiting values for the probability on a subset of agent characteristics. Section 8 sharpens the results of the previous section in the case of a finite-dimensional space of observations. Section 9 gives an application. Some open questions are listed in section 10.

*Presented at the Mathematical Social Science Board Colloquium on Mathematical Economics in August 1974 at the University of California, Berkeley. This research was supported by NSF grant SOC72-05551. I have benefited from discussions with R. Hall, A. Mas-Colell and M.K. Richter, but claim sole responsibility for errors.

2. The classical moment problem of Tchebyscheff and Markov

We paraphrase Krein's (1959) formulation of the problem of Markov. Let T be a closed interval $[a, b]$ in the real line. Let $h: T \to \mathbf{R}^l$ and $f: T \to \mathbf{R}$ be continuous functions, and $H \in \mathbf{R}^l$ be a specified vector. For two measures ν and η on T, define $\nu \leqq \eta$ to mean

$$\int_T f(t)\nu(\mathrm{d}t) \leqq \int_T f(t)\eta(\mathrm{d}t),$$

for all continuous non-negative functions $f: T \to \mathbf{R}$.[1] Suppose μ_- and μ_+ are non-negative measures on T satisfying

$$0 \leqq \mu_-(T) < 1 < \mu_+(T) \quad \text{and} \quad \mu_- \leqq \mu_+. \tag{1}$$

First problem. Does there exist a measure μ satisfying

$$H = \int_T h(t)\mu(\mathrm{d}t), \tag{2}$$

and

$$\mu_- \leqq \mu \leqq \mu_+, \qquad \mu(T) = 1. \tag{3}$$

Second problem. Suppose the previous problem has a solution. What are the limiting values F_- (infimium) and F_+ (supremium) of the integral

$$F = \int_T f(t)\mu(\mathrm{d}t), \tag{4}$$

for the μ satisfying (2) and (3).

These problems have been analyzed in detail by Krein (1959), Krein–Nydepman (1973), Akhiezer (1962), Akhiezer–Krein (1965), Karlin (1968), Karlin–Studden (1966), Mallow (1963), and Shohat (1943), particularly in the case that the functions h and f have the monotonicity properties necessary to define a 'Tchebyscheff system'. We shall be concerned only with the basic structure of these problems and the most elementary geometry used in their analysis. There is not to my knowledge a satisfactory formulation of Tchebyscheff systems in infinite-dimensional spaces which would allow us to carry over sharper results in our application.

3. The space of agent characteristics

Debreu (1969), Hildenbrand (1974) and Ichiishi (1974) have provided descriptions of the space of agent characteristics to which we adhere, except for assump-

[1]It is elementary that this implies $\nu(A) \leqq \eta(A)$ for each Borel set $A \subseteq T$. A measure on a compact metric space is regular. Hence, for $\varepsilon > 0$ there exist $K \subseteq A \subseteq U$ with K closed, U open, and $\nu(U \setminus K) < \varepsilon$, $\eta(U \setminus K) < \varepsilon$. Let f be a continuous function satisfying $f(T) \subseteq [0,1]$, $f(t) = 1$ for $t \in K$, and $f(t) = 0$ for $t \in T \setminus U$. Then $\nu(K) \leqq \int_T f(t)\nu(\mathrm{d}t) \leqq \int_T f(t)\eta(\mathrm{d}t) \leqq \eta(U)$, implying $\nu(A) \leqq \eta(A) + 2\varepsilon$. Hence, $\nu(A) \leqq \eta(A)$.

tions of a compact consumption set (to avoid auxiliary truncation or compactification arguments) and strictly convex preferences (to ensure continuous demand functions). An economy has l commodities, and the consumption set of each agent is a compact rectangle,

$$C = \{x \in \mathbf{R}^l \mid 0 \leqq x \leqq c\},$$

with c a strictly positive vector in $\mathbf{R}^l$.

A preference relation for an agent is defined by a closed set $r \subseteq C \times C$ with the properties

(i) [Completeness] $(\forall x, y \in C) \quad (x, y) \notin r \Rightarrow (y, x) \in r$;

(ii) [Transitivity] $(\forall x, y, z \in C)$

$$(x, y) \in r \quad \text{and} \quad (y, z) \in r \Rightarrow (x, z) \in r;$$

(iii) [Non-satiation] $(\forall x \in C) \quad x \neq c \Rightarrow (x, c) \notin r$;

(iv) [Strict convexity] $(\forall x, y \in C) \quad (\forall \theta \in (0, 1)) \quad x \neq y$ and

$$(x, y) \in r \Rightarrow (y, \theta x + (1-\theta)y) \notin r.$$

The class of these preference relations is denoted $\mathscr{P}(C \times C)$. Endowed with the Hausdorff set metric, $\mathscr{P}(C \times C)$ is a separable metric space.

The endowment of an agent is a vector ω in the interior C^0 of C. The space of agent characteristics is the product space $C^0 \times \mathscr{P}(C \times C)$ of pairs $t = (\omega, r)$ specifying the endowment and preferences of an agent. We endow this space with the product of the relative Euclidean topology for C^0 and the Hausdorff set metric topology for $\mathscr{P}(C \times C)$. Then the space of agent characteristics is a separable metric space.

Let S denote the non-negative unit simplex in $\mathbf{R}^l$, and define the demand function

$$h : S \times C^0 \times \mathscr{P}(C \times C) \rightarrow C$$

by the condition that $h(p, \omega, r)$ equal the unique r-maximal element in the set

$$\{x \in C \mid p \cdot x \leqq p \cdot \omega\}.$$

It is a standard result that h is continuous on $S \times C^0 \times \mathscr{P}(C \times C)$.

4. Existence and limiting value problems for the space of agent characteristics

We reformulate the classical moment problem for the space of agent characteristics. Consider a compact subset T of $C^0 \times \mathscr{P}(C \times C)$, let $t = (\omega, r)$ denote an element of T. Let S denote the non-negative unit simplex in $\mathbf{R}^l$. Let $h: S \times T \to \mathbf{R}^l$ and $\tilde{H}: S \to \mathbf{R}^l$ be continuous functions, and let $\tilde{f}: T \to \mathbf{R}^l$ be a bounded measurable function. Suppose μ_- is a non-negative measure on T. Suppose μ_+ is either a measure on T, or the content satisfying $\mu_+(A) = +\infty$ for $A \neq \emptyset$. [In the latter case, the constraint $\mu \leqq \mu_+$ in eq. (6) is absent.] Suppose μ_- and μ_+ satisfy

$$0 \leqq \mu_-(T) < 1 < \mu_+(T) \leqq +\infty \quad \text{and} \quad \mu_- \leqq \mu_+. \tag{5}$$

Existence problem. Find conditions for the existence of a probability measure μ satisfying

$$\mu(T) = 1, \quad \mu_- \leqq \mu \leqq \mu_+, \quad \text{and}$$

$$(\forall p \in S) \quad \tilde{H}(p) = \int_T h(p, t)\mu(\mathrm{d}t). \tag{6}$$

Limiting value problem. Find limiting values F_- (infimium) and F_+ (supremium) of the integral

$$\tilde{F} = \int_T \tilde{f}(t)\mu(\mathrm{d}t), \tag{7}$$

for μ satisfying the existence problem.

Following Krein, we first transform these problems to a simpler form. Define

$$v = (\mu - \mu_-)/(1 - \mu_-(T)), \tag{8}$$

and

$$v_+ = (\mu_+ - \mu_-)/(1 - \mu_-(T)). \tag{9}$$

Then eq. (6) can be written

$$(\forall p \in S) \quad H(p) = \int_T h(p, t)v(\mathrm{d}t), \tag{10}$$

where

$$H(p) \equiv \frac{\tilde{H}(p) - \int_T h(p, t)\mu_-(\mathrm{d}t)}{1 - \mu_-(T)}.$$

Eq. (5) becomes

$$v \leqq v_+ \quad \text{and} \quad v(T) = 1, \tag{11}$$

where either v_+ is a bounded measure or the constraint $v \leqq v_+$ is absent. We assume hereafter that T is the support of v_+.

The existence problem is to find conditions under which a measure v exists satisfying eq. (10) and eq. (11). The limiting value problem is to bound the integral

$$F = \int_T f(t)\nu(\mathrm{d}t), \tag{12}$$

where

$$f(t) = (1-\mu_-(T))\tilde{f}(t) \tag{13}$$

and

$$\tilde{F} = F + \int_T \tilde{f}(t)\mu_-(\mathrm{d}t). \tag{14}$$

We shall consider only the transformed problems.

5. The existence problem

We employ elementary convexity arguments to establish the following result; analogous arguments have been used by Krein (1959), Fan (1956), Freedman and Purves (1969), and McFadden and Richter (1971).

Theorem 1. There exists a probability measure ν *satisfying eqs.* (10) *and* (11) *if and only if for each finite sequence* $\{p_1, \ldots, p_n\} \subseteq S$ *and column vectors* $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbf{R}^l$,

$$\sum_{i=1}^{n} \lambda_i' H(p_i) \leqq \underset{\nu \leqq \nu_+}{\text{Max.}} \sum_{i=1}^{n} \lambda_i' \int_T h(p_i, t)\nu(\mathrm{d}t). \tag{15}$$

Corollary. If the condition of eq. (11) *is absent, then eq.* (15) *becomes*

$$\sum_{i=1}^{n} \lambda_i' H(p_i) \leqq \underset{t \in T}{\text{Max.}} \sum_{i=1}^{n} \lambda_i' h(p_i, t). \tag{16}$$

Proof. Consider the normal linear space $\mathscr{C}(S, \mathbf{R}^l)$ of continuous functions from S into $\mathbf{R}^l$. Define the set

$$A = \{H(\cdot) - \int_T h(\cdot, t)\nu(\mathrm{d}t) \mid 0 \leqq \nu \leqq \nu_+ \quad \text{and} \quad \nu(T) = 1\}. \tag{17}$$

Clearly A is a convex subset of $\mathscr{C}(S, \mathbf{R}^l)$. We next argue that A is compact. Since T is compact, the space $\mathscr{M}(T)$ of probability measures on T is compact in the topology of weak convergence [Parthasarathy (1967, Theorem 6.4)]. Consider the subset

$$M = \{\nu \in \mathscr{M}(T) \mid \nu \leqq \nu_+\}.$$

If the constraint $\nu \leqq \nu_+$ is absent, the conclusion follows. Otherwise, ν_+ is a bounded measures, and is therefore regular [Dunford and Schwartz (1964, III.9.22)]. Let ν_α be a net in M converging weakly to ν_0. For each Borel set $V \subseteq T$ and $\varepsilon > 0$, there exists an open set U containing V such that $\nu_+(U \backslash V) < \varepsilon$. Then

$$\nu_+(V)+\varepsilon \geqq \nu_+(U) \geqq \lim_\alpha \nu_\alpha(U) \geqq \nu_0(U) \geqq \nu_0(V), \tag{18}$$

the third inequality following from Billingsley (1968, Theorem 2.1). Hence, $\nu_+ \geqq \nu_0$, implying $\nu_0 \in M$ and M closed, hence compact. Since

$$\int_T h(\cdot, t)\nu(\mathrm{d}t)$$

is a continuous linear map from $\mathscr{M}(T)$ into $\mathscr{C}(S, \mathbf{R}^l)$, it follows [Kelley (1955, Theorem 8, p. 141)] that

$$\{\int_T h(\cdot, t)\nu(\mathrm{d}t) \mid \nu \in M\}$$

is compact. Hence, A is compact.

A separating hyperplane theorem [Dunford and Schwartz (1964, Theorem V.2.10)] establishes that the origin of $\mathscr{C}(S, \mathbf{R}^l)$ is not contained in A if and only if there exists a continuous linear functional Γ on $\mathscr{C}(S, \mathbf{R}^l)$ and a scalar $\varepsilon > 0$ such that

$$2\varepsilon \leqq \min_{y \in A}. \Gamma y.$$

Suppose separation is possible. By the Riesz representation theorem [Dunford and Schwartz (1964, Theorem IV.6.3)], there exists a regular countably additive set function γ from S into $\mathbf{R}^l$ such that

$$\Gamma y = \int_S y(p)\gamma(\mathrm{d}p), \tag{19}$$

for all $y \in \mathscr{C}(S, \mathbf{R}^l)$. The proof of Parthasarathy (1967, Theorem 6.3) can be applied without modification to establish the existence of a sequence of set functions γ_k with finite support converging weakly to γ. For k sufficiently large, the equicontinuity of the functions $y \in A$ implies

$$\varepsilon \leqq \min_{y \in A}. \int_S y(p)\gamma_k(\mathrm{d}p).$$

Choose k satisfying this condition, and let $(p_1, \ldots, p_n) \subseteq S$ be the locations of the point masses and $(\lambda_1, \ldots, \lambda_n)$ the vector weights. Then,

$$\varepsilon \leqq \min_{y \in A} . \sum_{i=1}^{n} y(p_i)\lambda_i,$$

or

$$\sum_{i=1}^{n} \lambda_i' H(p_i) > \max_{v \in M} . \sum_{i=1}^{n} \lambda_i' \int_T h(p_i, t) v(\mathrm{d}t). \tag{20}$$

Since this condition holds for some λ_i, p_i if and only if the existence problem has no solution, the theorem is proved.

The corollary follows by noting that when $M = \mathscr{M}(T)$, the right-hand side of eq. (20) is maximized when v is a point mass at a maximand of

$$\sum_{i=1}^{n} \lambda_i' h(p_i, t). \tag{21}$$

Q.E.D.

6. Limiting values for a continuous integrand

We consider the reduced problem of eq. (10) and eq. (11), or

$$v \in \mathscr{M}(T), \quad v \leqq v_+, \quad \text{and}$$

$$H(\cdot) = \int_T h(\cdot, t) v(\mathrm{d}t). \tag{22}$$

We assume the set of v satisfying eq. (22) to be non-empty. Consider a continuous function $f \colon T \to \mathbf{R}$, and

$$F = \int_T f(t) v(\mathrm{d}t). \tag{23}$$

We let $\mathscr{F}$ denote the set of values of the integral (23) attained by v satisfying eq. (22). We established in the proof of Theorem 1 that the set of $v \in \mathscr{M}(T)$ with $v \leqq v_+$ is compact in the weak topology. Then, $\mathscr{F}$ is compact. We seek the limiting values

$$F_- = \min . \mathscr{F} \quad \text{and} \quad F_+ = \max . \mathscr{F}. \tag{24}$$

Consider the augmented system of equations

$$\begin{aligned} &v \in \mathscr{M}(T), \quad v \leqq v_+, \\ &H(\cdot) = \int_T h(\cdot, t) v(\mathrm{d}t), \\ &F = \int_T f(t) v(\mathrm{d}t), \end{aligned} \tag{25}$$

for specified F. This system has a solution if and only if $F \in \mathscr{F}$. Hence, we can use the existence condition of Theorem 1 applied to the augmented system to obtain the limiting values.

Theorem 2. Suppose the set $\mathscr{F}$ is non-empty. Then,

$$F_+ = \inf_{\substack{\{p_1,\dots,p_n\} \subseteq S \\ \{\lambda_1,\dots,\lambda_n\} \subseteq \mathrm{R}^l \\ n=1,2,\dots}} \max_{\substack{\nu \in \mathscr{M}(T) \\ \nu \leqq \nu_+}} \left\{ \sum_{i=1}^{n} \lambda_i'\left(\int_T h(p_i, t)\nu(\mathrm{d}t) - H(p_i)\right) + \int_T f(t)\nu(\mathrm{d}t) \right\}, \tag{26}$$

$$F_- = \sup_{\substack{\{p_1,\dots,p_n\} \subseteq S \\ \{\lambda_1,\dots,\lambda_n\} \subseteq \mathrm{R}^l \\ n=1,2,\dots}} \min_{\substack{\nu \in \mathscr{M}(T) \\ \nu \leqq \nu_+}} \left\{ \sum_{i=1}^{n} \lambda_i'\left(\int_T h(p_i, t)\nu(\mathrm{d}t) - H(p_i)\right) + \int_T f(t)\nu(\mathrm{d}t) \right\}. \tag{27}$$

Proof. Let

$$L(\nu, (p_i), (\lambda_i), \lambda_0) = \sum_{i=1}^{n} \lambda_i'\left(\int_T h(p_i, t)\nu(\mathrm{d}t) - H(p_i)\right) + \lambda_0 \int f(t)\nu(\mathrm{d}t). \tag{28}$$

From Theorem 1, $F \in \mathscr{F}$ if and only if for all λ_0, (p_i), (λ_i),

$$\max. L(\nu, (p_i), (\lambda_i), \lambda_0) - \lambda_0 F \geqq 0. \tag{29}$$

Taking $\lambda_0 = 1$ implies

$$F_+ \leqq \inf_{(p_i),(\lambda_i)} \max_{\nu} L(\nu, (p_i), (\lambda_i), 1). \tag{30}$$

If $F > F_+$, then there exists λ_0, (λ_i), (p_i) such that

$$\max_{\nu} L(\nu, (p_i), (\lambda_i), \lambda_0) - \lambda_0 F < 0. \tag{31}$$

For the same λ_0, (λ_i), (p_i),

$$\max_{\nu} L(\nu, (p_i), (\lambda_i), \lambda_0) - \lambda_0 F_+ \geqq 0. \tag{32}$$

Subtracting,

$$\lambda_0(F-F_+) > 0, \tag{33}$$

implying $\lambda_0 > 0$. Normalize $\lambda_0 = 1$. Then eq. (31) implies equality in eq. (30), and eq. (26) holds. A similar argument establishes eq. (27). Q.E.D.

The limiting value F_+ (or F_-) in the preceding theorem can also be obtained as a solution of a linear program or its dual:

Corollary. Suppose $\mathscr{F}$ is non-empty. Then

$$F_+ = \max_{\nu\in \mathrm{ca}(T)} \int_T f(t)\nu(\mathrm{d}t) \tag{34a}$$

subject to

$$(\forall p \in S) \quad \int_T h(p,t)\nu(\mathrm{d}t) = H(p), \tag{34b}$$

$$\int_T \nu(\mathrm{d}t) = 1, \tag{34c}$$

$$\nu \leqq \nu_+, \tag{34d}$$

$$\nu \geqq 0, \tag{34e}$$

and

$$F_+ = \inf_{\substack{\lambda\in \mathrm{ca}(S)\\ \alpha\in R\\ \beta\in\mathscr{C}(T)}} \left\{\int_S H(p)\lambda(\mathrm{d}p)+\alpha+\int_T \beta(t)\nu_+(\mathrm{d}t)\right\}, \tag{35a}$$

subject to

$$(\forall t \in T) \quad \alpha+\beta(t)+\int_S h(p,t)\lambda(\mathrm{d}p) \geqq f(t), \tag{35b}$$

$$\beta \geqq 0. \tag{35c}$$

Proof. The linear program in eq. (34) is a restatement of eq. (26). Therefore, it is only necessary to show that the linear program of eq. (35) is dual to that in eq. (34). It is possible to appeal to general results on programming in linear spaces to obtain this result [see Hurwicz (1958), Ioffe–Tikhomirov (1968)]; however, we give a simple direct argument.

Let

$$\mathscr{X} = R\times R\times\mathscr{C}(S,\mathbf{R}^l)\times \mathrm{ca}\,(T),$$

and

$$\mathscr{Y} = R\times R\times \mathrm{ca}\,(S,\mathbf{R}^l)\times\mathscr{C}(T).$$

Give R the Euclidean topology, $\mathscr{C}(S, \mathbf{R}^l)$ the sup norm topology, and $\mathrm{ca}(T)$ the topology of weak convergence. Then $\mathscr{Y}$ is the space of continuous linear functionals on $\mathscr{X}$ [Dunford and Schwartz (1964, Theorem V.3.9)]. Define a set $A \subseteq \mathscr{X}$ by

$$\begin{aligned} A = \{(r, s, g, \eta) \in R \times R \times \mathscr{C}(S, \mathbf{R}^l) \times \mathrm{ca}\,(T) \mid r \\ \leqq \textstyle\int_T f(t)\nu(\mathrm{d}t), \quad s = \nu(T), \\ (\forall p \in S) \quad g(p) = \textstyle\int_T h(p, t)\nu(\mathrm{d}t), \quad \eta \geqq \nu \quad \text{for some} \\ \nu \in \mathrm{ca}\,(T), \quad \nu \geqq 0\}. \end{aligned}$$

Clearly, A is a convex cone. Suppose a net $\{r_\theta, s_\theta, g_\theta, \eta_\theta)\} \subseteq A$ converges to $(\bar{r}, \bar{s}, \bar{g}, \bar{\eta}) \in \mathscr{X}$, and suppose $\{\nu_\theta\} \subseteq \mathrm{ca}\,(T)$ satisfies $\nu_\theta \geqq 0$, $r_\theta = \int_T f(t)\nu_\theta(\mathrm{d}t)$, etc. Since $\nu_\theta(T) = s_\theta$ converges to $\bar{s}$, there exists a compact set

$$M = \{\nu \in \mathrm{ca}\,(T) \mid \nu \geqq 0 \quad \text{and} \quad \nu(T) \leqq \sup_\theta s_\theta\},$$

such that $\nu_\theta \in M$. Then, $\{\nu_\theta\}$ has a subset converging weakly to $\bar{\nu} \in M$, implying $\bar{r} = \int_T f(t)\bar{\nu}(\mathrm{d}t)$,

$$\nu(T) = \bar{s},$$

$$\bar{g} = \int h(\cdot, t)\bar{\nu}(\mathrm{d}t).$$

Hence, $(\bar{r}, \bar{s}, \bar{g}, \bar{\nu}) \in A$. Also, $\eta_\theta \geqq \nu_\theta$ implies $\bar{\eta} \geqq \bar{\nu}$. Hence, $(\bar{r}, \bar{s}, \bar{g}, \bar{\eta}) \in A$, and A is closed.

As noted in the proof of Theorem 1, the set of ν satisfying the constraints in eq. (34) is compact; hence, there exists $\bar{\nu} \in \mathrm{ca}\,(T)$ satisfying (34b) to (34e) and

$$F_+ = \int_T f(t)\bar{\nu}(\mathrm{d}t).$$

The definition of F_+ implies the points $(F_+ + 1/n,\ 1,\ H,\ \nu_+)$ are not in A. Hence there exist continuous linear functionals $(\delta^n, -\alpha^n, -\lambda^n; -\beta^n) \in \mathscr{Y}$ strictly separating these points from A; i.e., for all $\nu, \eta \in \mathrm{ca}\,(T)$, $\eta \geqq \nu \geqq 0$,

$$\begin{aligned} &\delta^n\left(F_+ + \frac{1}{n}\right) - \alpha^n - \textstyle\int_S H(p)\lambda^n(\mathrm{d}p) - \int_T \beta^n(t)\nu_+(\mathrm{d}t) \\ &> 0 \\ &\geqq \delta^n \textstyle\int_T f(t)\nu(\mathrm{d}t) - \alpha^n \nu(T) - \int_S \int_T h(p, t)\nu(\mathrm{d}t)\lambda^n(\mathrm{d}p) - \int_T \beta^n(t)\eta(\mathrm{d}t). \end{aligned} \tag{36}$$

Taking η large implies $\beta^n \geqq 0$. Substituting $\bar{\nu}$ for ν and η implies

$$\delta^n/n > \int_T \beta^n(t)[\nu_+(\mathrm{d}t) - \bar{\nu}(\mathrm{d}t)] \geqq 0,$$

and

$$0 \geqq \delta^n F_+ - \alpha^n - \int_S H(p)\lambda^n(\mathrm{d}p) - \int_T \beta^n(t)\nu_+(\mathrm{d}t) \geqq -\delta^n/n.$$

Normalize $\delta^n = 1$. Then,

$$\lim_n . \int_T \beta^n(t)[\nu_+(\mathrm{d}t) - \bar{\nu}(\mathrm{d}t)] = 0,$$

and

$$F_+ = \lim_n . \{\alpha^n + \int_S H(p)\lambda^n(\mathrm{d}p) + \int_T \beta^n(t)\nu_+(\mathrm{d}t)\}.$$

Further, taking $\nu = \eta$ to be a point mass at t in eq. (36) implies

$$(\forall t \in T) \quad f(t) - \alpha^n - \int_S h(p, t)\lambda^n(\mathrm{d}p) - \beta^n(t) \leqq 0,$$

and eq. (35b) is satisfied. Since for $\lambda \in \mathrm{ca}(S)$, $\alpha \in R$, $\beta \in \mathscr{C}(T)$ satisfying eqs. (34b) and (34c), we have

$$\begin{aligned}
&\alpha + \int H(p)\lambda(\mathrm{d}p) + \int \beta(t)\nu_+(\mathrm{d}t) \\
&\quad \geqq \alpha + \int H(p)\lambda(\mathrm{d}p) + \int \beta(t)\nu_+(\mathrm{d}t) \\
&\qquad + \int_T \{f(t) - \alpha - \int h(p, t)\lambda(\mathrm{d}p) - \beta(t)\}\bar{\nu}(\mathrm{d}t) \\
&\quad = \int_T f(t)\bar{\nu}(\mathrm{d}t) - \alpha(1 - \bar{\nu}(T)) - \int_S [H(p) - \int_T h(p, t)\bar{\nu}(\mathrm{d}t)]\lambda(\mathrm{d}p) \\
&\qquad + \int_T \beta(t)[\nu_+(\mathrm{d}t) - \bar{\nu}(\mathrm{d}t)] \geqq F_+,
\end{aligned}$$

we have established that the linear program (35) has the solution F_+. Q.E.D.

There are tractable computation algorithms for the linear programs above only when the space S of observed budgets is finite; this case is discussed later. However, when ν_+ is a (bounded) measure, the ν_+-continuity of ν and the Radon–Nikodym theorem [Dunford and Schwartz (1964, III.10.2)] imply the existence of a unique function $\psi \in L_1(T, \nu_+)$ such that

$$0 \leqq \psi(t) \leqq 1 \quad (\text{a.e. } \nu_+),$$

and

$$(\forall \text{ Borel } A \subseteq T) \quad \nu(A) = \int_A \psi(t)\nu_+(\mathrm{d}t).$$

Then, the limiting value problem becomes an optimal control problem on T,

$$F_{+} = \max_{\psi}.\ \int_T f(t)\psi(t)\nu_{+}(\mathrm{d}t), \tag{37}$$

subject to

$$H(p) = \int_T h(p,t)\psi(t)\nu_{+}(\mathrm{d}t) \quad (\forall p \in S),$$

and

$$0 \leqq \psi(t) \leqq 1.$$

7. Bounds on probabilities of given sets

Consider the reduced problem summarized in eq. (22), and suppose now that we wish to bound the measure $\nu(V)$ of a Borel set $V \subseteq T$ for ν satisfying (22). Defining $f: T \to \mathbf{R}$ to be the indicator function of the set V, this problem is the same as that of the previous section, except for the continuity of f. Thus, one approach to this question is to approximate f by sequences of continuous functions and apply the preceding analysis. We take an alternative approach here which exploits directly the structure of the problem.

Given the set V, let W denote the complement of V relative to T. For any scalar $\delta \in [0, 1]$, define

$$N_\delta = \{\eta \in \mathcal{M}(V) \mid \delta\eta \leqq \nu_{+}\},$$

$$M_\delta = \{\xi \in \mathcal{M}(W) \mid (1-\delta)\xi \leqq \nu_{+}\}. \tag{38}$$

Then, the problem of eq. (22) can be written as

$$(\forall p \in S) \quad H(p) = \delta\int_V h(p,t)\eta(\mathrm{d}t) + (1-\delta)\int_W h(p,t)\xi(\mathrm{d}t), \tag{39}$$

$$\delta \in [0, 1], \quad \eta \in N_\delta, \quad \xi \in M_\delta, \tag{40}$$

with $\delta = \nu(V)$. Thus, the problem can be restated as that of determining bounds on the values of δ for which eq. (39) has a solution. The following result gives bounds which may not be sharp.

Theorem 3. Define $D \subseteq [0, 1]$ to be the set of scalars δ such that for each finite sequence $\{p_1, \ldots, p_n\} \subseteq S$ and $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbf{R}^l$,

$$\sum_{i=1}^{n} H(p_i)\lambda_i \leqq \delta \sup_{\eta \in N_\delta}.\ \sum_{i=1}^{n} \lambda_i' \int_V h(p_i,t)\eta(\mathrm{d}t)$$

$$+(1-\delta) \sup_{\xi \in M_\delta}.\ \sum_{i=1}^{n} \lambda_i' \int_W h(p_i,t)\xi(\mathrm{d}t), \tag{41}$$

and

$$\sum_{i=1}^{n} H(p_i)\lambda_i \geqq \delta \inf_{\eta \in N_\delta} \sum_{i=1}^{n} \lambda_i' \int_V h(p_i, t)\eta(\mathrm{d}t)$$

$$+(1-\delta) \inf_{\xi \in M_\delta} \sum_{i=1}^{n} \lambda_i' \int_W h(p_i, t)\xi(\mathrm{d}t). \tag{42}$$

Then, each $\delta \in [0, 1]$ *such that eq.* (39) *and eq.* (40) *have a solution which satisfies*

$$\inf. D \leqq \delta \leqq \sup. D. \tag{43}$$

Proof. For $\delta \in [0, 1]$, define the set

$$A_\delta = \{H(\cdot)-\delta\int_V h(\cdot, t)\eta(\mathrm{d}t)$$
$$-(1-\delta)\int_W h(\cdot, t)\xi(\mathrm{d}t) \mid \eta \in N_\delta, \xi \in M_\delta\}.$$

Then A_δ is a convex subset of $\mathscr{C}(S, \mathbf{R}^l)$. If eq. (39) has a solution for a given δ, then the origin $0 \in A_\delta$, implying 0 cannot be strictly separated from A_δ. Then, for any continuous linear functional $\lambda \in \mathrm{ca}\,(S, \mathbf{R}^l)$,

$$\int_S H(p)\lambda(\mathrm{d}p) \leqq \delta \sup_{\eta \in N_\delta} \int_S \int_V h(p, t)\eta(\mathrm{d}t)\lambda(\mathrm{d}p)$$

$$+(1-\delta) \sup_{\xi \in M_\delta} \int_S \int_W h(p, t)\xi(\mathrm{d}t)\lambda(\mathrm{d}s). \tag{44}$$

Since the set of functions in $\mathrm{ca}(S, \mathbf{R}^l)$ with finite support is dense, this inequility is equivalent to eq. (41). A similar argument establishes that eq. (42) is satisfied. Therefore eq. (43) holds. Q.E.D.

Since the sets A_δ constructed in the proof above are not in general closed, the bounds obtained are not necessarily sharp. However, a corollary provides an important case where sharpness is guaranteed.

Corollary. Suppose V is a continuity set of V_+. Then, for each δ satisfying

$$\inf. D \leqq \delta \leqq \sup. D,$$

there exists a solution to eq. (39) *and eq.* (40).

Proof. We first consider a related problem. Let $\bar{V}$, $\bar{W}$ denote the closures of V, W, and define

$$\bar{N}_\delta = \{\eta \in \mathscr{M}(\bar{V}) \mid \delta\eta \leqq \nu_+\} \quad \text{and}$$

$$\bar{M}_\delta = \{\xi \in \mathscr{M}(\bar{W}) \mid (1-\delta)\xi \leqq \nu_+\}.$$

Define the set

$$A_\delta = \{H(\cdot)-\delta\textstyle\int_V h(\cdot, t)\eta(\mathrm{d}t)-(1-\delta)\int_W h(\cdot, t)\xi(\mathrm{d}t) \mid \eta \in \bar{N}_\delta,$$

$$\xi \in \bar{M}_\delta\}.$$

By the same arguments as in the proof of Theorem 2, $\bar{A}_\delta$ is convex and compact. Hence, a necessary and sufficient condition for the equation

$$H(\cdot) = \delta\textstyle\int_V h(\cdot, t)\eta(\mathrm{d}t)+(1-\delta)\int_W h(\cdot, t)\xi(\mathrm{d}t) \tag{45}$$

to have a solution is that for each $(p_1, \ldots, p_n) \subseteq S$ and $(\lambda_1, \ldots, \lambda_n) \subseteq \mathbf{R}^l$,

$$\sum_{i=1}^{n} \lambda_i H(p_i) \leqq \delta \max_{\eta\in \bar{N}_\delta}. \sum_{i=1}^{n} \lambda_i' \textstyle\int_V h(p_i, t)\eta(\mathrm{d}t)$$

$$+(1-\delta) \max_{\xi\in \bar{M}_\delta}. \sum_{i=1}^{n} \lambda_i \textstyle\int_W h(p_i, t)\xi(\mathrm{d}t). \tag{46}$$

Now, since V is a continuity set of ν_+, $\nu_+(\bar{V}\backslash V^0) = \nu_+(\bar{W}\backslash W^0) = 0$. Therefore, $\eta(V) = 1$ for all $\eta \in \bar{N}_\delta$, and $\xi(W) = 1$ for all $\xi \in \bar{M}_\delta$. Hence, eq. (45) becomes

$$H(\cdot) = \delta\textstyle\int_V h(\cdot, t)\eta(\mathrm{d}t)+(1-\delta)\int_W h(\cdot, t)\xi(\mathrm{d}t), \tag{47}$$

and eq. (46) becomes

$$\sum_{i=1}^{n} \lambda_i H(p_i) \leqq \delta \max_{\eta\in \bar{N}_\delta}. \sum_{i=1}^{n} \lambda_i \textstyle\int_V h(p_i, t)\eta(\mathrm{d}t)$$

$$+(1-\delta) \max_{\xi\in \bar{M}_\delta}. \sum_{i=1}^{n} \lambda_i \textstyle\int_W h(p_i, t)\xi(\mathrm{d}t)$$

$$\leqq \delta \sup_{n\in N_\delta}. \sum_{i=1}^{n} \lambda_i \textstyle\int_V h(p_i, t)\eta(\mathrm{d}t)$$

$$+(1-\delta) \sup_{\xi\in M_\delta}. \sum_{i=1}^{n} \lambda_i \textstyle\int_W h(p_i, t)\xi(\mathrm{d}t). \tag{48}$$

Hence, in this case the bound is necessary and sufficient for the original problem. Q.E.D.

8. Sharpenings when the number of observations is finite

The characterizations of measures achieving limiting values can be sharpened when the observations form a finite vector. Suppose $S = \{p_1, \ldots, p_K\}$ is a finite set, and define

$$H = \begin{bmatrix} H(p_1) \\ \vdots \\ H(p_K) \end{bmatrix} \quad \text{and} \quad h(t) = \begin{bmatrix} h(p_1, t) \\ \vdots \\ h(p_K, t) \end{bmatrix},$$

of dimension $N = K \cdot l$. Consider the problem

$$\max_{\nu \in M} . \int_T f(t)\nu(\mathrm{d}t),$$

subject to

$$H = \int_T h(t)\nu(\mathrm{d}t),$$

$M = \{\nu \in \mathcal{M}(T) \mid \nu \leqq \nu_+\}$, where f is continuous. From the proof of Theorem 2, M is compact. Consider the continuous linear map $\phi : M \to \mathbf{R}^{N+1}$ defined by

$$\begin{pmatrix} y \\ x \end{pmatrix} = \phi(\nu) = \begin{bmatrix} \int_T f(t)\nu(\mathrm{d}t) \\ \int_T h(t)\nu(\mathrm{d}t) \end{bmatrix}.$$

Then $\phi(M)$ is a convex compact set in $\mathbf{R}^{N+1}$. It is elementary to show that each extreme point in $\phi(M)$ is the image of an extreme point in M. Further, the maximum value of y for $(y, x) \in \phi(M)$ satisfying $x = H$ is a boundary point of $\phi(M)$, and can hence be written as a convex combination of at most $N+1$ extreme points of $\phi(M)$. Define a basis to consist of $N+1$ distinct extreme points of M. Then, a solution to the problem can be attained among the set of bases.

An extreme point of M is a measure ν characterized by a Borel set $V \subseteq T$, with $\nu(W) = \nu_+(W)$ for $W \subseteq T$ and $\nu(W) = 1$. In the case that the restriction $\nu \leqq \nu_+$ is non-binding, these extreme measures will be unit weights at single points. Then, the structure of the problem can be utilized to attain computational bounds. In particular, the linear program in eq. (34) has a finite basis, and it is possible to adapt the simplex algorithm to give an efficient procedure for improving bases and bounding solutions.

9. An example

Consider a two-good exchange economy with a continuum of consumers with identical endowments of one unit of each commodity. Suppose the first commodity is indivisible, so that the net trades in this commodity available to any consumer are -1, 0, $+1$, $+2$, etc., up to the largest integral purchase in the budget constraint. Two budgets are observed. At the first, only the trades -1, 0 are possible and a mean trade of -0.2 is observed. At the second, the trades -1, 0, $+1$, $+2$ are possible, and a mean trade of 1.0 is observed. It is desired to bound the possible mean trade when the budget makes the trades -1, 0, 1 possible.

The space T of agent characteristics is in this application an eight-element set enumerating the possible rankings of these trades by strictly convex preferences, as follows:

(1)	-1,	0,	1,	2,	v_1,
(2)	2,	1,	0	-1,	v_2,
(3)	0,	-1,	1,	2,	v_3,
(4)	0,	1,	-1,	2,	v_4,
(5)	0,	1,	2,	-1,	v_5,
(6)	1,	2,	0,	-1,	v_6,
(7)	1,	0,	2,	-1,	v_7,
(8)	1,	0,	-1,	2,	v_8.

The conditions for consistency with the observations are then

$$-0.2 = -v_1,$$

$$1.0 = -v_1+2v_2+v_6+v_7+v_8,$$

where v_i are the probability weights, and the expression to be bounded is

$$F = -v_1+v_2+v_6+v_7+v_8.$$

Maximizing or minimizing this expression is a linear programming problem whose dual corresponds to the bounds given in Theorem 2. In this example, the solutions by inspection are

$$F_+ = 0.6, \quad \text{achieved by} \quad v_1 = 0.2, \quad v_6 = 0.8,$$

$$F_- = 0.2, \quad \text{achieved by} \quad v_1 = 0.2, \quad v_2 = 0.4, \quad v_3 = 0.4.$$

A second example is given in McFadden (1974).

10. Open questions

This paper has applied only elementary methods to the problem of limiting values of integrals over the space of agent characteristics. In the case of finite vectors of observations, much sharper characterizations of extremands can be obtained, and these greatly facilitate empirical application. A general open question is whether analogous techniques can be applied on the space of agent characteristics. In particular, are there structures in which optimal control methods are useful?

Empirical experience suggests that bounds such as those in Theorem 2 can be approximated with reasonable accuracy employing relatively few alternative (p_i), (λ_i) vectors. Can approximation theorems be established to estimate rate of convergence to the exact bounds?

The analysis in this note has been applied solely to strictly convex preferences yielding continuous demand functions. Because this set of preference relations is not compact in the Hausdorff set metric topology, we restricted our attention to a compact subset of these preferences. It would be desirable to consider the entire class of convex acyclic preferences, since this set is compact. However, this would require that we consider the space $\mathscr{H}(T)$ of upper hemicontinuous closed convex-valued correspondences on T and the $\mathscr{H}(T)$-topology of the space of measures on T. Can analogues of the standard theorems on $\mathscr{C}(T)$ and $\mathscr{M}(T)$ be established for this space?

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