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EXISTENCE CONDITIONS FOR THEIL-TYPE PREFERENCES

I. Professor H. Theil[1] has suggested a differential specification of a system of demand equations which he adopts for empirical analysis. This note establishes conditions for the existence of these demand functions, and for their consistency with utility maximization. The following points are made:

(1) The only system of demand functions which is consistent with the differential specification globally (for all positive prices and incomes) is a system of Cobb-Douglas demand functions, derived by maximization of log-additive utility.

(2) In the case of three or more goods whose demands are not identically zero in a neighborhood of a specified price-income point, the only system of demand functions which is consistent with the differential specification locally (in this neighborhood) is a system of Cobb-Douglas demand functions, derived by maximization of log-additive utility.

(3) In the case that exactly two goods have demands which are not identically zero in a neighborhood of a specified price-income point, a system of demand equations slightly more general than Cobb-Douglas demand functions are possible, and can be derived by utility maximization. The Slutsky terms, Hicksian (income-compensated) demand functions, and expenditure function are displayed for this class.

(4) In all the above cases, the existence of a system of demand functions requires strong restrictions on Theil's "specific substitution effect" parameters.

II. Consider n goods, indexed $i=1,\ldots,n$. Let m denote income, p_i denote the price of good i, and q_i the quantity demanded of good i. Professor Theil makes the following differential specification of the system of demand equations

$$(6.7) \quad w_i d(\log q_i) = B_i \left[d(\log m) - \sum_{k=1}^{n} w_k \, d(\log p_k)\right] + \sum_{j=1}^{n} C_{ij} \left[d(\log p_j) - \sum_{k=1}^{n} B_k \, d(\log p_k)\right]$$

For $i=1,\ldots,n$; where the B_i and C_{ij} are constants satisfying

$$(6.5) \quad C_{ij} = C_{ji}$$

$$(6.3) \quad \sum_{i=1}^{n} B_i = 1$$

$$(6.6) \quad \sum_{j=1}^{n} C_{ij} = \phi B_i \qquad (\phi \text{ constant}).$$

The system (6.7) reduces, using (6.6), to the system of partial differential equations

$$(1) \quad \partial q_i / \partial m = B_i / p_i \qquad i=1,\ldots,n$$

$$(2) \quad \partial q_i / \partial p_j = - B_i q_j / p_i + [C_{ij} - \phi B_i B_j] m / p_i p_j \qquad i,j=1,\ldots,n$$

III. Solutions of (1) must be of the functional form

(3) $$q_i = B_i\, m / p_i + b_i, \qquad i=1,\ldots,n,$$

where b_i is a function of prices.

Consider the requirement that the differential specification hold for all positive prices and incomes. In this case, since for given positive prices, demand will approach zero when income approaches zero, the specification (3) requires $b_i = 0$. Then, the resulting demand system

(4) $$q_i = B_i m / p_i \qquad i=1,\ldots,n,$$

requires $B_i \geqq 0$ for all goods, and is of the Cobb-Douglas form. Such demand functions derive from maximization of a utility function

(5) $$u = \sum_{i=1}^{n} B_i \,(\log q_i).$$

IV. We now consider the question of the existence of the demand functions locally. Substitution of (3) into (2) for j=i yields

$$- B_i m / p_i^2 + \partial b_i / \partial p_i = - B_i [B_i m / p_i + b_i] / p_i + [C_{ii} - \phi B_i^2] m / p_i^2$$

or

(6) $$\partial b_i / \partial p_i = - B_i b_i / p_i + [B_i + C_{ii} - (1 + \phi) B_i^2] m / p_i^2.$$

Now, (6) is independent of income m in the neighborhood of the specified price-income point. Hence, the last term in (6) must be zero, or

(7) $$C_{ii} = -B_i + (1 + \phi)B_i^2.$$

Then, (6) becomes

(8) $$\partial b_i \big/ \partial p_i + B_i \, b_i \big/ p_i = 0.$$

Solutions of (8) must have the functional form

(9) $$b_i = a_i \, p_i^{-B_i}, \qquad i=1,\ldots,n,$$

where a_i is a function of all prices except p_i.

Substituting (9) and (3) into (2) for $j \neq i$ yields

$$p_i^{-B_i} (\partial a_i \big/ \partial p_j) = - B_i [B_j m \big/ p_j + a_j \, p_j^{-B_j}] \big/ p_i$$

$$+ [C_{ij} - \phi B_i B_j] m \big/ p_i p_j ,$$

or

(10) $$\partial a_i \big/ \partial p_j = - B_i \, a_j \, p_j^{-B_j} \, p_i^{B_i - 1}$$

$$+ [C_{ij} - (1 + \phi)B_i B_j] m \big/ p_j p_i^{1-B_i}$$

Now, (10) is independent of income m in the neighborhood, implying

(11) $$C_{ij} = (1 + \phi) \, B_i B_j , \qquad (j \neq i)$$

Further, from (9), equation (10) must also be independent of p_i. Hence, either $a_j = 0$ or a_j contains p_i as a multiplicative term raised to the $1-B_i$ power.

Let X denote the set of indices j with $a_j = 0$, Y denote the remaining indices, and n_Y the number of indices in Y. Then, the demand functions can be written

(12) $$q_i = B_i\, m \Big/ p_i \qquad \text{For } i \text{ in } X$$

(13) $$q_i = B_i\, m \Big/ p_i + A_i\,(1/p_i) \prod_{j\in Y} p_j^{\,1-B_j} \qquad \text{For } i \text{ in } Y$$

Imposing the requirement of homogeneity of degree zero in prices and income on (13),

$$n_Y - \sum_{j\in Y} B_j - 1 = 0,$$

or, with (6.3),

(14) $$\sum_{j\in X} B_j = 2 - n_Y$$

From (12), $B_j \geqq 0$ for j in X. Hence, $n_Y = 0, 1,$ or 2. If $n_Y = 0$, we have Cobb-Douglas demand functions. The supposition $n_Y = 1$ leads in (12) to demand functions with constant income shares for all but one good, giving a contradiction. The case $n_Y = 2$ implies $q_j = 0$ for j in X, and only two goods have demands which are not identically zero. We conclude that the case of three or more goods with demands locally not identically zero requires Cobb-Douglas demand functions.

V. In the case where only two goods have demands which are not identically zero in a neighborhood of a specified price-income point, the demand functions (13) reduce to

$$q_1 = B_1 m / p_1 + A (p_2 / p_1)^{B_1}$$

(15) $$q_2 = B_2 m / p_2 - A (p_1 / p_2)^{B_2}$$

which satisfy the budget constraint for an arbitrary constant A.

The Slutsky substitution terms corresponding to (15) are

$$S_{ii} = - B_1 B_2 m/p_i^2 \qquad i=1,2$$

(16) $$S_{12} = B_1 B_2 m/p_1 p_2$$

From the Slutsky terms we have the partial differential equations

$$\partial^2 m / \partial p_i^2 + B_1 B_2 m / p_i^2 = 0 \qquad i=1,2$$

(17) $$\partial^2 m / \partial p_1 \partial p_2 - B_1 B_2 m / p_1 p_2 = 0$$

where m is the expenditure function, giving the income necessary to attain a given utility level with given commodity prices.

Solutions of (17) must be of the functional form, using the linear homogeneity of m,

$$m = (p_1 p_2)^{\frac{1-r}{2}} [E_1(u)\, p_1{}^r + E_2(u)\, p_2{}^r] \qquad B_1 \neq \tfrac{1}{2}$$

(18)
$$m = (p_1 p_2)^{\frac{1}{2}} [E_1(u) + E_2(u) \log(p_1/p_2)] \qquad B_1 = \tfrac{1}{2}$$

where $r = \sqrt{1 - 4B_1 B_2}$, and E_1 and E_2 are increasing functions of the utility level u.

A necessary condition for the demand functions (15) to be determined (locally) by utility maximization is that the own Slutsky terms be negative. In (16), this requires $0 < B_1 < 1$, which implies r real and positive in (18) for $B_1 \neq 1/2$. In this case, the expenditure function (18) satisfies the conditions of the Shephard-Uzawa duality theorem, and the demand functions (15) are derived (locally) by maximization of a classical utility function. We note further that the expenditure function gives a complete description of the individual's preferences, and can be taken as the fundamental specification of preferences from which a utility function can be derived.

Finally, the Hicksian (income-compensated) demand curves, given by the price derivatives of the expenditure function, are from (18)

(19)
$$q_1^* = \begin{cases} E_1(u) \left(\frac{1+r}{2}\right) (p_2/p_1)^{\frac{1-r}{2}} + E_2(u) \left(\frac{1-r}{2}\right) (p_2/p_1)^{\frac{1+r}{2}} & \text{For } B_1 \neq \tfrac{1}{2} \\ \tfrac{1}{2} (p_2/p_1)^{\frac{1}{2}} [E_1(u) + 2\, E_2(u) + E_2(u) \log(p_1/p_2)] & \text{For } B_1 = \tfrac{1}{2} \end{cases}$$

(20)
$$q_2^* = \begin{cases} E_1(u)\left(\frac{1-r}{2}\right)(p_1/p_2)^{\frac{1+r}{2}} + E_2(u)\left(\frac{1-r}{2}\right)(p_1/p_2)^{\frac{1-r}{2}} & \text{For } B_1 \neq \frac{1}{2} \\ \frac{1}{2}(p_1/p_2)^{\frac{1}{2}}\left[E_1(u) - 2E_2(u) + E_2(u)\log(p_1/p_2)\right] & \text{For } B_1 = \frac{1}{2} \end{cases}$$

We have found in the two-good case a class of expenditure functions slightly more general than the Cobb-Douglas expenditure function, which do not require constant income share. However, this class yields a Slutsky matrix which is identical to the Cobb-Douglas Slutsky matrix, and requires the same restrictions on Theil's parameters.

(21)
$$C_{ij} = -B_i\,\delta_{ij} + (1+\phi)B_i\,B_j \qquad i,j=1,\ldots,n$$

that the Cobb-Douglas demands impose.

Reference: H. Theil "Simultaneous Estimation of Complete Systems of Demand Equations," preliminary lecture notes.

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