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A ROBUST TEST FOR STOCHASTIC DOMINANCE

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ABSTRACT: Statistical tests are developed for the hypotheses that first- or second-degree stochastic dominance holds between some pair of random variables in a set. The tests are applicable when the distributions of the random variables are unknown, but samples from each distribution are observed. These tests do not require that the distributions be in parametric families, and do permit some statistical dependence of the random variables within an observation period, and across periods. Thus, the tests are robust with respect to functional form and some patterns of statistical dependence. The paper applies the tests to data on daily returns from closed-end mutual funds, and concludes that some funds are second-degree stochastically dominated.

KEYWORDS: STOCHASTIC DOMINANCE, PERMUTATION TESTS, PORTFOLIO ANALYSIS

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1. INTRODUCTION

Let $\mathcal{A} = \{X_1,\ldots,X_K\}$ denote a set of K distinct random variables, or _prospects_.[2] Let F_k denote the (marginal) cumulative distribution function of prospect X_k.[3] A prospect X_1 _first-degree weakly stochastically dominates_ a prospect X_2 if, for all x, $F_2(x) \geq F_1(x)$, and _second-degree weakly stochastically dominates_ X_2 if, for all x, $\int_{-\infty}^{x} F_2(y)dy \geq \int_{-\infty}^{x} F_1(y)dy$. The set $\mathcal{A}$ is _first-degree_ [resp., _second-degree_] _stochastically maximal_ if no prospect in $\mathcal{A}$ is first-degree [resp., second-degree] weakly stochastically dominated by another prospect in $\mathcal{A}$. First-degree dominance implies second-degree dominance, and second-degree maximality implies first-degree maximality.

This paper develops a statistical test for the hypotheses that $\mathcal{A}$ is _not first-degree_ [resp., _second-degree_] _maximal_; i.e., first-degree [resp., second-degree] weak stochastic dominance holds between _some_ pair of prospects in $\mathcal{A}$. These tests are applicable when the distributions F_k are unknown, but realizations $(x_{1n},\ldots,x_{Kn})$ of the prospects are observed in periods n = 1,...,N. The tests we develop _do not_ require that the distributions F_k be in parametric families, and _do_ permit some statistical dependence between different prospects within an observation period, and across periods. Thus, the tests are robust with respect to the form of the F_k, some patterns of statistical dependence in the sample, and some conditioning on common information.

The concept of first-degree stochastic dominance is due to Lehmann (1955); that of second-degree stochastic dominance is due to Hadar and Russell

(1969, 1971, 1974a,b, 1978), and Hanoch and Levy (1969). McFadden (1989) proposes a statistical test in the case of independent observations on two statistically independent prospects; the test in this paper extends his results.

Historic economic interest in stochastic dominance arose from its relationship to maximization of expected utility. Hadar and Russell (1969) established that X_1 first-degree weakly stochastically dominates X_2 if and only if, for all increasing continuous functions u such that the expectations exist, $Eu(X_1) \geq Eu(X_2)$, and that X_1 second-degree weakly stochastically dominates X_2 if and only if, for all increasing continuous <u>strictly</u> <u>concave</u> functions u such that the expectations exist, $Eu(X_1) \geq Eu(X_2)$. Then, except possibly for ties, no (risk-averse) expected utility maximizer should choose a prospect in an available set $\mathcal{A}$ that is first-degree (second-degree) weakly stochastically dominated by another prospect in $\mathcal{A}$.

A common economic application identifies the prospects X_k with <u>investment strategies</u>, giving the returns from a unit of investment. The economic hypothesis is that investors are expected utility maximizers who choose stochastically maximal strategies. An approach to testing this hypothesis is to test for stochastic dominance among mutual funds; see Porter (1978), who has carried out an empirical study of stochastic dominance treating the distribution of realized returns as exact. There are several reasons to be cautious about the economic interpretation of such tests.

First, mutual fund managers act as agents for shareholders with heterogeneous VNM utility functions who can move between funds. Then, the strategy for a fund does not necessarily maximize a single VNM utility function, even if all shareholders are VNM maximizers. Nevertheless, a manager following a Pareto-satisfactory strategy should not operate a dominated fund.

In general, consumers hold mutual fund shares as part of broader portfolios that contain other financial and real assets. The returns on these assets are typically not independent of the mutual fund return, and the consumer's positions will take these interdependencies into account. Even if the consumer maximizes VNM utility of combined yield, there is no guarantee that choices will be consistent with stochastic dominance measured in terms of the marginal returns from the mutual funds alone. For example, suppose there are four equally probable states of nature, X_1 yields (0,1,9,10) in the respective states, X_2 yields (0,0,10,10), and that a consumer holding one share of either X_1 or X_2 also holds another asset Z with yields (9,10,0,1). Then, X_1 strictly second-degree dominates X_2, but X_2+Z strictly second-degree dominates X_1+Z. Thus, marginal dominance of X_1 over X_2 does not imply that X_1 is preferred to X_2. To avoid this problem, it is necessary to examine comprehensive asset portfolios of consumers rather than stock portfolios alone.

Consumers who are life-cycle optimizers solve a dynamic stochastic program in which the criterion is a discounted stream of VNM utilities. This problem produces optimal behavior that depends on available information and remaining life, and contains strategic elements. Then, behavior cannot be characterized in terms of optimization of a stationary function of wealth, except in special cases; see Hakanson (1971). Further, the manager is the agent of a population of share-holders, with aging, entry, and exit changing the distribution of tastes and holdings. For VNM utility maximization to imply stochastic maximality in mutual fund daily returns, it is necessary that transactions cost be zero and that maximality comparisons be made conditional on current information.

Other economic applications where stochastic dominance may be of interest are choice of a geographical or occupational labor market to search for a wage offer, or choice of strategy in a repeated market game against anonomous opponents.

2. STOCHASTIC MAXIMALITY

Without loss of generality for economic applications, we can assume that prospects are bounded above and below. By an increasing affine transformation, if necessary, we can assume that they are contained in the open unit interval. We will assume that von Neumann-Morgenstern (VNM) utility functions are continuous, increasing functions on [0,1], and again by an increasing affine transformation if necessary will assume that their range is the unit interval. Then, expected utility is always well-defined and finite. These assumptions together will be called the mathematical regularity conditions.

The theorem of Hadar and Russell (1969) characterizing first and second degree stochastic maximality is restated below, with a simplified proof; see also Hanoch and Levy (1969) and McFadden (1989).

Theorem 1. *Consider a set of distinct prospects* $\mathcal{A} = \{X_1,\ldots,X_K\}$. *Assume the mathematical regularity condition.*

The prospects in $\mathcal{A}$ *are first-degree stochastically maximal; i.e.,*

$$(1) \qquad d^* = \min_{i \neq j} \max_{x} \; [F_i(x) - F_j(x)] > 0,$$

if and only if for each i *and* j, *there exists a continuous increasing function* u *such that* $Eu(X_i) > Eu(X_j)$.

The prospects in $\mathcal{A}$ *are second-degree stochastically maximal; i.e.,*

$$(2) \qquad s^* = \min_{i \neq j} \max_{x} \int_{-\infty}^{x} [F_i(y) - F_j(y)]dy > 0,$$

if and only if for each i *and* j, *there exists a continuous increasing strictly concave function* u *such that* $Eu(X_i) > Eu(X_j)$.

Proof: The argument uses the following formula for integration by parts (Dunford and Schwartz, 1966, III.6.22): If v(x) and w(x) are functions of bounded variation on (a,b), one is continuous, and the other is right-continuous, then $\int_a^b v(x)w(dx) = v(b-)w(b-)-v(a+)w(a+)-\int_a^b w(x)v(dx)$. Define $w(x) = \int_0^x [F_i(y)-F_j(y)]dy$. Then w(x) is continuous and differentiable, and w'(x) is right-continuous. Since the support of X_i and X_j is contained in (0,1), $w'(x) \equiv 0$ in neighborhoods of x = 0 and x = 1. Let u be a continuous VNM utility function. Since monotone functions and differences of monotone functions are of bounded variation, u, w, and w' have this property. Applying the integration by parts formula, and using u(0) = 0 and w'(1) = 0, we obtain

$$\text{(3)} \qquad Eu(X_i) - Eu(X_j) \equiv \int_0^1 u(x)[F_i(dx)-F_j(dx)] \equiv \int_0^1 u(x)w'(dx)$$
$$= - \int_0^1 w'(x)u(dx).$$

Then, (3) is non-negative for all VNM functions u if and only if $w'(x) = F_i(x)-F_j(x)$ is non-positive (since (3) holds for a sequence of VNM functions u converging weakly to a step function at any specified x_o). This proves (1).

Next, assume u is a strictly concave VNM utility function. Then the right-side derivative u'(x) exists for all $x \in (0,1)$, and is decreasing and right-continuous. Then $\int_0^1 w'(x)u(dx) = \int_0^1 w'(x)u'(x)dx \equiv \int_0^1 u'(x)w(dx)$, by Dunford and Schwartz, (1966, III.10.2 and III.12.8). Again apply the integration by parts formula, and combine this with (3) and the condition $w(0+) \equiv 0$ to obtain

$$Eu(X_i) - Eu(X_j) \equiv - \int_0^1 w'(x)u(dx) \equiv - \int_0^1 u'(x)w(dx)$$
$$= -u'(1)w(1) + \int_0^1 w(x)u'(dx).$$

Since u' is decreasing, this expression is non-negative if and only if w(x) is non-positive for all x. (Again, a sequence of u' converging weakly to a step function yields the result.) This proves (2). □

We establish that second-degree stochastic maximality is an implication of classical maximization of atemporal expected utility.

Theorem 2. *Suppose each* X_k *in the set of distinct prospects* $\mathcal{A} = \{X_1,\ldots,X_K\}$ *is the strategy of an economic agent that maximizes an increasing, strictly concave von Neumann-Morgenstern utility function. Suppose that it is feasible for each agent to mix available strategies; i.e., to hold a prospect that is a lottery in the prospects in* $\mathcal{A}$. *Assume the mathematical regularity conditions. Then, each prospect in* $\mathcal{A}$ *is second-degree stochastically maximal.*

Proof: Suppose X_1 is not second-degree stochastically maximal. Then there is a second prospect, say X_2, such that $Eu(X_2) \geq Eu(X_1)$ for all increasing, strictly concave u. Let u be the utility function of a manager that chooses X_1. Since X_1 and X_2 are distinct, strict concavity implies $Eu((X_1+X_2)/2) > Eu(X_1)/2 + Eu(X_2)/2 \geq Eu(X_1)$, yielding the contradiction that X_1 is not optimal for this manager. □

Using the characterization of first-degree stochastic dominance in Theorem 1, one can establish a first-degree analog of Theorem 2.

Theorem 3. *Suppose each* X_k *in the set of distinct prospects* $\mathcal{A} = \{X_1,\ldots,X_K\}$ *is the strategy of an economic agent that maximizes an increasing von Neumann-Morgenstern utility function. Suppose each prospect has a distinct distribution function. Assume the mathematical regularity conditions. Then, each prospect in* $\mathcal{A}$ *is first-degree stochastically maximal.*

Proof: Suppose X_1 is not first-degree stochastically maximal. Then, by (1), there is a second prospect, say X_2, such that $F_1(x) \geq F_2(x)$ for all x. Let u be the utility function of a consumer that chooses X_1. Then,

$$Eu(X_2) - Eu(X_1) = \int_0^1 u(x)[F_2(dx)-F_1(dx)] = \int_0^1 [F_1(x)-F_2(x)]u(dx) > 0,$$

using the integration by parts argument from the proof of Theorem 1. The last inequality is strict since u is increasing and $F_1 \neq F_2$. This contradicts the optimality of X_1. □

The assumptions and conclusions of Theorems 1-3 are invariant under increasing affine transformations of returns or of the VNM utility functions. Therefore, in developing statistical tests, (1) and (2) can be used without requiring that returns be in the unit interval.[4]

A sufficient condition for the maintained hypothesis of distinct prospects in Theorem 2 is that the N×K array of sample returns be of rank K; this would normally be guaranteed (with probability one) from the historical record as part of the process of defining $\mathcal{A}$, prior to drawing the sample. However, the maintained hypothesis of distinct distributions in Theorem 3 is not easily verified, and would require a multivariate Kolmogoroff-Smirnov test or comparable test, with some of the same issues in calculating significance levels that we encounter in the tests for stochastic maximality described below.

3. THE TEST STATISTICS

Suppose realized returns are observed over N periods for each of the prospects in $\mathcal{A} = \{X_1,\ldots,X_K\}$. Let x_{kn} denote the realized return of prospect k in period n, and F_{kN} denote the empirical distribution of returns for prospect k. Consider the null hypothesis that the prospects in $\mathcal{A}$ are not first-degree stochastically maximal. An empirical analog of d^* in (1) that is suitable for a test statistic is

$$d^*_{2N} = \min_{i \neq j} \max_x \, [F_{iN}(x)-F_{jN}(x)]; \tag{4}$$

this is simply a multivariate version of the Kolmogorov-Smirnoff statistic. An empirical analog of s^* in (2),

$$s^*_{2N} = \min_{i \neq j} \max_x \int_0^x [F_{iN}(y)-F_{jN}(y)]dy, \tag{5}$$

is suitable for testing the hypothesis that the prospects in $\mathcal{A}$ are not second-degree stochastically maximal. In both (4) and (5), the null hypothesis will be rejected when the statistic is large. We next describe an efficient computational algorithm for (4) and (5), and later a monte carlo procedure that associates a "significance level" with each statistic. These algorithms are well-defined no matter what the patterns of statistical dependence in the data. However, for (1) and (2) to be defined, and the limits $d^*_{2N} \xrightarrow{p} d^*$ and $s^*_{2N} \xrightarrow{p} s^*$ to hold, we assume the time series of observations are strictly stationary and α-mixing. For the monte carlo calculation of "significance levels" to approximate the true significance levels of the tests, we assume statistical dependence in an observation period must satisfy the following <u>generalized</u> <u>exchangeability</u> property: The random variables $Y_1 = F_1(X_1), \ldots, Y_K = F_K(X_K)$ are exchangeable. Obviously, if the X_k are statistically independent, then they satisfy generalized exchangeability. An example of generalized exchangeability with statistical dependence is $X_k = \alpha_k + \beta_k(\sqrt{\rho}Z_0 + \sqrt{1-\rho}Z_k)$, where $Z_0, Z_1, \ldots, Z_K$ are independent random variables with mean zero and variance one; $Z_1, \ldots, Z_K$ are identically distributed; and α_k, β_k, and $\rho \in [0,1)$ are parameters. Then, $\mathbf{E}X_k = \alpha_k$, $\mathbf{Var}(X_k) = \beta_k^2$, and $\mathbf{Corr}(X_k, X_j) = \rho$. This example is related to the Capital Asset Pricing Model encountered in finance, where Z_0 is a general market factor and the Z_k are factors specific to the prospects; see Sharpe (1963).

Computation of the statistics (4) and (5) can be carried out economically using the following algorithm: For each pair X_i and X_j with $i < j$, form a vector z of length 2N containing the observations from X_i, followed by the observations from X_j. Form a vector l of length 2N containing the indices of the elements of z in ascending order; i.e., $z_{l_m} \le z_{l_{m+1}}$. When i and j are not clear from the context, z, l are denoted z^{ij}, l^{ij}. For each $i < j$, define

(6) $d_0 = d_0^+ = d_0^- = s_0 = s_0^+ = s_0^- = 0,$

and recursively, for m = 1,...,2N,

(7) $\delta_m = \begin{cases} -1 & \text{if } l_m > N \\ +1 & \text{if } l_m \leq N \end{cases}$

(8) $d_m = d_{m-1} + \delta_m,$

(9) $d_m^+ = \text{Max}\ \{d_{m-1}^+, d_m\},$ $\quad d_m^- = \text{Max}\ \{d_{m-1}^-, -d_m\}$

(10) $s_m = s_{m-1} + d_{m-1}\cdot(z_{l_m} - z_{l_{m-1}}),$

(11) $s_m^+ = \text{Max}\ \{s_{m-1}^+, s_m\},$ $\quad s_m^- = \text{Max}\ \{s_{m-1}^-, -s_m\},$

Again, a superscript "ij" is added to the expressions in (6)-(11) when necessary to identify the pair of prospects i,j being evaluated. Theorem 4 establishes that (4) and (5) satisfy

(12) $d_{2N}^* = N^{-1}\cdot\text{Min}_{i<j}\ \text{Min}\ \{d_{2N}^{+ij}, d_{2N}^{-ij}\},$

(13) $s_{2N}^* = N^{-1}\cdot\text{Min}_{i<j}\ \text{Min}\ \{s_{2N}^{+ij}, s_{2N}^{-ij}\}.$

One pass through the data is sufficient to calculate (6)-(13).

Theorem 4. d_{2N}^* *and* s_{2N}^*, *defined in* (4) *and* (5), *are given by* (6)-(13).

Proof: It is sufficient to show that $d_{2N}^{+ij} = N\cdot\text{Max}_x\ [F_{iN}(x)-F_{jN}(x)]$ and $s_{2N}^{+ij} = N\cdot\text{Max}_x \int_{-\infty}^{x} [F_{iN}(y)-F_{jN}(y)]dy$; the remaining expressions follow by reversing the order of i and j. For an i,j pair, define $G_{2N}(x)$ to be the empirical distribution function formed from the pooled observations in z. Note that δ_m is an indicator that is +1 if z_{l_m} is an observation from X_i and -1 if it is an observation from X_j. Note that $2N\cdot G_{2N}(x) = m$ implies $z_{l_m} \leq x < z_{l_{m+1}}$. We establish by induction that

(14) $$N\cdot\underset{x\le z_{l_m}}{\mathrm{Max}}\ [F_{iN}(x)-F_{jN}(x)] = d_m^+$$

and

(15) $$N\cdot\underset{x\le z_{l_m}}{\mathrm{Max}}\int_{-\infty}^{x}[F_{iN}(y)-F_{jN}(y)]dy = s_m^+.$$

These equalities are obvious for m = 1. Suppose they hold for all $m' \le m$. Then $F_{iN}(x)-F_{jN}(x)$ is constant for $z_l \le x < z_{l_{m+1}}$, and jumps by δ_{m+1}/N at $x = z_{l_{m+1}}$. Hence, the left-hand-side of (14) for m+1 is the larger of d_m^+ and d_{m+1}, and thus, using (8) and the left equation in (9), is d_{m+1}^+.

The expression $\int_{-\infty}^{x}[F_{iN}(y)-F_{jN}(y)]dy$ is linear for $z_{l_m} \le x \le z_{l_{m+1}}$, with slope d_m/N, and is continuous. Therefore, its variation over the interval is $d_m(z_{l_{m+1}} - z_{l_m})/N$, and it achieves its maximum or minimum on this interval at an end point. The expression s_m in (10) has the same variation, except for the scale factor N. Then, (15) holds at step m+1. This completes the induction argument. Finally, note that $d_{2N} = 0$, implying that the maximands in (14) and (15) are constant for $x > z_{l_{2N}}$. Then, the maximum values of these equations for $-\infty < x < +\infty$ are achieved in the interval $z_{l_1} \le x \le z_{l_{2N}}$. □

We next consider the statistical behavior of s_{2N}^* and d_{2N}^*. The first result establishes that these statistics converge in probability to s^* and d^*, respectively, under weak conditions.

Theorem 5. *Assume that* $(X_{1n},\ldots,X_{Kn})$, *viewed as a stochastic process indexed by* n = 1,2,..., *with values in* $[0,1]^K$, *is strictly stationary*[5] *and* α*-mixing with* $\alpha(j) = O(j^{-\delta})$ *for some* $\delta > 1$.[6] *Then* $s_{2N}^* \xrightarrow{p} s^*$ *and* $d_{2N}^* \xrightarrow{p} d^*$.

Proof: Consider the random variables

(16) $$Z_n = g(x,x_{in},x_{jn}) \equiv \int_{-\infty}^{x} [1(y>x_{in}) - 1(y>x_{jn})]dy.$$

Since g is a continuous function, the Z_n are again α-mixing with $\alpha(j) = O(j^{-\delta})$ for some $\delta > 1$. Since $|Z_n| \le 1$, all its moments exist and are uniformly bounded by one. Therefore, the strong law of large numbers for mixing processes (White and Domowitz, 1984) implies $s_{2N}^{ij}(x) \equiv \sum_{n=1}^{N} Z_n/N \xrightarrow{as} EZ_n \equiv s_{\infty}^{ij}(x) \equiv \int_{-\infty}^{x} [F_i(y) - F_j(y)]dy$. The Lipschitz property $|s_{2N}(x)-s_{2N}(y)| \le |x-y|$ for x,y $\in [0,1]$ implies a uniform strong law, $\sup_x |s_{2N}^{ij}(x)-s_{\infty}^{ij}(x)| \xrightarrow{as} 0$. The continuous mapping theorem (Pollard, 1984, p. 70) implies $\max_x s_{2N}^{ij}(x) \xrightarrow{as} s_{\infty}^{ij*}$. The Glivenko-Cantelli theorem (Loève, 1960, p.20) implies $\sup_x |d_{2N}^{ij}(x)-d_{\infty}^{ij}(x)| \xrightarrow{as} 0$, where $d_{2N}^{ij}(x)$ and $d_{\infty}^{ij}(x)$ are the first-degree analogs of $s_{2N}^{ij}(x)$ and $s_{\infty}^{ij}(x)$. Then, $\max_x d_{2N}^{ij}(x) \xrightarrow{as} d_{\infty}^{ij*}$ by application of the continuous mapping theorem. The finite minimum of these statistics for $i \ne j$ gives the conclusion of the theorem. □

The following theorem implies that $N^{1/2}s_{2N}^{*}$ has an asymptotic distribution when $s^{*} \le 0$, and that this distribution is nondegenerate in the "least favorable" case of identical marginals. From the form of (2) and (4), if the statistical properties of s_{2N}^{12*} are obtained in the limiting case $F_1 \equiv F_2$, then the behavior of s_{2N}^{*} follows immediately from the joint distribution of a finite number of such pairs. An analogous result holds for $N^{1/2}d_{2N}^{*}$.

Theorem 6. *Assume that* (X_{1n},X_{2n}) *is a strictly stationary stochastic process, taking values in* $[0,1]\times[0,1]$, *such that the process is* α*-mixing with* $\alpha(j) = O(j^{-\delta})$ *for some* $\delta > 1$. *Define* $\psi_1(x) = \int_0^w F_1(y)dy$. *From* (16), $g(w,x_{1n},x_{2n}) = \max(w-x_{1n},0) - \max(w-x_{2n},0)$, *so* $Eg(w,x_{1n},x_{2n}) = \psi_1(w) - \psi_2(w)$. *Define* $\tilde{g}(w,x_{1n},x_{2n}) = g(w,x_{1n},x_{2n}) - Eg(w,x_{1n},x_{2n})$,

$$(17)\qquad S_{2N}(w) = N^{1/2}\left[\int_0^w [F_{1N}(y)-F_{2N}(y)]dy\right] \equiv N^{1/2}s_{2N}(w) \equiv N^{-1/2}\sum_{n=1}^{N} g(w,x_{1n},x_{2n}),$$

$$(18)\qquad \tilde{S}_{2N}(w) = S_{2N}(w) - ES_{2N}(w) \equiv N^{-1/2}\sum_{n=1}^{N}\tilde{g}(w,x_{1n},x_{2n}).$$

Then $E\tilde{S}_{2N}(w) = 0$; $\tilde{S}_{2N}(0) \equiv 0$; *there exists* $M > 0$ *such that*

$$(19)\qquad E(\tilde{S}_{2N}(v) - \tilde{S}_{2N}(w))^2 \le M(v-w)^2;$$

and there exists a covariance function $\rho(w,v)$ *that is uniformly Lipschitz on* $[0,1]\times[0,1]$ *such that* $E\tilde{S}_{2N}(w)\tilde{S}_{2N}(v) \to \rho(u,v)$ *uniformly.*[7]

Assume $0 < \rho(1,1) = \lim N^{-1}\sum_{n,m=1}^{N} \mathrm{cov}\{(X_{2n}-X_{1n}),(X_{2m}-X_{1m})\}$. *Then the sequence of processes* $\tilde{S}_{2N}(.)$ *for* $N \to \infty$ *is tight: i.e., it has the stochastic boundedness property that for each* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$(20)\qquad \sup_N P(\sup_x |\tilde{S}_{2N}(x)| > \delta) < \varepsilon,$$

and the stochastic equicontinuity property that for each positive η *and* ε, *there exists* $\delta > 0$ *such that for all* N,

$$(21)\qquad P(\sup_{|w-v|<\delta} |\tilde{S}_{2N}(w)-\tilde{S}_{2N}(v)| > \eta) < \varepsilon.$$

Also, the $\tilde{S}_{2N}$ *converge in distribution to a Gaussian process* $\tilde{S}_\infty(.)$ *that has continuous sample paths with probability one, and the covariance function* ρ.

If $\psi_1(x) \le \psi_2(x)$, *with equality holding for* $x \in \mathbf{A}$, *then* $N^{-1/2}s_{2N}^{+12} \equiv \max_x S_{2N}(x)$ *converges in distribution to* $S_\infty^* \equiv \max_{x\in\mathbf{A}} \tilde{S}_\infty(x)$. *If* $\psi_1(x) > \psi_2(x)$ *for some* x, *then* $P(\max_x S_{2N}(x) < \varepsilon) \to 0$ *for every* $\varepsilon > 0$.

Proof: The results $\tilde{S}_{2N}(0) \equiv 0$ and $E\tilde{S}_{2N}(w) = 0$ are immediate from the definition of the process. The function $\tilde{g}(w,x_{1n},x_{2n})$ is continuous and piecewise linear in (w,x_{1n},x_{2n}), with $|\tilde{g}(w,x_1,x_2)| \le 2$, and the values of $g(w,x_{1n},x_{2n}) - g(v,x_{1n},x_{2n}) \equiv \min\{v,\max\{x_{1n},w\}\} - \min\{v,\max\{x_{2n},w\}\}$ are bounded in magnitude by $|v-w|$. For any mean-zero process Z_n that is strictly stationary, bounded by C, and α-mixing, Hall and Heyde (1980, Theorem A.5) establish $|EZ_nZ_m - EZ_nEZ_m| \le 4C^2\alpha(n-m)$. But $g(w,x_{1n},x_{2n}) - g(v,x_{1n},x_{2n})$ has this property with $C = |v-w|$ and $\alpha(j) = O(j^{-\delta})$, $\delta > 1$. Let $M = 4\sum_{n=1}^{\infty}\alpha(n) < +\infty$. Then

$$(22)\quad E(\tilde{S}_{2N}(w)-\tilde{S}_{2N}(v))^2$$
$$\equiv N^{-1}\sum_{n,m=1}^{N}[\tilde{g}(w,x_{1n},x_{2n}) - \tilde{g}(v,x_{1n},x_{2n})][\tilde{g}(w,x_{1m},x_{2m}) - \tilde{g}(v,x_{1m},x_{2m})]$$
$$\le 4(v-w)^2\sum_{n=1}^{N}\alpha(n) = M(v-w)^2.$$

By the same argument,

$$(23)\quad |E(\tilde{S}_{2N}(w_1)-\tilde{S}_{2N}(v_1))(\tilde{S}_{2N}(w_2)-\tilde{S}_{2N}(v_2))|$$
$$\le M|v_1-w_1||v_2-w_2|.$$

The stationarity and mixing properties of $\tilde{S}_{2N}(.)$ imply that its second moments converge uniformly as $N \to \infty$. Define

$$(24)\quad \rho(w,v) = \lim_{N\to\infty} E(\tilde{S}_{2N}(w)\tilde{S}_{2N}(v));$$

from (23), $\rho(w,v)$ is uniformly Lipschitz. Then, the hypotheses of the central limit theorem of Hall and Heyde (1980) are satisfied, implying for any finite vector $(w_1,\ldots,w_k)$ that $(\tilde{S}_{2N}(w_1),\ldots,\tilde{S}_{2N}(w_k))$ converges in distribution to a multivariate normal with covariances $\rho(w_i,w_j)$. From (22), a theorem of Billingsley (1968, Thm. 12.3, with condition (12.51)) establishes that the sequence $\tilde{S}_{2N}(.)$ is tight. Then the stochastic boundedness and equicontinuity conditions (20) and (21), and convergence in distribution to $\tilde{S}_{\infty}(.)$, are implied by Billingsley's Theorems 8.2 and 8.1, respectively.

Assume $\psi_1(x) \le \psi_2(x)$, with equality holding for $x \in \mathbf{A}$. From (21), given λ and $\gamma > 0$, there exists $\delta > 0$ such that $P(\sup_{|w-v|<\delta,\, v\in\mathbf{A}} |\tilde{S}_{2N}(w)-\tilde{S}_{2N}(v)| > \lambda) < \gamma$. Note that $S_{2N}(x) = \tilde{S}_{2N}(x)$ for $x \in \mathbf{A}$. Define $\mathbf{B} = \{x\in[0,1] \,|\, x\notin\mathbf{A}\ \&\ |x-y|<\delta$ for some $y\in\mathbf{A}\}$ and $\mathbf{D} = \{x\in[0,1] \,|\, x\notin\mathbf{A}\ \&\ |x-y|\ge\delta$ for all $y\in\mathbf{A}\}$. Given $\varepsilon > 0$, define the events

$$\mathcal{B}:\quad \max_x S_{2N}(x) \le \varepsilon$$

$$\mathcal{C}:\quad \max_x S_{2N}(x) \le \varepsilon+\lambda$$

$$\mathcal{D}:\quad \sup_{x\in\mathbf{A}\cup\mathbf{B}} S_{2N}(x) > \varepsilon+\lambda$$

$$\mathcal{F}:\quad \max_{x\in\mathbf{D}} S_{2N}(x) > \varepsilon+\lambda.$$

$$\mathcal{G}:\quad \max_{x\in\mathbf{A}} \tilde{S}_{2N}(x) \le \varepsilon$$

Then, $\mathcal{B} \subseteq \mathcal{G}$, implying $P(\mathcal{B}) \le P(\mathcal{G})$. Also, $\mathcal{G} \subseteq \mathcal{C}\cup\mathcal{F}\cup\mathcal{D}$, implying $P(\mathcal{G}) \le P(\mathcal{C})+P(\mathcal{F})+P(\mathcal{G}\cap\mathcal{D})$. But

$$\mathcal{G}\cap\mathcal{D} \Longrightarrow \lambda < \sup_{\substack{|w-v|<\delta \\ v\in\mathbf{A}}} (S_{2N}(w)-\tilde{S}_{2N}(v)) \le \sup_{\substack{|w-v|<\delta \\ v\in\mathbf{A}}} (\tilde{S}_{2N}(w)-\tilde{S}_{2N}(v)),$$

implying by the stochastic equicontinuity condition that $P(\mathcal{G}\cap\mathcal{D}) < \gamma$. Next, from (20), there exists δ' such that $P(\sup_{x\in\mathcal{D}} \tilde{S}_{2N}(x) \ge \delta') < \gamma$. Noting that $\psi_1(x)-\psi_2(x)$ is continuous and negative on $\mathcal{D}$, there exists N_0 such that for $N \ge N_0$, $N^{1/2}(\psi_1(x) - \psi_2(x)) < -\delta'$. Then, for $N \ge N_0$,

$$P(\mathcal{F}) = P(\max_{x\in\mathcal{D}} S_{2N}(x) = \max_{x\in\mathcal{D}} \{\tilde{S}_{2N}(x) + N^{1/2}(\psi_1(x) - \psi_2(x))\} > \varepsilon+\lambda)$$

$$\le P(\max_{x\in\mathcal{D}} \tilde{S}_{2N}(x) > \varepsilon+\lambda+\delta') < \gamma.$$

Therefore, $P(\mathcal{B}) \le P(\mathcal{G}) \le P(\mathcal{C}) + 2\gamma$. Taking γ and λ small, and using the right-continuity of distribution functions, this establishes that $\max_x S_{2N}(x)$ converges in distribution to $S_\infty^* \equiv \max_{x\in\mathbf{A}} \tilde{S}_\infty(x)$.

Assume $\lambda = \psi_1(x_0) - \psi_2(x_0) > 0$ for some x_0. Note that $\max_x S_{2N}(x) \geq \tilde{S}_{2N}(x_0) + N^{1/2}\lambda$. Then, given $\gamma > 0$, (20) implies there exists δ' such that

$$P(\max_x S_{2N}(x) \leq N^{1/2}\lambda - \delta') < \gamma. \quad \square$$

The scale of the statistic $N^{1/2}s_{2N}^{*12}$ is linear homogeneous in the scales of X_1 and X_2. Then, trivially, for each $\varepsilon > 0$, there are distributions satisfying the null for which $P(N^{1/2}s_{2N}^{*12} > \varepsilon)$ can be made as close to one as we please. Thus, some scaling or conditioning that makes ε a function of the sample is needed to enable us to identify a least favorable case within the null that determines the critical level for the test. Our approach is to condition on the distribution of the pooled realizations from each pair of prospects; e.g. $F_{12}(x) = (F_1(x)+F_2(x))/2$. When the assumption of generalized exchangeability is satisfied, F_{12} is the distribution of points obtained by drawing a realization from (X_1,X_2), and then drawing one of the two components at random. The following theorem establishes an asymptotic least favorable case under the null.

Theorem 7. *Suppose the assumptions and the definition* $S_{2N}^{12*} = N^{1/2}s_{2N}^{12*}$ *from Theorem 6. Suppose generalized exchangeability, and let* $H(F_1(x_1),F_2(x_2))$ *denote the joint density of* (X_1,X_2). *If* (X_{1n},X_{2n}) *are second-degree stochastically maximal, then for N sufficiently large,*

$$(25) \qquad P(S_{2N}^{12*} > \varepsilon | (X_{1n},X_{2n}) \sim H(F_1,F_2)) > P(S_{2N}^{12*} > \varepsilon | (X_{1n},X_{2n}) \sim H(F_{12},F_{12})).$$

If, on the other hand, X_1 *second-degree weakly stochastically dominates* X_2, *then for N sufficiently large,*

$$(26) \qquad P(S_{2N}^{12*} > \varepsilon | (X_{1n},X_{2n}) \sim H(F_1,F_2)) \leq P(S_{2N}^{12*} > \varepsilon | (X_{1n},X_{2n}) \sim H(F_{12},F_{12})).$$

Proof: If $F_1 \equiv F_2$, then the result is trivial. Suppose $F_1 \not\equiv F_2$. From Theorem 6, the right-hand-side of (25) or (26) is strictly between zero and one for small positive ε. Suppose (X_{1n}, X_{2n}) are second-degree stochastically maximal. The last result in Theorem 6 implies that the left-hand-side of (25) goes to one. Hence, (25) holds for N large. A corollary is that the test is consistent.

Alternately, suppose X_1 second-degree weakly stochastically dominates X_2. Then $\psi_1(x) \le \psi_2(x)$, with equality for $x \in \mathbf{A}$. Then the penultimate result in Theorem 6 implies

$$\begin{aligned} &P(S_{2N}^{12*} \le \varepsilon | (X_{1n}, X_{2n}) \sim H(F_1, F_2)) \\ &\quad \longrightarrow P(\max_{x \in \mathbf{A}} \tilde{S}_\infty(x) \le \varepsilon | (X_{1n}, X_{2n}) \sim H(F_1, F_2)) \\ &\quad\quad = P(\max_{x \in \mathbf{A}} \tilde{S}_\infty(x) \le \varepsilon | (X_{1n}, X_{2n}) \sim H(F_{12}, F_{12})) \\ &\quad\quad \ge P(\max_{x} \tilde{S}_\infty(x) \le \varepsilon | (X_{1n}, X_{2n}) \sim H(F_{12}, F_{12})), \end{aligned}$$

and

$$\begin{aligned} &P(S_{2N}^{12*} \le \varepsilon | (X_{1n}, X_{2n}) \sim H(F_{12}, F_{12})) \\ &\quad \longrightarrow P(\max_{x} \tilde{S}_\infty(x) \le \varepsilon | (X_{1n}, X_{2n}) \sim H(F_{12}, F_{12})). \quad \square \end{aligned}$$

In general, the statistic s_{2N}^{*} has neither a tractable finite-sample distribution, nor an asymptotic distribution for which there are convenient computational approximations. Except for special cases, neither does d_{2N}^{*}.[8] However, we argue that empirical distributions of the statistics s_{2N}^{*} and d_{2N}^{*}, calculated in a monte carlo simulation by randomly switching observations from each period between each pair of distributions F_i and F_j, yield approximations to the significance levels of the test statistics under the null hypotheses of not second-degree or not first-degree stochastically maximal. The steps in this simulation for s_{2N}^{*} and d_{2N}^{*} are as follows:

Step 1. Calculate s^{*}_{2N} for the observed data; denote this value $\bar{s}^{*}_{2N}$. Similarly, let $\bar{d}^{*}_{2N}$ denote the value of d^{*}_{2N} for the observed data.

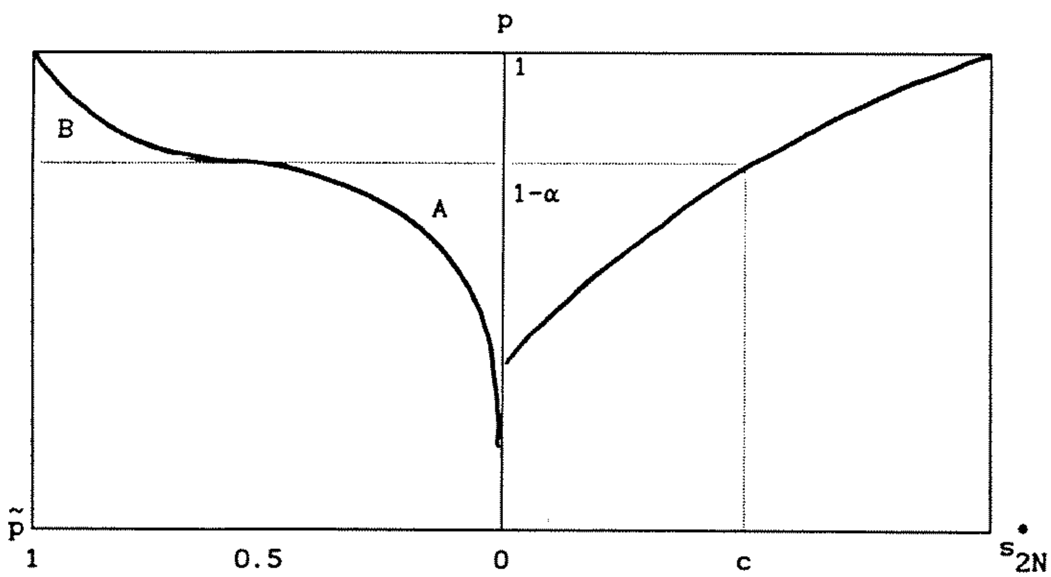
Step 2. Iterate in a series of monte carlo simulations. In each simulation, for each $i < j$, form $\delta' = (q,-q)$, where q is an N-vector of random signs. (Interpreting δ as indicators of membership, this construction randomly permutes observations within each period between F_1 and F_2, so that they can be treated as draws from the pooled distribution F_{12}.) Use the resulting δ to compute the expressions (8)-(11). Repeat this for all $i < j$. Then calculate simulated values of the statistics d^{*}_{2N} and s^{*}_{2N} in (12) and (13), and count the fractions of these simulated statistics in the monte carlo simulations that are smaller in magnitude than the corresponding observed values $\bar{d}^{*}_{2N}$ and $\bar{s}^{*}_{2N}$. We claim that these fractions approximate the significance levels of these statistics in tests of not first-degree and not second-degree stochastic maximality. Then, a criterion that rejects the null hypothesis when the calculated significance level is below a nominal level α will yield a test with actual size near α. Alternately, the simulated empirical distribution of these statistics can be used to calculate approximate critical levels for a test of specified size.

The empirical distributions in Step 2 used to calculate d^{*}_{2N} and s^{*}_{2N} are

$$(27) \qquad G^{ij}_{2N}(x_i,x_j) = (2N)^{-1}\sum_{n=1}^{N}\{1(x_{in}\le x_i \& x_{jn}\le x_j) + 1(x_{jn}\le x_i \& x_{in}\le x_j)\};$$

the associated empirical marginal distributions are both $F_{ij,N}$. If one enumerated the possible permutations in Step 2, one would obtain the exact distribution of d^{*}_{2N} and s^{*}_{2N}, conditioned on G^{ij}_{2N}, and from this the exact significance levels of $\bar{d}^*_{2N}$ and $\bar{s}^*_{2N}$ when the true distributions are given by (27). The monte carlo procedure that rejects the null hypothesis when the simulated significance level is below a specified level α such as $\alpha = 0.05$ has an exact rejection probability that can be bounded by the following argument:

Let c denote the critical level for the test. The right panel in the figure below shows schematically the cumulative distribution function of s^*_{2N} given (27). This distribution will usually have a positive probability p that the statistic is zero, and be continuous for positive s^*_{2N}. Randomize if necessary to break ties in case the distribution jumps over an assigned significance level α. Then, the distribution of significance levels 1-p for random drawings from the distribution of s^*_{2N} is uniform on [0,1].



The probability $\tilde{p}$ that a value of the statistic with exact significance level p will yield a simulated significance level less than α is given by the probability that a binomially distributed frequency with parameters (p,M) will exceed $1-\alpha$, where M is the number of monte carlo trials. The left panel in this figure depicts this probability. Area A for $p < 1-\alpha$ is the probability of rejection by the monte carlo method when the exact significance level calculation would lead to acceptance. Area B is the probability of acceptance by the monte carlo method when the exact calculation would lead to rejection. Then,

Area A + Area B = Probability of Misclassification,

Area A - Area B = Actual significance level - Nominal significance level

Because the variance of the binomial is lower in Area B than in Area A in the usual testing situation where $\alpha < 1/2$, one will have Area A > Area B, so that the actual rejection probability exceeds the nominal one. One can choose M sufficiently large to make Area A acceptably small. To bound Area A, note first that the frequency f of acceptance using the simulation when the true probability is p satisfies $Ef = p$, $E(f-p)^2 = p(1-p)/M$, and $|f-p| \le 1$. Then, Bernstein's inequality (Pollard, 1984, Appendix B) implies

(28) $$P(f-p > 1-\alpha-p) \le \exp(-M(1-\alpha-p)^2).$$

Then,

(29) $$\text{Area A} \le \int_0^{1-\alpha} P(f-p > 1-\alpha-p)dp \le \int_0^{1-\alpha} \exp(-M(1-\alpha-p)^2)dp \le (\pi/M)^{1/2}/2.$$

Therefore, choosing $M \ge \pi/(2\theta)^2$ guarantees that Area A will be less than θ. For example, $\theta = 0.01$, so that the probability of misclassification is less than 0.01 and the actual rejection probability is less than 0.06, is achieved when $M \ge 7854$.

An alternative to using a fixed number of permutations to accept or reject the null hypothesis would be to use a (truncated) sequential probability ratio test: Let $i = 1,\ldots,r$ index the trials. Let W_i denote the number of permutations yielding a statistic exceeding that observed in the data through trial i, less $i\cdot\alpha$. If on a trial, W_i exceeds a parameter B, terminate the trials and reject the null hypothesis. Otherwise, at the end of r trials, accept the null hypothesis. Siegmund (1985) gives the distribution theory for such methods.

The next theorem establishes that asymptotically as N and the number of monte carlo iterations go to infinity, the actual sizes of the tests approach their nominal sizes.

Theorem 8. *Suppose the assumptions of Theorem 6 hold for each pair of random variables X_i and X_j. Suppose the joint distribution of these random variables satisfies generalized exchangeability. Then, the significance level for $N^{1/2}\bar{s}^{*}_{2N}$ calculated in monte carlo Steps 1 and 2 approaches the probability given by (26), $P(S^{12*}_{2N} \le N^{1/2}\bar{s}^{*}_{2N} | (X_{1n},X_{2n}) \sim H(F_{12},F_{12}))$, as N and the number of monte carlo iterations approach infinity.*

Proof: From Theorem 7, a limiting least favorable case of distributions satisfying the hypothesis of not second-degree stochastically maximal is that each pair of prospects X_i and X_j have the identical marginal distribution $F_{ij}(x) = (F_1(x)+F_2(x))/2$. Under the assumption of generalized exchangeability, the bivariate marginal distribution for prospects i and j satisfies $H(F_i(x_{in}),F_j(x_{jn})) = H(F_j(x_{jn}),F_i(x_{in}))$, implying that the pair (x_{in},x_{jn}) and its permutation (x_{jn},x_{in}) are equally probable. Thus, all values of the statistic s^{*}_{2N} obtained by random permutations of all pairs are equally likely. Thus, the empirical distribution of the realizations of s^{*}_{2N} over a sample of independent iterations approximates the distribution of s^{*}_{2N} in the population of observations generated by random permutations. But sampling from this population is equivalent for a pair of prospects i,j to sampling from the joint empirical distribution $H(F_{N,ij}(x_i),F_{N,ij}(x_j))$. Thus, letting the number of monte carlo iterations approach infinity yields a probability given by the right-hand-side of (25) conditioning on this joint empirical distribution. Then, letting N approach infinity, the joint empirical distribution converges to the population joint distribution $H(F_{ij},F_{ij})$, and the probability on the right-hand-side of (25) converges to the significance level of the test. □

4. A MONTE CARLO STUDY

To assess the usefulness of the statistics discussed in this paper, we start with an extremely simple case where the finite-sample distributions are easily calculated. This will shed light on the finite-sample accuracy of the limiting least favorable case construction, and on the power of the tests.

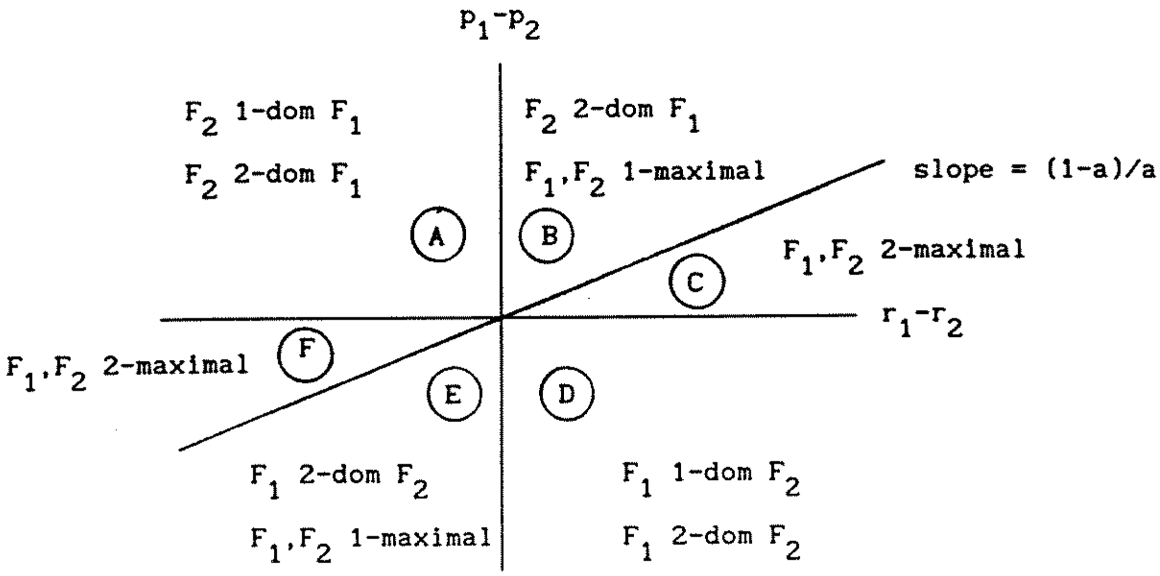
Consider 3-point distributions with probability mass p at x = 0, mass r at x = 1, and mass 1-r-p at x = a ∈ (0,1). Then,

$$F(x) = p\cdot\mathbf{1}(0\le x<a) + (1-r)\cdot\mathbf{1}(a\le x<1) + \mathbf{1}(x\ge 1) \tag{30}$$

and

$$\int_{-\infty}^{x} F(y)dy = p\cdot\max(x,0) + (1-r-p)\cdot\max(x-a,0) + r\cdot\max(x-1,0). \tag{31}$$

The following figure gives the cases of dominance and maximality, for X_1 and X_2 with probabilities (p_1,r_1) and (p_2,r_2), and common support {0,a,1}.



Suppose X_1 and X_2 independent, with N independent draws from each density. Let (k_i, l_i, m_i) denote cell counts at (0,a,1) for X_i. These counts are multinomially distributed. For N > 40 and (p,r) $\in$ [.15,.4], the multinomial is well approximated by

$$\text{(32)} \qquad \begin{bmatrix} (k_i - Np_i)/\sqrt{N} \\ (m_i - Nr_i)/\sqrt{N} \end{bmatrix} \overset{a}{\sim} N(0,\Sigma) \qquad \text{with} \qquad \Sigma = \begin{bmatrix} p_i(1-p_i) & -p_i r_i \\ -p_i r_i & r_i(1-r_i) \end{bmatrix}.$$

The figure above with k_i/N in place of p_i and m_i/N in place of r_i defines regions A-F corresponding to cases for calculation of the test statistics:

A. $k_1 - k_2 \geq 0$ and $m_1 - m_2 \leq 0$ --

$d^{*12}_{2N} = \max(k_1-k_2, m_2-m_1)/N$ $\quad d^{*21}_{2N} = 0$ $\quad d^{*}_{2N} = 0$

$s^{*12}_{2N} = ((k_1-k_2)a+(m_2-m_1)(1-a))/N$ $\quad s^{*12}_{2N} = 0$ $\quad s^{*12}_{2N} = 0$

B. $k_1 - k_2 \geq 0$ and $0 \leq m_1 - m_2 \leq (k_1 - k_2)a/(1-a)$ --

$d^{*12}_{2N} = (k_1-k_2)/N$ $\quad d^{*21}_{2N} = (m_1-m_2)/N$ $\quad d^{*}_{2N} = \min(k_1-k_2, m_1-m_2)/N$

$s^{*12}_{2N} = (k_1-k_2)a/N$ $\quad s^{*21}_{2N} = 0$ $\quad s^{*}_{2N} = 0$

C. $k_1 - k_2 \geq 0$ and $m_1 - m_2 \geq (k_1 - k_2)a/(1-a)$ --

$$d_{2N}^{*12} = (k_1-k_2)/N \qquad d_{2N}^{*21} = (m_1-m_2)/N \qquad d_{2N}^{*} = \min(k_1-k_2, m_1-m_2)/N$$

$$s_{2N}^{*12} = (k_1-k_2)a/N \qquad s_{2N}^{*21} = ((k_2-k_1)a+(m_1-m_2)(1-a))/N$$

$$s_{2N}^{*} = \min((k_1-k_2)a, (k_2-k_1)a+(m_1-m_2)(1-a))/N$$

Regions A, B, C, are mirrored by regions D, E, F, respectively, with the subscripts reversed.

Define $U = [(k_1-k_2)/\sqrt{N} - \sqrt{N}(p_1-p_2)]/\sigma_1$ and $V = [(m_1-m_2)/\sqrt{N} - \sqrt{N}(r_1-r_2) - \rho\sigma_2 U]/\sigma_3$, where $\sigma_1 = [p_1(1-p_1)+p_2(1-p_2)]^{1/2}$, $\sigma_2 = [r_1(1-r_1)+r_2(1-r_2)$ $\rho = -(p_1r_1+p_2r_2)/\sigma_1\sigma_2$, and $\sigma_3 = \sigma_2[1-\rho^2]^{1/2}$. Then U and $V|U$ are approximately standard normal. The distributions then satisfy

$$P(\sqrt{N}d_{2N}^{*12} > \varepsilon) = 1 - P(U \leq [\varepsilon-\sqrt{N}(p_1-p_2)]/\sigma_1 \ \&\ V \geq [-\varepsilon-\sqrt{N}(r_1-r_2)]/\sigma_3)$$

$$\stackrel{a}{=} 1 - \int_{-\infty}^{[\varepsilon-\sqrt{N}(p_1-p_2)]/\sigma_1} \phi(u)\Phi(([\varepsilon+\sqrt{N}(r_1-r_2)+\rho\sigma_2 u]/\sigma_3)du$$

$$P(\sqrt{N}d_{2N}^{*21} > \varepsilon) = 1 - P(U \leq [\varepsilon-\sqrt{N}(p_2-p_1)]/\sigma_1 \ \&\ V \geq [-\varepsilon-\sqrt{N}(r_2-r_1)]/\sigma_3)$$

$$\stackrel{a}{=} 1 - \int_{-\infty}^{[\varepsilon-\sqrt{N}(p_2-p_1)]/\sigma_1} \phi(u)\Phi(([\varepsilon+\sqrt{N}(r_2-r_1)+\rho\sigma_2 u]/\sigma_3)du$$

$$P(\sqrt{N}d_{2N}^{*} > \varepsilon) = P(U < [-\varepsilon-\sqrt{N}(p_1-p_2)]/\sigma_1 \ \&\ V < [-\varepsilon-\sqrt{N}(r_1-r_2)]/\sigma_3)$$

$$+ P(U > [\varepsilon-\sqrt{N}(p_1-p_2)]/\sigma_1 \ \&\ V > [\varepsilon-\sqrt{N}(r_1-r_2)]/\sigma_3)$$

$$\stackrel{a}{=} \int_{-\infty}^{[-\varepsilon-\sqrt{N}(p_1-p_2)]/\sigma_1} \phi(u)\Phi(([-\varepsilon-\sqrt{N}(r_1-r_2)-\rho\sigma_2 u]/\sigma_3)du$$

$$+ \int_{[\varepsilon-\sqrt{N}(p_1-p_2)]/\sigma_1}^{+\infty} \phi(u)\Phi(([-\varepsilon+\sqrt{N}(r_1-r_2)+\rho\sigma_2 u]/\sigma_3)du.$$

Define random variables

$$W = [(k_1-k_2)a/\sqrt{N} - \sqrt{N}(p_1-p_2)]/\tau_1$$

and

$$Y = [(k_1-k_2)a/\sqrt{N} - (m_1-m_2)(1-a)/\sqrt{N} - \sqrt{N}(p_1-p_2)a + \sqrt{N}(r_1-r_2)(1-a) - \lambda\tau_2 W]/\tau_3,$$

linear transformations of U and V with $\tau_1 = a\sigma_1$,

$$\tau_2 = [a^2\sigma_1^2 + (1-a)^2\sigma_2^2 - 2a(1-a)\sigma_1\sigma_2\rho]^{1/2},$$

$$\lambda = (a^2\sigma_1^2 - a(1-a)\rho\sigma_1\sigma_2)/\tau_1\tau_2,$$

and $\tau_3 = \tau_2[1-\lambda^2]^{1/2}$. Then, the distributions of the statistics for no second-degree maximality satisfy

$$P(\sqrt{N}s_{2N}^{*12} > \varepsilon)$$

$$= 1 - P(W \le [\varepsilon-\sqrt{N}(p_1-p_2)a]/\tau_1 \ \& \ Y \le [\varepsilon-\sqrt{N}((p_1-p_2)a+\sqrt{N}(r_1-r_2)(1-a)]/\tau_3)$$

$$\overset{a}{=} 1 - \int_{-\infty}^{[\varepsilon-\sqrt{N}(p_1-p_2)]/\tau_1} \phi(u)\cdot$$

$$\cdot\Phi(([\varepsilon-\sqrt{N}(p_1-p_2)a + \sqrt{N}(r_1-r_2)(1-a) - \lambda\tau_2 u]/\tau_3)du$$

$$P(\sqrt{N}s_{2N}^{*21} > \varepsilon)$$

$$= 1 - P(W \le [\varepsilon-\sqrt{N}(p_2-p_1)a]/\tau_1 \ \& \ Y \le [\varepsilon-\sqrt{N}((p_2-p_1)a+\sqrt{N}(r_2-r_1)(1-a)]/\tau_3)$$

$$\overset{a}{=} 1 - \int_{-\infty}^{[\varepsilon-\sqrt{N}(p_2-p_1)]/\tau_1} \phi(u)\cdot$$

$$\cdot\Phi(([\varepsilon-\sqrt{N}(p_2-p_1)a + \sqrt{N}(r_2-r_1)(1-a) - \lambda\tau_2 u]/\tau_3)du P(\sqrt{N}s_{2N}^{*} < \varepsilon)$$

$$P(\sqrt{N}s_{2N}^{*} > \varepsilon)$$

$$= P(W < [-\varepsilon-\sqrt{N}(p_1-p_2)a]/\tau_1 \ \& \ Y > [\varepsilon-\sqrt{N}((p_1-p_2)a+\sqrt{N}(r_1-r_2)(1-a)]/\tau_3)$$

$$+ P(W > [\varepsilon-\sqrt{N}(p_1-p_2)a]/\tau_1 \ \& \ Y < [-\varepsilon-\sqrt{N}((p_1-p_2)a+\sqrt{N}(r_1-r_2)(1-a)]/\tau_3)$$

$$\overset{a}{=} \int_{-\infty}^{[-\varepsilon-\sqrt{N}(p_2-p_1)]/\tau_1} \phi(u)\cdot$$

$$\cdot\Phi(([-\varepsilon + \sqrt{N}(p_2-p_1)a - \sqrt{N}(r_2-r_1)(1-a) + \lambda\tau_2 u]/\tau_3)du$$

$$+ \int_{-\infty}^{[-\varepsilon-\sqrt{N}(p_2-p_1)]/\tau_1} \phi(u)\cdot$$

$$\Phi(([-\varepsilon + \sqrt{N}(p_2-p_1)a - \sqrt{N}(r_2-r_1)(1-a) + \lambda\tau_2 u]/\tau_3)du.$$

Table 1 gives the results of calculating these expressions in various cases.

The first experiment considers a case where X_2 strictly second-degree dominates X_1, but the prospects are first-degree maximal. The table gives the probabilities at various sample sizes that tests will reject the null hypotheses that d^{*12}, d^{*21}, d^*, s^{*12}, s^{*21}, or s^* are zero. The first four null hypotheses are false in this case, and the last two are true. The statistics d^{*12}_{2N}, d^{*21}_{2N} are the Smirnov statistics, and d^*_{2N} is the Kolmogorov-Smirnov statistic. Table 1 gives their power against this case. Note that relatively large sample sizes are needed in this case to achieve Type I and Type II error probabilities of comparable magnitude, illustrating the relatively poor power characteristics of these tests. The second-degree test s^*_{2N} has an actual size that substantially exceeds its nominal size for samples below 800. Thus, the pooled distribution yielding the limiting least favorable case in Theorem 7 is a poor approximation for smaller samples, and use of the procedure for such smaller samples is not recommended.

Experiment 2 considers a case where X_2 first-degree dominates X_1, and therefore also second-degree dominates X_1. The first and second degree statistics are comparable in power characteristics, and do not show any large bias in nominal size for small samples.

Experiment 3 considers a case where X_1 and X_2 are second-degree maximal, and hence first-degree maximal. The power of the second-degree test against this case is a little lower than the power of the first-degree test.

Taken together, the experiments suggest that when X_1 and X_2 are either second-degree maximal, or one first-degree dominates the other, so that the first-degree and second-degree tests should either both accept or both reject, the power characteristics of the tests are comparable. In the case where one second-degree dominates the other, but they are first-degree maximal, large samples appear to be required to achieve actual sizes near the nominal size.

Our final monte carlo experiments utilize normally distributed random variables that are correlated across prospects and over time. Suppose X_1,X_2 satisfy

(33) $$X_{kn} = (1-\lambda)[\alpha_k+\beta_k(\sqrt{\rho}Z_{0n}+\sqrt{1-\rho}Z_{kn})] + \lambda X_{k,n-1},$$

where the Z_{kn} are standard normal random variables with mean zero and variance one that are independent across k and n; and α_k, β_k, ρ, λ are parameters satisfying $\rho \in [0,1)$ and $\lambda \in (-1,1)$. Then, $EX_{kn} = \alpha_k$, $Var(X_{kn}) = \beta_k^2$, and $Corr(X_{kn},X_{jm}) = \rho\lambda^{|n-m|}$, and the processes are α-mixing and generalized exchangeable.

Table 2 gives the results of the experiments. In each experiment, the samples are of size 50, with 1000 permutations used to determine significance. Each experiment is repeated 100 times. Experiment 1 considers a case in which X second-degree dominates Y, but the two distributions are first-degree maximal, so that we expect both the one-sided and two-sided first-degree tests to reject, as well as the one-sided second-degree test for Y dominating X. The remaining second-degree tests are expected to accept. The first panel in the table gives the averages over the 100 trials of the statistics observed in the sample, the critical levels for 10, 5 and one percent tests, and the significance levels of the sample statistics. The last entry gives the percentage of the trials that reject at the 10 percent significance level. Standard errors for the monte carlo experiment are given in parentheses. One finds that for these contrasting densities, the test rejects the false nulls with reasonably high power, even for the small sample size. The second-degree tests reject the hypothesis that Y second-degree dominates X with high power; and accept the remaining hypotheses with low type I errors.

Experiment 2 has Y first-degree dominating X, and hence also second-degree dominating X. Here, the tests reject first or second degree dominance of Y by X with reasonable power, and in the remaining tests accept the null hypotheses as expected. Experiment 3 has X and Y second-degree maximal, so that they are also first-degree maximal. In this case, all the tests are expected to reject. We find this to be the case, although the tests for Y dominating X have only moderate power.

Figures 1 and 2 illustrate the cumulative distribution functions of the first and second-degree two-sided test statistics for a sample of size N = 800 from each of the densities $N(0,1)$ and $N(0.5,2)$. These statistics are not normalized by $\sqrt{N}$. There is a substantial probability that the second-degree statistic has value zero; this occurs because the slowly varying integrated empirical process has a positive probability of univalence over its domain.

Further experiments, not reported, find that as in the case of the three-point distribution, the actual sizes of the tests may exceed nominal sizes when sample size is small and the distributions are close to the boundary of the null. For example, the case $X \sim N(0,1)$, $Y \sim N(-0.5,16)$, and N = 50 has X second-degree dominating Y, so that one expects acceptance of the null in the two-sided test. However, the actual rejection rate in a 10 percent significance level test is 13 percent, slightly larger than the nominal significance level. This suggests caution in applying the tests to small sample sizes.

5. AN EMPIRICAL APPLICATION

We apply the tests for not first or second degree maximal to data on daily returns of closed-end mutual funds. These funds hold relatively diversified portfolios of stocks, and are often the sole stock market purchase of consumers. Unless these stocks are held to diversify risk in a broader portfolio that includes non-stock assets, we would expect rational consumers to avoid dominated funds. Then, successfully marketed funds should be maximal.

We study five funds, observed over 4685 trading days from June 15, 1971 through December 29, 1989. Table 3 lists the funds and their salient characteristics. Table 4 gives the means and variances of the daily returns from these funds: ASA is a high-mean, high-variance fund, JHI is a low-mean, low variance fund, and the remainder are in between. Table 5 gives the within-period correlations of the funds; these are small, but there does appear to be some contemporaneous correlation between TY and the remaining funds. Tables 6 gives vector autoregressions of the returns on a trend and on five lags of the returns on these five funds. There appears to be some serial correlation at low lags, contradicting some versions of the efficient markets hypothesis. Table 7 gives autoregressions for each return on 25 lags. There is little evidence of correlations at long lags, although TY has some correlation at lags 22 and 23, approximately one trading month. There is no significant trend in either Table 6 or Table 7. These results are at least not obviously inconsistent with the maintained assumptions that the series are stationary and α-mixing.

Table 8 gives the results of applying the stochastic dominance tests to each pair of funds. First degree maximality is supported in all the comparisons. Second degree dominance of GAM over NGS and of TY over NGS is accepted at the 10 percent level, the significance level of the statistic is between 10 percent and 1 percent for GAM versus TY, JHI vs NGS, and JHI vs TY, and second-degree dominance is rejected for the remaining pairs. Of course, these pair-wise tests are not independent.

In summary, these results suggest that TY may dominate all the remaining general funds, and that NGS may be dominated by all the general funds. ASA is maximal relative to all the general funds. This in turn suggests that consumers who hold only NGS in their asset portfolios are either not von Neumann-Morgenstern utility maximizers, or are using NGS to provide diversification against non-stock-market assets that is not attainable using the dominating TY fund. Used as a prescriptive tool, the tests could be employed by consumers to screen dominated funds such as NGS out of their portfolios unless holding them can be justified on grounds of diversification, or by portfolio managers to evaluate the relative attractiveness of alternative investment strategies.

REFERENCES

Adler, R. and G. Samorodnitsky (1987) "Tail Behavior for the Suprema of Gaussian Processes with Applications to Empirical Processes", The Annals of Probability, 15: 1339-1351.

Arrow, K. (1971), Essays in the Theory of Risk Bearing, Markham, Chicago.

Billingsley, P. (1968), Convergence of Probability Measures, Wiley, New York.

Dunford, N. and J. Schwartz (1964), Linear Operators 1, Interscience, New York

Durbin, J. (1973), Distribution Theory for Tests Based on the Sample Distribution Function, SIAM, Philadelphia.

Durbin, J. (1985), "The First Passage Density of a Continuous Gaussian Process to a General Boundary", Journal of Applied Probability 22: 99-122.

Fishburn, P. and R. Vickson (1978), "Theoretical Foundations of Stochastic Dominance", in G. Whitmore and M. Findlay (eds.) Stochastic Dominance, Lexington Books, Lexington, MA, 39-113.

Gnedenko, B. and V. Koroyuk (1961), "On the Maximum Discrepancy between Two Empirical Distributions", Selected Translations in Mathematical Statistics and Probability 1: 13-16.

Hadar, J. and W. Russell (1969), "Rules for Ordering Uncertain Prospects", American Economic Review 59: 25-34.

Hadar, J. and W. Russell (1971), "Stochastic Dominance and Diversification", Journal of Economic Theory 3: 288-305.

Hadar, J. and W. Russell (1974a), "Diversification of Interdependent Prospects", Journal of Economic Theory 7: 231-240.

Hadar, J. and W. Russell (1974b), "Stochastic Dominance in Choice under Uncertainty", in M. Balsch, D. McFadden, S. Wu (ed), Essays on Economic Behavior under Uncertainty, North Holland, Amsterdam, 134-150.

Hadar, J. and W. Russell (1978), "Applications in Economic Theory and Analysis", in G. Whitmore and M. Findlay (eds.) Stochastic Dominance, Lexington Books, Lexington, MA., 295-333.

Hakanson, N. (1971) "On Optimal Myopic Portfolio Policies, with and without Serial Correlation of Yields", Journal of Business 44:324-334.

Hall, P. and C. Heyde (1980) Martingale Limit Theory and Its Application, Academic Press, New York.

Hanoch, G. and H. Levy (1969) "The Efficiency Analysis of Choices Involving Risk", Review of Economic Studies 36:335-346.

Joy, O. and R. Porter (1974), "Stochastic Dominance and Mutual Fund Performance", Journal of Finance and Quantitative Analysis 9: 25-31.

Kroll, Y., H. Levy, and H. Markowitz (1984), "Mean-variance versus Direct Utility Maximization", Journal of Finance 39: 47-62.

McFadden, D. (1989) "Testing for Stochastic Dominance", in T. Fomby and T. K. Seo (eds) Studies in the Economics of Uncertainty, Springer, New York.

Meyer, J. and R. Rasche (1988), "Kolmogorov-Smirnov Tests for the Location and Scale Hypothesis with Applications to Portfolios of Common Stock", Working Paper, Michigan State University.

Pollard, D. (1984), Convergence in Stochastic Processes, Springer, New York.

Porter, R. (1978), "Portfolio Applications: Empirical Studies", in G. Whitmore and M. Findlay (eds.) Stochastic Dominance, Lexington Books, Lexington, 117-161.

Press, W., B. Flannery, S. Teukolsky, W. Vetterling (1986), Numerical Recipes, Cambridge Univ. Press, Cambridge.

Sharpe, W., (1963) "A Simplified Model for Portfolio Analysis", Management Science, 9:277-293.

Siegmund, D. (1985) Sequential Analysis, Springer-Verlag, New York.

White, H. and I. Domowitz (1984) "Nonlinear Regression with Dependent Observations", Econometrica, 52, 643-61.

FOOTNOTES

1. We are indebted to Robert Hall, Lars Hansen, Donald Katzner, Deborah Nolan, Rulon Pope, Danny Quah, Paul Ruud, Ken Singleton, and Bob Young for useful discussions, but are solely responsible for errors.

2. Prospects X_1 and X_2 are <u>distinct</u> if $\Pr(X_1 = X_2) < 1$. Note that X_1 and X_2 may be distinct, but have common marginal distributions $F_1 \equiv F_2$.

3. The distributions F_k will be conditioned on an information set $\mathcal{I}$ in some applications, and stochastic dominance will be defined conditional on $\mathcal{I}$.

4. The boundedness of the domain of returns is not required for the definition of test statistics, or computation of the statistics and their significance levels in finite samples, but compactness is used in establishing asymptotic properties of the tests.

5. A process X_n, $n = 1,2,\ldots$, is *stationary* if the joint distribution of the X_n for $j \le n \le i$ equals the joint distribution of the X_n for $j+k \le n \le i+k$, for any $k > 0$ and $0 \le j \le i$.

6. A process X_n, $n = 1,2,\ldots$ is *α-mixing* if there exists a sequence $\alpha(j)$ such that $\lim_{j\to\infty} \alpha(j) = 0$ and $\alpha(j) \ge |P(B\cap A)-P(B)P(A)|$ for each event A regarding the behavior of the process up to time n and each event B regarding the behavior of the process after time n+j.

7. In the case of independent prospects, and independent realizations across periods, the covariance function for $w \le v$ is

$$\rho(v,w) = \int_{-\infty}^{w} (w-y)^2(F_1(dy)+F_2(dy))$$
$$- (v-w)(\psi_1(w)+\psi_2(w)) - \psi_1(w)\psi_1(v) - \psi_2(w)\psi_2(v).$$

8. In the special case K = 2, X_1 and X_2 statistically independent, and observations statistically independent across periods, d^*_{2N} is the classical two-sample Kolmogorov-Smirnov statistic. If the two prospects have the same distribution, then this statistic has an analytic exact finite sample distribution (Durbin, 1973, p. 43) and a large sample approximation (Durbin, 1973, p. 22,44):

$$\Pr(d^*_{2N} \le d | F_1 \equiv F_2) \approx \frac{\sqrt{2\pi}}{d} \sum_{j=1}^{\infty} e^{-(2j-1)^2\pi^2/8d^2}.$$

As in the general case, we argue that this is the limiting least favorable case of those satisfying the null hypothesis, and hence that the distribution above can be used to approximate significance levels.

Since the process $\tilde{S}_{2N}(w)$ from Theorem 6 converges in distribution to a Gaussian process, results on the tail behavior of Gaussian processes provide a guide to approximations to the limiting distribution of s^*_{2N}. Adler and Samorodnitsky (1987) find that under general regularity conditions tail probabilities $P(s^*_{2N} > s)$ for the suprema of standard Gaussian processes have bounds for large s of the form $Cs^{\alpha}\exp(-s^2/2\sigma^2)$, where σ^2 is the maximum variance of the process. For the case of prospects that are independent of each other and across periods, McFadden (1989) established by a direct chaining argument the asymptotic large-s bounds $0.25\cdot\exp(-s^2/\pi\sigma^4) < P(s^*_{2N} > s) < 3\cdot\exp(-s^2/8\sigma^2)$. These results suggest that a good empirical approximation to the tail probability can be obtained using the functional form $P(s^*_{2N} > s) \approx Cs^{\alpha}\exp(-\beta s^2)$.

APPENDIX A. A GAUSS PROGRAM TO CALCULATE THE STOCHASTIC DOMINANCE STATISTICS

SD2 is a GAUSS procedure that calculates the first and second degree stochastic dominance statistics and their significance levels for two prospects. It takes as inputs the vectors of returns x and y for the two prospects, and the number of permutations r to be used to approximate the distribution of the statistic. Its output is a 5 by 6 array:

Column	Description
1	First-degree statistic, one-sided for x 1-dom y
2	First-degree statistic, one-sided for y 1-dom x
3	First-degree statistic, two-sided for not 1-maximal
4	Second-degree statistic, one-sided for x 2-dom y
5	Second-degree statistic, one-sided for y 2-dom x
6	Second-degree statistic, two-sided for not 2-maximal

Row	
1	Statistic calculated from observations
2	Approximate 10 percent critical level
3	Approximate 5 percent critical level
4	Approximate 1 percent critical level
5	Approximate significance level of observed statistic

To use the comparative advantage of GAUSS in matrix over recursive operations, this program calculates the first and second degree processes as vectors, and then forms the statistics from these vectors, rather than using a recursive loop. In a FORTRAN version of this procedure, it is computationally advantageous to reverse this, and do all calculations in a single loop through the data, as described in the text.

```
proc 1 = sd2(x,y,r);
local z,n,n2,l,rr,zz,f,d,s,ss,sss,rrr,ri,ff,i,p,r10,r05,r01;

@preliminaries@

z = x|y;                                 @stack data@
n = rows(x);                             @number of observations@
n2 = rows(z);
l = sortind(z);                          @index of data in ascending order@

rr = round(20000/n);                     @divide repetitions into chunks@
                                         @that fit the computer@
zz = ((z[l]|100000)-(-100000|z[l]));     @vector of increments in z@

@calculation of SD statistics for the observations@

f = ones(n,1);
f = f|(-f);                              @indicator for observed membership@
d = cumsumc(f[l,1]);                     @first-degree empirical process@
s = cumsumc((0|d).*zz);                  @second-degree empirical process@
                                         @zeros take care of tails of zz@
d = maxc(d)~maxc(-d);                    @the statistics@
```

```
d = d~minc(d');
s = maxc(s)~maxc(-s);
s = s~minc(s');
ss = d~s;
sss = ss;

@calculation of SD statistics for random permutations@

rrr = r;
do while rrr > 0;                        @loop until all chunks are done@
  ri = minc(rr|rrr);
  rrr = rrr-ri;
  ff = 2*(rndu(n,ri) .> 0.5)-1;
  ff = ff|(-ff);                            @random permutation of membership@
  d = cumsumc(ff[1,.]);                     @first-degree empirical process@
  s = (zeros(1,ri)|d).*(zz*ones(1,ri));     @second-degree empirical process@
  s = cumsumc(s);                           @column vectors of statistics@
  d = maxc(d)~maxc(-d);
  d = d~minc(d');
  s = maxc(s)~maxc(-s);
  s = s~minc(s');
  ss = ss|(d~s);
endo;

@summary statistics@

ss = ss[2:r+1,.];                          @statistics from permutations@
i = 0;
do while i < 6;                            @sort each column of ss@
  i = i+1;
  ss[.,i] = sortc(ss[.,i],1);
endo;

p = meanc(sss .< ss);                      @significance level calculation@
r10 = round(.90*r);                        @10 percent critical level@
r05 = round(.95*r);
r01 = round(.99*r);
sss = sss|ss[r10,.]|ss[r05,.]|ss[r01,.];
sss = sss|(p');
retp(sss);
endp;
```

APPENDIX B. A FORTRAN PROGRAM TO CALCULATE THE STOCHASTIC DOMINANCE STATISTICS

The following program calculates the first and second degree statistics for tests that a pair of prospects are not stochastically maximal, and using random permutation of the ownership of realizations within each period, calculates the significance level of the statistics.

```
c ****************** START OF SDOM PROCEDURE ************************
c
c  The FORTRAN subroutine given below considers the hypothesis that a
c  distribution G (weakly) stochastically dominates a distribution F,
c  calculates test statistics for first and second degree stochastic
c  dominance, and calculates the significance levels of these values.
c  The program uses the following variables and parameters:
c
c     n    The size of the sample from each of the distributions F and G,<4001.
c     n2   = n*2
c     z    A vector containing the n observations from F, followed by the
c          n observations from G, so that z(i),z(i+n) are the pair of
c          observations from period i, for i=1,...n.  The length of z is n2.
c     sd1  The test statistic for first-degree stochastic dominance.
c     p1   The probability of a statistic larger than sd1 if the null holds.
c     q1   The probability of a statistic smaller than sd1 if the null holds.
c     sd2  The test statistic for second-degree stochastic dominance.
c     p2   The probability of a statistic larger than sd2 if the null holds.
c     q2   The probability of a statistic smaller than sd2 if the null holds.
c
c     randu     A function that returns a uniform pseudo-random number.
c     irbit2    A function that returns a pseudo-random .true. or .false.
c     indexx    A function that indexes the components of a vector in
c               ascending order.
c     nit       A parameter controlling the number of random permutations
c               used to calculate the significance level,
c               nit = multiple of 1000, 999< nit < 8001
c
      subroutine sdom(n,n2,nit,z,sd1,p1,q1,sd2,p2,q2,iseed)
      real z,sd1,sd2,p1,p2,t1,t2,q1,q2
      integer n,n2,nit,it,i,j,l,m
      logical d
      dimension l(8000),m(4000),d(8000),z(8000)
      common/ldata/ d,/ndata/ l,m
c
c  the internal vectors dimensioned and defined:  l(n2) is the vector
c  of indices of the elements of z in ascending order; m(n) is the vector
c  of indices of the elements of l <= n; d(n2) is the vector of indicators
c  that are .true. for a point from G and .false. for a point from F.
c
c  Parameter nit is the number of iterations to approximate significance
c  level.  A value of 5000 permits a test at the nominal 0.05 level with
c  a true size between .045 and .055, and at most prob. .01 of misclassif.
c  The precision of other nit values can be calculated from the normal
```

```
c  approximation to the binomial distribution.
c
c
      n2=n+n
      rnit=1./float(nit)
c  index the data in ascending order, and define d and m
      call indexx(n,n2,z)
      j=1
      do 10 i=1,n2
        if(l(i).le.n) then
          m(j)=i
          j=j+1
        endif
   10 continue
c  compute the test statistics
        call sdf(n,n2,z,sd1,sd2)
c
c
c  for nit iterations, randomly permute the data between F and G in each
c  period, and count the number of outcomes exceeding the values of
c  the statistics from the original data
c
      p1=0.
      p2=0.
      q1=0.
      q2=0.
c
      do 100 i=1,nit
        call rperm(n,n2,iseed)
        call sdf(n,n2,z,t1,t2)
c
c  accumulate statistics
        if(t1.gt.sd1) p1=p1+1.
        if(t1.lt.sd1) q1=q1+1.
        if(t2.gt.sd2) p2=p2+1.
        if(t2.lt.sd2) q2=q2+1.
c
  100 continue
c
c  compute the final statistics and significance levels
c
        p1=p1*rnit
        p2=p2*rnit
        q1=q1*rnit
        q2=q2*rnit
        sd1=sd1*sqrt(rnit)
        sd2=sd2*sqrt(rnit)
c
      return
      end
c
c
c  subroutine to calculate the test statistics sd1 and sd2
c  note:  the returned statistics are not normalized by
c  dividing by sqrt(n)
c
```

```
      subroutine sdf(n,n2,z,sd1,sd2)
      real z,sd1,sd2,dd,ss,dpos,dneg,spos,sneg
      integer n,n2,i,l,m
      logical d
      dimension l(8000),m(4000),d(8000),z(8000)
      common/ldata/ d,/ndata/ l,m
c  initializations
c
      dpos=0.
      dneg=0.
      spos=0.
      sneg=0.
      if(d(1)) then
        dd=1.
      else
        dd=-1.
      endif
      ss=0.
c  loop through data
      do 10 i=2,n2
c
        ss=ss+dd*(z(l(i))-z(l(i-1)))
        if(d(i)) then
          dd=dd+1.
        else
          dd=dd-1.
        endif
        if(dd.gt.dpos) dpos=dd
        if(dd.lt.dneg) dneg=dd
        if(ss.gt.spos) spos=ss
        if(ss.lt.sneg) sneg=ss
c
   10 continue
      sd1=-dneg
      sd2=-sneg
      if(sd1.gt.dpos) sd1=dpos
      if(sd2.gt.spos) sd2=spos
      return
      end
c
c
c  Subroutine to randomly permute the values d for each pair of observations.
c
      subroutine rperm(n,n2,iseed)
      integer n,n2,j,k,l,m
      logical irbit2,d
      dimension l(8000),m(4000),d(8000)
      common/ldata/ d,/ndata/ l,m
c
      do 10 j=1,n
        if(irbit2(iseed)) then
          d(m(j)) = .not.d(m(j))
          do 15 k=1,n2
             if (l(k).eq.l(m(j))+n) d(k) = .not.d(k)
   15     continue
        endif
```

```
   10 continue
      return
      end
c
c
c  function to return random .true. or .false., a modification of Press et.
c  al, Numerical Recipes, p. 213.  The C language bit-modifying functions
c  IAND (bitwise AND), IOR (bitwise OR), NOT (bitwise complement),
c  ISHFT(j,k) (leftshift j by k bits), IEOR (bitwise exclusive OR) are used.
c
      logical function irbit2(iseed)
      parameter (ib1=1,ib2=2,ib5=16,ib18=131072,mask=ib1+ib2+ib5)
      if(iand(iseed,ib18).ne.0) then
        iseed=ior(ishft(ieor(iseed,mask),1),ib1)
        irbit2=.true.
      else
        iseed=iand(ishft(iseed,1),nor(ib1))
        irbit2=.false.
      endif
      return
      end
c
c
c  subroutine to sort in place, adapted from Numerical Recipes, p. 253
c  z contains the input vector of length n2, l contains the indices of
c  the elements of z, in ascending order.  d contains .true. if the
c  corresponding element of l exceeds n, .false. otherwise.
c
      subroutine indexx(n,n2,z)
      real z,q
      integer n,n2,j,k,ir,lt,mm,i,l,m
      logical d
      dimension d(8000),l(8000),m(4000),z(8000)
      common /ldata/ d,/ndata/ l,m
      do 11 j=1,n2
        l(j)=j
   11 continue
      k=n+1
      ir=n2
   10 continue
        if(k.gt.1) then
          k=k-1
          lt=l(k)
          q=z(lt)
        else
          lt=l(ir)
          q=z(lt)
          l(ir)=l(1)
          ir=ir-1
          if(ir.eq.1) then
            l(1)=lt
            do 5 mm=1,n2
              if(l(mm).gt.n) then
                d(mm) = .true.
              else
                d(mm) = .false.
```

```
          endif
 5      continue
        return
      endif
    endif
    i=k
    j=k+k
20  if(j.le.ir) then
      if(j.lt.ir) then
        if(z(l(j)).lt.z(l(j+1))) j=j+1
      endif
      if(q.lt.z(l(j))) then
        l(i)=l(j)
        i=j
        j=j+j
      else
        j=ir+1
      endif
    go to 20
    endif
    l(i)=lt
    go to 10
c     return
      end
c
c  The following C-language functions perform elementary bit operations
c
int nor_(ia)
int *ia ;
{      return(~*ia);
  }
c
int iand_(ia,ib)
int *ia; int *ib ;
{      return(*ia & *ib);
  }
c
int ior_(ia,ib)
int *ia, *ib;
{      return (*ia|*ib);
  }
c
int ieor_(ia,ib)
int *ia, *ib ;
{      return(*ia^*ib);
  }
c
int ishft_(ia,k)
int *ia, *k ;
{      return(*ia<<*k) ;
  }
```

TABLE 1. POWER OF TESTS FOR STOCHASTIC DOMINANCE

Experiment 1: X_2 2-dom X_1 and X_1, X_2 1-maximal

$p_1 = .3$ $r_1 = .3$ $p_2 = .2$ $r_2 = .25$

Probability of Rejecting the Null that the Associated Population Quantity is Zero

Nominal Size 10%

Sample Size	d_{2N}^{*12}	d_{2N}^{*21}	d_{2N}^{*}	s_{2N}^{*12}	s_{2N}^{*21}	s_{2N}^{*}
50	32.1%	15.5%	33.6%	22.1%	5.2%	18.1%
100	50.2%	21.6%	50.2%	33.5%	3.9%	20.6%
200	75.3%	32.3%	69.5%	56.6%	2.5%	19.5%
400	95.2%	50.1%	85.5%	86.5%	1.3%	13.7%
800	100.0%	74.5%	93.5%	99.5%	0.6%	6.6%
1600	100.0%	94.4%	99.3%	100.0%	0.3%	1.8%
3200	100.0%	99.8%	99.7%	100.0%	0.0%	0.2%

Nominal Size 5%

Sample Size	d_{2N}^{*12}	d_{2N}^{*21}	d_{2N}^{*}	s_{2N}^{*12}	s_{2N}^{*21}	s_{2N}^{*}
50	34.5%	16.1%	23.9%	22.0%	5.0%	12.6%
100	54.8%	22.8%	39.8%	34.6%	3.6%	15.9%
200	81.7%	34.7%	60.9%	63.7%	2.3%	16.2%
400	99.0%	54.5%	80.3%	100.0%	1.2%	13.7%
800	100.0%	80.6%	93.5%	100.0%	0.5%	5.5%
1600	100.0%	98.2%	99.0%	100.0%	0.3%	1.4%
3200	100.0%	100.0%	99.7%	100.0%	0.0%	0.2%

Experiment 2: X_2 2-dom and 1-dom X_1

$p_1 = .3$ $r_1 = .25$ $p_2 = .2$ $r_2 = .3$

Probability of Rejecting the Null that the

Associated Population Quantity is Zero

Nominal Size 10%

Sample Size	d^{*12}_{2N}	d^{*21}_{2N}	d^{*}_{2N}	s^{*12}_{2N}	s^{*21}_{2N}	s^{*}_{2N}
50	38.9%	2.0%	8.6%	40.6%	1.2%	5.6%
100	56.7%	1.2%	7.1%	58.1%	0.5%	3.1%
200	79.5%	0.6%	4.5%	79.9%	0.3%	0.9%
400	96.2%	0.4%	1.8%	96.0%	0.0%	0.1%
800	100.0%	0.3%	0.3%	99.9%	0.0%	0.0%
1600	100.0%	0.0%	0.0%	100.0%	0.0%	0.0%
3200	100.0%	0.0%	0.0%	100.0%	0.0%	0.0%

Nominal Size 5%

Sample Size	d^{*12}_{2N}	d^{*21}_{2N}	d^{*}_{2N}	s^{*12}_{2N}	s^{*21}_{2N}	s^{*}_{2N}
50	42.0%	1.7%	4.7%	44.3%	1.0%	2.8%
100	62.1%	1.1%	4.1%	64.3%	0.5%	1.6%
200	86.3%	0.6%	2.7%	87.9%	0.3%	0.9%
400	100.0%	0.0%	1.0%	100.0%	0.0%	0.1%
800	100.0%	0.0%	0.0%	100.0%	0.0%	0.0%
1600	100.0%	0.0%	0.0%	100.0%	0.0%	0.0%
3200	100.0%	0.0%	0.0%	100.0%	0.0%	0.0%

Experiment 3: X_1, X_2 2-maximal and 1-maximal

$p_1 = .3$ $r_1 = .3$ $p_2 = .25$ $r_2 = .2$

Probability of Rejecting the Null that the

Associated Population Quantity is Zero

Nominal Size 10%

Sample Size	d_{2N}^{*12}	d_{2N}^{*21}	d_{2N}^{*}	s_{2N}^{*12}	s_{2N}^{*21}	s_{2N}^{*}
50	15.5%	32.1%	33.6%	8.2%	17.1%	26.6%
100	21.5%	50.2%	50.1%	10.3%	20.9%	38.7%
200	32.2%	75.3%	69.4%	16.3%	27.1%	55.1%
400	50.1%	95.0%	85.4%	29.9%	37.3%	73.0%
800	74.6%	99.9%	95.6%	55.1%	53.2%	88.0%
1600	94.5%	100.0%	99.4%	85.5%	74.3%	96.6%
3200	100.0%	100.0%	99.7%	100.0%	98.7%	99.4%

Nominal Size 5%

Sample Size	d_{2N}^{*12}	d_{2N}^{*21}	d_{2N}^{*}	s_{2N}^{*12}	s_{2N}^{*21}	s_{2N}^{*}
50	16.2%	34.5%	23.9%	7.1%	17.9%	19.1%
100	22.8%	54.8%	39.7%	8.8%	22.2%	30.8%
200	34.7%	81.6%	60.8%	14.8%	29.3%	48.0%
400	54.6%	98.9%	80.2%	30.3%	40.9%	68.0%
800	80.6%	100.0%	93.5%	62.3%	59.1%	85.5%
1600	98.4%	100.0%	99.1%	100.0%	82.1%	95.8%
3200	100.0%	100.0%	99.7%	100.0%	98.7%	99.3%

TABLE 2. MONTE CARLO SIZE AND POWER CALCULATIONS, NORMAL PROSPECTS

Sample Size N = 50, Intra-period Correlation ρ = 0.1, Autocorrelation λ = 0.1,

Monte Carlo permutations R = 1000, Experimental trials K = 100

Experiment 1: X second-degree dominates Y, but X,Y first-degree maximal

$X \sim N(0,1)$ and $Y \sim N(-1,16)$

First-Degree	X 1-Dom Y?	Y 1-Dom X?	X,Y not 1-Maximal?
Observation	11.580 (0.332)	22.020 (0.357)	11.500 (0.316)
10% critical level	10.410 (0.055)	10.420 (0.051)	5.980 (0.014)
5% critical level	11.910 (0.060)	11.930 (0.062)	6.760 (0.043)
1% critical level	14.640 (0.082)	14.740 (0.088)	8.100 (0.044)
Significance level	0.097 (0.013)	0.000 (0.000)	0.011 (0.005)
Percent rejections in 10% test	0.570 (0.050)	1.000 (0.000)	0.960 (0.020)

Second-Degree	X 2-Dom Y?	Y 2-Dom X?	X,Y not 2-Maximal?
Observation	0.187 (0.136)	87.026 (1.929)	0.187 (0.136)
10% critical level	36.862 (0.400)	36.874 (0.416)	5.638 (0.094)
5% critical level	45.812 (0.513)	45.681 (0.502)	7.633 (0.117)
1% critical level	62.230 (0.705)	62.221 (0.659)	11.423 (0.177)
Significance level	0.744 (0.005)	0.002 (0.001)	0.491 (0.008)
Percent rejections in 10% test	0.000 (0.000)	1.000 (0.000)	0.020 (0.014)

Experiment 2: Y first-degree dominates X (⇒ Y second-degree dominates X)

$X \sim N(0,16)$ and $Y \sim N(1,16)$

First-Degree	X 1-Dom Y?	Y 1-Dom X?	X,Y not 1-Maximal?
Observation	9.960 (0.383)	2.630 (0.232)	2.430 (0.204)
10% critical level	9.970 (0.043)	10.000 (0.047)	5.650 (0.048)
5% critical level	11.490 (0.061)	11.470 (0.066)	6.210 (0.041)
1% critical level	14.030 (0.089)	14.070 (0.084)	7.540 (0.050)
Significance level	0.169 (0.019)	0.746 (0.024)	0.588 (0.033)
Percent rejections in 10% test	0.430 (0.050)	0.000 (0.000)	0.060 (0.024)

Second-Degree	X 2-Dom Y?	Y 2-Dom X?	X,Y not 2-Maximal?
Observation	51.304 (3.389)	3.437 (0.640)	1.718 (0.295)
10% critical level	46.710 (0.402)	46.225 (0.424)	7.230 (0.110)
5% critical level	58.498 (0.513)	58.050 (0.511)	9.582 (0.137)
1% critical level	80.261 (0.783)	79.465 (0.800)	14.256 (0.195)
Significance level	0.187 (0.022)	0.669 (0.014)	0.420 (0.017)
Percent rejections in 10% test	0.510 (0.050)	0.000 (0.000)	0.060 (0.024)

Experiment 3: X,Y second-degree maximal (⇒ first-degree maximal)

$X \sim N(0,1)$ and $Y \sim N(1,16)$

First-Degree	X 1-Dom Y?	Y 1-Dom X?	X,Y not 1-Maximal?
Observation	21.820 (0.386)	11.320 (0.310)	11.160 (0.286)
10% critical level	10.470 (0.050)	10.420 (0.055)	5.980 (0.014)
5% critical level	11.880 (0.057)	11.930 (0.055)	6.780 (0.041)
1% critical level	14.520 (0.076)	14.720 (0.085)	8.110 (0.046)
Significance level	0.001 (0.003)	0.101 (0.012)	0.009 (0.004)
Percent rejections in 10% test	1.000 (0.000)	0.550 (0.050)	0.950 (0.022)

Second-Degree	X 2-Dom Y?	Y 2-Dom X?	X,Y not 2-Maximal?
Observation	53.449 (2.702)	32.675 (1.174)	25.532 (1.048)
10% critical level	35.666 (0.354)	35.595 (0.353)	7.456 (0.097)
5% critical level	44.851 (0.452)	44.720 (0.457)	9.909 (0.123)
1% critical level	61.129 (0.626)	61.371 (0.622)	14.633 (0.186)
Significance level	0.103 (0.019)	0.149 (0.010)	0.030 (0.012)
Percent rejections in 10% test	0.750 (0.043)	0.360 (0.048)	0.950 (0.022)

TABLE 3. CLOSED-END MUTUAL FUNDS

ASA	ASA Ltd. is a closed-end investment company with at least 50% of its funds in South African gold stocks.
GAM	General American Investment is a closed-end regulated management company investing primarily in medium and high-quality growth stocks, with the aim of long-term capital appreciation.
JHI	John Hancock Investors Trust is a closed-end diversified investment company whose main objective is income distribution to shareholders, and main holdings are in debt securities, up to 50 percent of which are direct placements.
NGS	Niagara Share Corporation is a closed-end management company investing primarily in common stocks, seeking high earnings and dividend potential.
TY	Tri-Continental Corp. is a closed-end diversified management company investing in common stocks and equivalents with the aims of long-term appreciation and growth in income.

TABLE 4. MEANS AND STANDARD DEVIATIONS OF DAILY RETURNS

Stock	Daily Mean	Daily Std. Dev.	Annualized Mean
ASA	0.00090465	0.0255878	0.26504
GAM	0.00063212	0.0174761	0.17857
JHI	0.00047009	0.0132406	0.12997
NGS	0.00058492	0.0183271	0.16420
TY	0.00054928	0.0137133	0.15347

TABLE 5. INTRA-PERIOD CORRELATIONS

	ASA	GAM	JHI	NGS
GAM	-0.0042			
JHI	0.0030	0.0655		
NGS	0.0513	0.0500	0.0926	
TY	0.0602	0.1186	0.1346	0.1925

TABLE 6. VECTOR AUTOREGRESSIONS ON TREND AND FIVE LAGS

INDEPENDENT VARIABLES	DEPENDENT VARIABLES: ASA		GAM		JHI		NGS		TY	
	COEFF	T-STAT	COEFF	T-STAT	COEFF	T-STAT	COEFF	T-STAT	COEFF	T-STAT
CONSTANT	0.001	1.634	0.000	0.817	0.000	1.090	0.000	0.831	0.000	0.176
TREND	-0.000	-0.731	0.000	0.634	0.000	0.582	0.000	0.027	0.000	0.766
ASA LAG 1	-0.010	-0.669	0.023	2.297	-0.015	-1.993	0.021	2.118	0.003	0.461
GAM LAG 1	0.069	3.117	-0.124	-8.322	0.077	6.933	0.251	16.924	0.238	21.328
JHI LAG 1	0.088	2.988	-0.002	-0.100	-0.224	-15.084	0.008	0.423	0.078	5.242
NGS LAG 1	0.005	0.245	0.022	1.483	0.003	0.303	-0.257	-17.287	0.017	1.527
TY LAG 1	-0.000	-0.004	0.061	3.030	0.045	2.989	0.126	6.273	-0.119	-7.907
ASA LAG 2	-0.031	-2.104	-0.017	-1.712	0.006	0.835	0.013	1.287	-0.019	-2.498
GAM LAG 2	0.003	0.119	-0.078	-4.838	0.011	0.882	0.134	8.357	0.106	8.840
JHI LAG 2	0.019	0.611	0.015	0.722	-0.129	-8.441	-0.049	-2.387	0.040	2.568
NGS LAG 2	0.090	3.973	0.018	1.196	0.016	1.375	-0.140	-9.164	-0.002	-0.181
TY LAG 2	0.021	0.687	-0.002	-0.076	0.012	0.790	0.065	3.203	-0.106	-6.959
ASA LAG 3	-0.004	-0.244	-0.003	-0.342	0.005	0.617	0.012	1.168	0.024	3.282
GAM LAG 3	-0.018	-0.754	-0.038	-2.322	0.021	1.741	0.045	2.760	0.074	6.022
JHI LAG 3	0.017	0.561	0.010	0.477	-0.070	-4.529	-0.036	-1.754	0.058	3.714
NGS LAG 3	0.017	0.725	0.021	1.349	-0.003	-0.243	-0.085	-5.532	0.008	0.655
TY LAG 3	0.022	0.730	0.039	1.903	0.015	1.014	0.004	0.202	-0.062	-4.061
ASA LAG 4	-0.013	-0.903	0.005	0.475	-0.008	-1.068	-0.002	-0.235	0.014	1.888
GAM LAG 4	0.012	0.496	-0.054	-3.297	0.013	1.108	0.035	2.194	0.054	4.427
JHI LAG 4	-0.046	-1.511	0.016	0.782	-0.047	-3.080	-0.020	-0.992	-0.019	-1.217
NGS LAG 4	0.042	1.855	0.011	0.687	0.009	0.795	-0.065	-4.242	-0.013	-1.119
TY LAG 4	-0.082	-2.763	0.056	2.744	-0.001	-0.036	0.028	1.401	-0.053	-3.498
ASA LAG 5	-0.005	-0.313	-0.011	-1.101	-0.003	-0.400	0.032	3.278	0.011	1.430
GAM LAG 5	0.074	3.136	0.018	1.105	0.003	0.291	0.050	3.161	0.053	4.442
JHI LAG 5	0.012	0.420	-0.003	-0.143	-0.020	-1.365	0.010	0.484	-0.015	-1.025
NGS LAG 5	-0.013	-0.593	-0.007	-0.476	-0.001	-0.131	-0.056	-3.850	-0.005	-0.449
TY LAG 5	0.027	0.943	-0.008	-0.402	0.009	0.618	0.036	1.882	-0.025	-1.706
ROOT MEAN SQUARE OF LAGS > 2	0.03547		0.02635		0.02399		0.04121		0.03953	

TABLE 7. AUTOREGRESSIONS ON TREND AND 25 LAGS

INDEPENDENT VARIABLES	DEPENDENT VARIABLES									
	ASA		GAM		JHI		NGS		TY	
	COEFF	T-STAT	COEFF	T-STAT	COEFF	T-STAT	COEFF	T-STAT	COEFF	T-STAT
CONSTANT	0.001	1.565	0.000	0.929	0.000	1.172	0.001	1.375	0.000	0.808
TREND	-0.000	-0.556	0.000	0.874	0.000	0.910	0.000	0.427	0.000	1.027
LAG 1	-0.011	-0.738	-0.113	-7.729	-0.204	-13.901	-0.182	-12.438	-0.020	-1.366
LAG 2	-0.028	-1.884	-0.054	-3.660	-0.113	-7.574	-0.093	-6.230	-0.039	-2.675
LAG 3	-0.004	-0.247	-0.018	-1.245	-0.057	-3.796	-0.049	-3.276	-0.018	-1.210
LAG 4	-0.012	-0.849	-0.031	-2.106	-0.036	-2.372	-0.030	-2.035	-0.021	-1.413
LAG 5	-0.002	-0.167	0.036	2.437	-0.014	-0.952	-0.040	-2.655	0.000	0.008
LAG 6	-0.012	-0.800	-0.044	-2.999	-0.000	-0.001	-0.061	-4.084	0.006	0.412
LAG 7	0.021	1.413	-0.003	-0.228	-0.004	-0.247	-0.017	-1.121	0.018	1.262
LAG 8	0.020	1.347	0.003	0.184	0.019	1.247	-0.010	-0.649	-0.028	-1.888
LAG 9	0.015	1.018	-0.010	-0.691	0.008	0.504	0.023	1.527	-0.038	-2.629
LAG 10	0.008	0.556	-0.023	-1.569	-0.004	-0.253	0.001	0.057	0.014	0.932
LAG 11	0.044	3.013	0.007	0.443	-0.007	-0.493	-0.024	-1.570	0.026	1.784
LAG 12	0.027	1.874	-0.014	-0.979	-0.008	-0.527	-0.042	-2.829	-0.021	-1.467
LAG 13	0.003	0.200	-0.009	-0.623	-0.034	-2.257	-0.017	-1.128	0.025	1.687
LAG 14	0.022	1.525	0.001	0.077	-0.024	-1.609	-0.021	-1.418	0.004	0.246
LAG 15	0.006	0.383	0.017	1.121	-0.013	-0.890	-0.001	-0.088	0.004	0.283
LAG 16	0.023	1.554	-0.006	-0.401	-0.022	-1.457	-0.020	-1.350	0.014	0.953
LAG 17	-0.025	-1.684	0.012	0.842	0.015	0.995	-0.015	-0.971	-0.002	-0.103
LAG 18	0.014	0.942	-0.004	-0.261	-0.019	-1.285	-0.017	-1.116	-0.022	-1.481
LAG 19	-0.002	-0.106	-0.018	-1.199	-0.003	-0.206	-0.023	-1.518	0.010	0.675
LAG 20	-0.035	-2.409	-0.004	-0.289	-0.003	-0.172	0.016	1.052	-0.013	-0.891
LAG 21	-0.001	-0.057	-0.018	-1.204	-0.024	-1.600	0.004	0.235	-0.029	-1.978
LAG 22	-0.018	-1.237	-0.006	-0.428	0.014	0.921	-0.019	-1.270	-0.033	-2.233
LAG 23	0.019	1.305	-0.020	-1.369	-0.031	-2.036	-0.002	-0.108	-0.061	-4.178
LAG 24	0.006	0.432	-0.009	-0.592	0.004	0.281	0.011	0.767	0.001	0.042
LAG 25	0.007	0.491	-0.013	-0.892	0.013	0.888	0.024	1.616	-0.001	-0.096
ROOT MEAN SQUARE OF LAGS > 6	0.02012		0.01218		0.01694		0.01881		0.02418	

TABLE 8. TESTS FOR STOCHASTIC DOMINANCE BETWEEN PAIRS OF FUNDS

(4685 Observations; the number of permutations to approximate significance is given in parentheses for each pair of funds; acceptances at 5 percent are boxed.)

	First-Degree Dominance			Second-Degree Dominance		
ASA vs GAM (1000 per)	1≻2?	2≻1?	not max?	1≻2?	*2≻1?*	not max?
Observation	541.	405.	405.	14.3	*1.28*	1.28
10% Critical Level	106.	105.	57.0	2.91	*2.88*	0.532
5% Critical Level	121.	122.	63.0	3.56	*3.83*	0.687
1% Critical Level	142.	149.	75.0	4.98	*5.55*	0.927
Significance Level	0.00	0.00	0.00	0.00	*0.328*	0.00
ASA vs JHI (1000 per)	1≻2?	2≻1?	not max?	1≻2?	*2≻1?*	not max?
Observation	627.	584.	584.	19.8	*2.04*	2.04
10% Critical Level	108.	103.	59.0	2.83	*2.65*	0.499
5% Critical Level	118.	118.	66.0	3.37	*3.34*	0.671
1% Critical Level	141.	139.	77.0	4.52	*4.47*	1.06
Significance Level	0.00	0.00	0.00	0.00	*0.179*	0.00
ASA vs NGS (1000 per)	1≻2?	2≻1?	not max?	1≻2?	*2≻1?*	not max?
Observation	484.	384.	384.	12.3	*1.50*	1.50
10% Critical Level	103.	101.	59.0	2.85	*2.81*	0.528
5% Critical Level	117.	117.	65.0	3.69	*3.56*	0.702
1% Critical Level	145.	145.	77.0	4.99	*4.74*	1.03
Significance Level	0.00	0.00	0.00	0.00	*0.306*	0.00100
ASA vs TY (1000 per)	1≻2?	2≻1?	not max?	1≻2?	*2≻1?*	not max?
Observation	682.	637.	637.	20.3	*1.66*	1.66
10% Critical Level	102.	107.	59.0	2.63	*2.73*	0.548
5% Critical Level	118.	120.	67.0	3.27	*3.42*	0.671
1% Critical Level	140.	149.	79.0	4.33	*4.54*	1.01
Significance Level	0.00	0.00	0.00	0.00	*0.252*	0.00
GAM vs JHI (1000 per)	1≻2?	2≻1?	not max?	1≻2?	*2≻1?*	not max?
Observation	306.	312.	306.	6.10	*0.759*	0.759
10% Critical Level	104.	99.0	57.0	2.03	*1.93*	0.399
5% Critical Level	118.	115.	63.0	2.52	*2.39*	0.502
1% Critical Level	153.	145.	75.0	3.44	*3.17*	0.646
Significance Level	0.00	0.00	0.00	0.00	*0.370*	0.00500

	First-Degree Dominance			Second-Degree Dominance		
GAM vs NGS (1000 per)	1>2?	2>1?	not max?	1>2?	2>1?	not max?
Observation	181.	218.	181.	0.00	2.01	0.00
10% Critical Level	100.	100.	58.0	2.22	2.27	0.426
5% Critical Level	118.	112.	65.0	2.85	2.80	0.580
1% Critical Level	149.	135.	81.0	3.67	3.83	0.801
Significance Level	0.00100	0.00	0.00	0.851	0.129	0.701

GAM vs TY (6000 per)	1>2?	2>1?	not max?	1>2?	2>1?	not max?
Observation	410.	389.	389.	6.58	0.388	0.388
10% Critical Level	101.	102.	58.0	1.97	2.00	0.370
5% Critical Level	115.	116.	64.0	2.44	2.46	0.471
1% Critical Level	140.	144.	76.0	3.46	3.44	0.675
Significance Level	0.00	0.00	0.00	0.00	0.519	0.0893

JHI vs NGS (6000 per)	1>2?	2>1?	not max?	1>2?	2>1?	not max?
Observation	448.	464.	448.	0.538	7.78	0.538
10% Critical Level	101.	100.	60.0	1.98	1.97	0.404
5% Critical Level	117.	114.	66.0	2.50	2.46	0.516
1% Critical Level	143.	141.	78.0	3.41	3.35	0.764
Significance Level	0.00	0.00	0.00	0.470	0.00	0.0452

JHI vs TY (1000 per)	1>2?	2>1?	not max?	1>2?	2>1?	not max?
Observation	238.	164.	164.	0.848	0.661	0.661
10% Critical Level	100.	102.	58.0	1.59	1.69	0.361
5% Critical Level	113.	115.	63.0	2.04	2.14	0.430
1% Critical Level	136.	146.	74.0	2.88	2.79	0.644
Significance Level	0.00	0.00400	0.00	0.288	0.373	0.0100

NGS vs TY (1000 per)	1>2?	2>1?	not max?	1>2?	2>1?	not max?
Observation	618.	533.	533.	8.50	0.167	0.167
10% Critical Level	98.0	99.0	59.0	1.99	2.01	0.401
5% Critical Level	114.	109.	64.0	2.43	2.45	0.502
1% Critical Level	142.	128.	79.0	3.22	3.19	0.700
Significance Level	0.00	0.00	0.00	0.00	0.690	0.348