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EQUILIBRIUM IN CONTINUOUS-TIME FINANCIAL MARKETS: ENDOGENOUSLY DYNAMICALLY COMPLETE MARKETS

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EQUILIBRIUM IN CONTINUOUS-TIME FINANCIAL MARKETS: ENDOGENOUSLY DYNAMICALLY COMPLETE MARKETS

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We prove existence of equilibrium in a continuous-time securities market in which the securities are potentially dynamically complete: the number of securities is at least one more than the number of independent sources of uncertainty. We prove that dynamic completeness of the candidate equilibrium price process follows from mild exogenous assumptions on the economic primitives of the model. Our result is universal, rather than generic: dynamic completeness of the candidate equilibrium price process and existence of equilibrium follow from the way information is revealed in a Brownian filtration, and from a mild exogenous nondegeneracy condition on the terminal security dividends. The nondegeneracy condition, which requires that finding one point at which a determinant of a Jacobian matrix of dividends is nonzero, is very easy to check. We find that the equilibrium prices, consumptions, and trading strategies are well-behaved functions of the stochastic process describing the evolution of information. We prove that equilibria of discrete approximations converge to equilibria of the continuous-time economy.

KEYWORDS: Dynamic completeness, convergence of discrete-time finance models, continuous-time finance, general equilibrium theory.

1. INTRODUCTION

IN AN ARROW–DEBREU MARKET, agents are allowed to shift consumption across states and times by trading a complete set of Arrow–Debreu contingent claims. Mas-Colell and Richard (1991), Dana (1993), and Bank and Riedel (2001) proved existence of equilibrium in a continuous-time Arrow–Debreu market.

By contrast, in a securities market, agents are restricted to trading a prespecified set of securities. The securities market is said to be dynamically complete

if agents can, by rapidly trading the given set of securities, achieve all the consumption allocations they could achieve in an Arrow–Debreu market. When the uncertainty is driven by a Brownian motion, markets are said to be potentially dynamically complete if the number of securities is at least one more than the number of independent sources of uncertainty.[2] A securities market which is potentially dynamically complete may or may not be dynamically complete.

Existence of equilibrium in continuous-time finance models with a single agent has been established in a number of papers (Bick (1990), He and Leland (1995), Cox, Ingersoll and Ross (1985), Duffie and Skiadas (1994), Raimondo (2002, 2005)[3]).

Surprisingly, the existing literature does not contain a satisfactory theory of the existence of equilibrium in continuous-time securities markets with more than one agent. With dynamic incompleteness, essentially nothing is known. While we hope that some of the techniques developed in this paper will help in addressing the dynamically incomplete case, we do not here present a result in that case.

This paper and the existing literature all deal with the case in which markets are potentially dynamically complete. Every previous paper *assumes*, in one form or another, that the candidate equilibrium price process is dynamically complete and shows that this implies that the candidate equilibrium is, in fact, an equilibrium; see Basak and Cuoco (1998), Cuoco (1997), Dana and Jean-blanc (2002), Detemple and Karatzas (2003), Duffie (1986, 1991, 1995, 1996), Duffie and Huang (1985), Duffie and Zame (1989), Karatzas, Lehoczky, and Shreve (1990), and Riedel (2001). The form of the assumption varies. Some assume directly that the dispersion matrix of the candidate equilibrium price process is almost surely nonsingular, which is well known to be equivalent to dynamic completeness. Others assume that the dispersion matrix of the candidate equilibrium price process is uniformly positive definite; this is stronger than dynamic completeness and it fails for an open set of primitives in our model. Still others assume that the candidate equilibrium price of consumption is uniformly bounded above and uniformly bounded away from zero, but this is inconsistent with unbounded consumption (found in the Black–Scholes and most other continuous-time finance models) and standard utility functions. Finally, some *construct* securities which suffice to complete the markets, rather than taking the securities' dividends as given by the model.

But the candidate equilibrium price process is determined from the economic primitives of the model by a fixed point argument, which makes it impossible, except in knife-edge special cases, to determine from the primitives whether or not the dynamic completeness assumption is satisfied. Thus, if we

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[2]With non-Brownian processes, such as Lévy processes that allow jumps, the number of securities needed for potential dynamic completeness may be larger.

[3]All of these papers except Raimondo (2002, 2005) require one or more endogenous assumption(s) that is (are) not expressed solely in terms of the primitives of the model.

consider one of these models and choose specific utility functions, endowment, and dividend processes, except in the rare cases where we can solve explicitly for the candidate equilibrium, we cannot apply any of the previous theorems to determine whether or not an equilibrium exists. Indeed, the previous results do not rule out the possibility that equilibrium generically *fails* to exist; see the example in Section 3.

While dynamic completeness plays a role in proving the existence of equilibrium, its main application in finance is to derivative pricing. Given a dynamically complete securities price process, options and other derivatives can be uniquely priced by arbitrage arguments and can be replicated by trading the underlying securities. With dynamic incompleteness, arbitrage considerations do not determine a unique option price and replication is not possible. The previous results provide no guarantee that equilibrium prices will support the standard theory of pricing and replicating options, which depends on dynamic completeness.

In this paper, we *prove* that the candidate equilibrium price process is, *in fact*, dynamically complete and that the candidate equilibrium is, in fact, an equilibrium.[4] Dynamic completeness of the candidate equilibrium price process and the existence of equilibrium follow from the way information is revealed by a Brownian motion and from a mild exogenous nondegeneracy condition on the terminal security dividends.

To motivate our nondegeneracy condition, suppose we are given a market with K independent sources of uncertainty and $K + 1$ securities labeled $j = 0, \ldots, K$, so the market is potentially dynamically complete. Now suppose that two of the securities are perfect clones of each other, in that they pay exactly the same dividends at all nodes. Clearly, this is the same as a market with K sources of uncertainty and K securities, so it cannot possibly be dynamically complete. Similarly, if the dividend processes of the securities were linearly dependent, we could not possibly have dynamic completeness. Thus, we need to assume that the dividend processes are not linearly dependent. In our model, the dividend of security j at the terminal date T is given as a function $G_j(\beta(T, \omega))$ of the terminal realization $\beta(T, \omega)$ of the Brownian motion in state ω. In the important special case in which security 0 is a zero-coupon bond, our condition requires that $G_1, \ldots, G_K$ be C^1 functions on some open set $V \subset \mathbf{R}^K$ and that the Jacobian determinant of $(G_1, \ldots, G_K)$ be nonzero at one point $x \in V$. In particular, our assumption depends only on the securities

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[4]More specifically, we show that any consumption which is adapted to the Brownian filtration and has finite value at the candidate equilibrium price process can be replicated by an admissible self-financing trading strategy. We allow the filtration in the original continuous-time economy to be larger than the Brownian filtration, and the filtration in the Loeb-measure economy we construct is *always* larger than the Brownian filtration. This does not pose a problem for the existence of equilibrium because any consumption adapted to the larger filtration is a mean-preserving spread of a consumption adapted to the Brownian filtration, so agents' demands are always adapted to the Brownian filtration.

dividends at the terminal date, not on the other economic primitives; changing the utility function or endowment of an agent or the initial ownership of the securities has no effect on the existence of equilibrium. Clearly, the nondegeneracy condition is generically satisfied. Moreover, one can easily tell whether the condition is satisfied for any particular value of the economic primitives simply by checking whether a determinant is nonzero at least at one point; this contrasts with the situation in most generic results, in which one knows that the result holds except on a small set of primitives, but it is hard to tell whether the result holds for any specific value of the primitives.

If there are just enough securities for potential dynamic completeness, then some form of linear independence of the securities dividends is a necessary condition for dynamic completeness of the Arrow–Debreu securities prices. Thus, some form of linear independence of the dividends is implicitly assumed in all the previous papers. We chose to place our nondegeneracy assumption on the lump terminal dividends because it is convenient to do so. Not all of the previous papers have lump terminal dividends; we believe it should be possible to place the assumption instead on the intermediate flow dividends, although the statement of the assumption would be more complex.

We obtain explicit formulas for the equilibrium price process and its dispersion matrix, each trader's equilibrium securities wealth, and the dispersion matrix for each traders' equilibrium trading strategy in terms of the equilibrium consumptions; each trader's equilibrium trading strategy can then be calculated using linear algebra. These formulas are expressed in terms of the equilibrium consumptions, which are not known a priori. However, since the equilibrium is Pareto optimal, there is a vector of Negishi (1960) utility weights λ such that, at each node, the equilibrium consumptions maximize the weighted sum of the utilities of the agents. Thus, the key features of the equilibrium can be calculated explicitly from knowledge of the primitives of the model (endowments and utility functions of the individuals, and the dividends of the securities) and the equilibrium utility weights. Moreover, even if the equilibrium utility weights are not known, the explicit nature of the formulas can potentially be used to establish general properties of equilibria, such as comparative statics results.

We prove that all key elements of equilibria of discrete approximations converge to the corresponding elements of equilibria of the continuous-time model.[5] This is important for two reasons: First, many people regard the discrete models as the appropriate models for "real" economies. In particular, financial transaction data are very high frequency, but nonetheless discrete.

[5]When the continuous-time model has multiple equilibria, it is possible that the equilibria of the discrete approximations will be near one continuous-time equilibrium for some approximations and near a different continuous-time equilibrium for other approximations, so the sequence of equilibria of the discrete approximations converges to the *set* of equilibria of the continuous-time economy, rather than to a single equilibrium of the continuous-time economy.

The theoretical literature has focused on continuous time because the formulas are much simpler in continuous time than in discrete time. Cox, Ross, and Rubinstein (1979) showed that one can compute the prices of options when the price of a stock is given by a geometric binomial random walk and showed that the discrete option prices converge to those given by the Black–Scholes formula. Our convergence theorems tell us that the equilibrium pricing formulas from the continuous-time model apply asymptotically to equilibria of the discrete models as the discretization gets finer. This shows that the formulas are applicable to discrete models. It also shows that the formulas can be used in the econometric analysis of data which are high frequency but nonetheless discrete.[6] Second, since we show that the equilibria of the discretizations are close to equilibria of the continuous-time economy, algorithms to compute equilibria for discrete economies[7] provide a means to compute equilibria of the continuous-time economy.

Our starting point is a continuous-time model, on which we state our existence theorem, Theorem 2.1, and our characterization of the elements of the equilibrium, Proposition 2.2. A function is said to be real analytic if, at every point in its domain, there is a power series which converges to the function on an open set containing the point. We assume that the primitives of the economy are given by real analytic functions of time and the current value of the Brownian motion for times $t \in (0, T)$, where T is the terminal date.[8] The assumption that utility functions are analytic is not problematic; most of the utility functions commonly studied are in fact analytic. Option payoffs are not analytic because of the kink when the stock price equals the exercise price of the option. However, options can be handled under certain conditions; see the two paragraphs preceding Equation (1) in Section 2. The assumption that the dividends at time t are a *function* of t and the value of the Brownian motion at time t is discussed in Section 6.

[6]As noted above, if the continuous-time economy has multiple equilibria, equilibria of the discrete approximation economies converge to the set of continuous-time equilibria, rather than to a single equilibrium. Nonetheless, this tells us that the formulas given in Proposition 2.2 apply approximately to the equilibria of the discretizations. For example, Equation (6) gives a formula for the dispersion matrix of the securities prices in terms of the Brownian motion, the future dividends, and the endogenously determined prices of consumption. The convergence results in Theorem 4.2 imply that Equation (6) holds asymptotically over any sequence of equilibria of the discrete approximations, using the prices of consumption in the given discrete approximations.

[7]See, for example, Brown, DeMarzo, and Eaves (1996) and Judd, Kubler, and Schmedders (2000, 2002, 2003).

[8]We also require that the utility functions satisfy Inada conditions (ruling out Constant Absolute Risk Aversion (CARA) utility) and be differentiably strictly concave (ruling out risk neutrality). It is important to emphasize that the primitives of the model, and as we will find, the equilibrium, are analytic functions of time and the Brownian motion, not analytic functions of time alone. Brownian motion is almost surely nowhere differentiable and almost surely of unbounded variation on every interval of time. Consequently, the equilibrium prices and trading strategies are nowhere differentiable, and of unbounded variation, as functions of time.

We discretize the continuous-time model to construct a sequence of models; in each, we replace the Brownian motion β by a random walk $\hat{\beta}$.[9] A naive discretization would approximate the K-dimensional Brownian motion by a K-dimensional random walk in which each node has 2^K successors, and the random walk moves independently in each direction at each node. However, in a discrete random walk in which each node has 2^K successors, one needs at least 2^K securities to obtain dynamic completeness; for $K > 1$, $K + 1 < 2^K$ and the discrete model cannot possibly be dynamically complete. So that the discrete approximations correspond closely to the continuous-time model it is critical that the discrete model have the same number of securities as the continuous-time model and that it be dynamically complete.[10] Thus, we construct a K-dimensional random walk $\hat{\beta}$ in which each nonterminal node has exactly $K + 1$ successors. Endowments, dividends, and utility functions are induced on the discrete economies from the specification of the continuous-time economy. These discrete economies are general equilibrium incomplete markets (GEI) models; the book by Magill and Quinzii (1996) is an excellent reference on GEI models. Endowments and dividends are perturbed as necessary to ensure the existence of a dynamically complete equilibrium under the Duffie–Shafer (1985, 1986) or Magill–Shafer (1990) theorems. We then prove our convergence theorem, Theorem 4.2, that equilibria of the discrete approximations converge to equilibria of the continuous-time economy.

Since equilibrium prices are arbitrage-free, the equilibrium securities prices at the terminal date in the discrete model must be given by the exogenously specified dividends at the terminal date. It is well known that the equilibrium securities price process at time t is the conditional expectation of future dividends, valued at the Arrow–Debreu equilibrium prices of consumption; this comes from the first-order conditions for utility maximization (see, for example, Magill and Quinzii (1996) in the discrete case). We show that because of the smoothness properties of the Gaussian distribution, such conditional expectations are given by real analytic functions of $(t, \hat{\beta}(t, \omega)) \in (0, T) \times \mathbf{R}^K$, so the equilibrium securities price process is a real analytic function of $(t, \hat{\beta}(t, \omega))$. Moreover, the dispersion matrix of the securities prices and its associated determinant is a real analytic function of $(t, \hat{\beta}(t, \omega))$; this is a property of the way information is revealed by a Brownian motion and does not depend on any specific assumptions on the functional form of the primitives.

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[9]Zame (2001) considered a model in which time is discretized but the probability space is not, so each step of the random walk is normally distributed. In that setting, he found that the discrete-time approximation does not converge to the continuous-time limit. Raimondo (2001) studied the existence of equilibrium in models with discrete time and a continuum of states.

[10]If one were to extend these methods to the analysis of dynamically incomplete continuous-time models, it would be critical to ensure that the discrete approximation has the same degree of incompleteness as the continuous-time model, so the random walk would have to be chosen carefully in that context also.

We use nonstandard analysis to project the discrete model back into the continuous-time model and find that the dispersion matrix of the continuous-time securities prices is the projection of the dispersion matrix in the discrete model, and hence is real analytic. A real analytic function cannot be zero on a set of positive measure unless it is identically zero. Since the determinant associated with the dispersion matrix is nonzero on a set of positive measure, it must be nonzero except on a set of measure zero. This implies that the dispersion matrix has rank K except on a set of measure zero, but it is well known that this condition is equivalent to dynamic completeness; thus, we have shown that the candidate equilibrium prices in the continuous-time model are dynamically complete. We verify that the projected prices are equilibrium prices in the continuous-time model.

Our proof depends heavily on nonstandard analysis and, in particular, on the nonstandard theory of stochastic processes. Nonetheless, the statements of our main results, Theorems 2.1 and 4.2 and Proposition 2.2, can be understood without any knowledge of nonstandard analysis.

Nonstandard analysis provides powerful tools to move from discrete to continuous time, and from discrete distributions like the binomial to continuous distributions like the normal; in particular, it provides the ability to transfer computations back and forth between the discrete and continuous settings. Our sequence of discrete approximations extends to a hyperfinite approximation, one which is infinite but has all the formal properties of finite approximations. In particular, the hyperfinite approximation has a GEI equilibrium which is dynamically complete in the hyperfinite model. We then use nonstandard analysis to produce a candidate equilibrium in the continuous-time model, show that the equilibrium in the hyperfinite model is infinitely close to the candidate equilibrium in the continuous-time model, verify that the candidate prices are dynamically complete, and are in fact equilibrium prices.

Anderson (1976) provided a construction for Brownian motion and Brownian stochastic integration using Loeb (1975) measure—a measure in the usual standard sense produced by a nonstandard construction. Anderson's Brownian motion is a hyperfinite random walk which can simultaneously be viewed as being a standard Brownian motion in the usual sense of probability theory. While the standard stochastic integral is motivated by the idea of a Stieltjes integral, the actual standard definition of the stochastic integral is of necessity rather indirect because almost every path of Brownian motion is of unbounded variation, and Stieltjes integrals are only defined with respect to paths of bounded variation. However, a hyperfinite random walk is of hyperfinite variation, and hence a Stieltjes integral with respect to it makes perfect sense. Anderson showed that the standard stochastic integral can be obtained readily from this hyperfinite Stieltjes integral.

We modify Anderson's construction of the hyperfinite random walk to a random walk with branching number equal to $K + 1$ and extend the results on stochastic integration to that random walk. We show that equilibrium consumptions are nonzero at all times and states. Consequently, we can use the

first-order conditions to characterize the equilibrium prices. Then we use the Loeb measure construction to produce a candidate equilibrium of the original continuous-time model. The central limit theorem then allows us to explicitly describe the candidate equilibrium prices as integrals with respect to a normal distribution; however, with more than one agent, the prices depend on the terminal distributions of wealth, which are not described in closed form. We show that the hyperfinite equilibrium is infinitely close to the candidate equilibrium, which implies that the equilibria of the discrete approximations converge to candidate equilibria of the continuous-time model. Finally, in a process analogous to that first used in Brown and Robinson (1975), we show that the candidate equilibrium is an equilibrium of the continuous-time economy.[11]

2. THE MODEL

In this section we define the continuous-time model.

There is a single consumption good. Trade and consumption occur over a compact time interval $[0, T]$, endowed with a measure ν which agrees with Lebesgue measure on $[0, T)$ and such that $\nu(\{T\}) = 1$. Consumption and dividends on $[0, T)$ are flows; consumption at the terminal date T is a lump. We choose this formulation to give a finite-horizon model in which the securities will have positive value at the terminal date T; if securities paid only a flow dividend on $[0, T]$, they would expire worthless at time T and it would not be possible to formulate our nondegeneracy condition on their dividends. An alternative would be to take an infinite-horizon model, but this would require addressing certain additional technical problems. We think of our finite-time horizon T as a truncation of an infinite-horizon model. The lump of consumption at time T aggregates the flow consumption on the interval (T, ∞) in the infinite-horizon model, conditional on the information available at time T. Our nondegeneracy assumption will be imposed on the lump dividend at the terminal date T.

The uncertainty in the model is described by a standard K-dimensional Brownian motion β on a probability space $(\Omega, \mathcal{F}, \mu)$; the components of β are independent of each other and the variance of $\beta_k(t, \cdot)$ equals t. We define $\mathcal{I}(t, \omega) = (t, \beta(t, \omega))$. The primitives of the economy—dividends, endowments and utility functions—will be described as functions of $\mathcal{I}(t, \omega)$.

We are given a right-continuous filtration $\{\mathcal{F}_t : t \in [0, T]\}$ on $(\Omega, \mathcal{F}, \mu)$ such that $\mathcal{F}_0$ contains all null sets and β is adapted to $\{\mathcal{F}_t\}$, that is, for all t, $\beta(t, \cdot)$ is measurable with respect to $\mathcal{F}_t$.

Let $J = K$. There are $J + 1$ securities, indexed by $j = 0, \ldots, J$; security j is in net supply $\eta_j \in \{0, 1\}$. Security j pays dividends (measured in consumption units) at a flow rate $A_j(t, \omega) = g_j(\mathcal{I}(t, \omega))$ at times $t \in [0, T)$, and a lump

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[11]The argument is more complicated here because our continuous-time economy is more complicated than the economy in Brown and Robinson (1975).

dividend $A_j(T, \omega) = G_j(\mathcal{I}(T, \omega))$ at time T. We assume that $g : [0, T] \times \mathbf{R}^K \to \mathbf{R}_+^K$ is real analytic on $(0, T) \times \mathbf{R}^K$ and that G_j is continuous almost everywhere on $\{T\} \times \mathbf{R}^K$. For example, A_0 could be a zero-coupon bond ($g_0(t, \omega) = 0$ for $t \in [0, T)$, $g_0(T, \omega)$ is constant) or $A_j(t, \omega) = e^{\sigma_j \cdot \beta(t, \omega)}$, where σ_j is the jth row of a $J \times J$ matrix σ, for $j = 1, \ldots, J$.

There are I agents $i = 1, \ldots, I$. Agent i has a flow rate of endowment $e_i(t, \omega) = f_i(\mathcal{I}(t, \omega))$ at times $t \in [0, T)$, and a lump endowment $e_i(T, \omega) = F_i(\mathcal{I}(T, \omega))$, where f_i is analytic on $(0, T) \times \mathbf{R}^K$ and F_i is continuous almost everywhere on $\{T\} \times \mathbf{R}^K$. Let $e(t, \omega) = \sum_{i=1}^I e_i(t, \omega)$ denote the aggregate endowment.

The utility functions are von Neumann–Morgenstern utility functions, expectations of functions of the consumption and the process $\mathcal{I}$ which are analytic on $(0, T) \times \mathbf{R}^K$. More formally, given a measurable consumption function $c_i : [0, T] \times \Omega \to \mathbf{R}_{++}$, the utility function of the agent is

$$U_i(c) = E_\mu \left[\int_0^T h_i(c_i(t, \cdot), \mathcal{I}(t, \cdot)) \, dt + H_i(c_i(T, \cdot), \mathcal{I}(T, \cdot)) \right],$$

where the functions $h_i : \mathbf{R}_+ \times ([0, T) \times \mathbf{R}^K) \to \mathbf{R} \cup \{-\infty\}$ and $H_i : \mathbf{R}_+ \times (\{T\} \times \mathbf{R}^K) \to \mathbf{R} \cup \{-\infty\}$ are analytic on $\mathbf{R}_{++} \times ((0, T) \times \mathbf{R}^K)$ and C^2 on $\mathbf{R}_{++} \times (\{T\} \times \mathbf{R}^K)$, respectively, and satisfy

$$\lim_{c \to 0_+} \frac{\partial h_i}{\partial c} = \infty \quad \text{uniformly over } ([0, T] \times \mathbf{R}^K),$$

$$\lim_{c \to 0_+} \frac{\partial H_i}{\partial c} = \infty \quad \text{uniformly over } \{T\} \times \mathbf{R}^K,$$

$$\lim_{c \to \infty} \frac{\partial h_i}{\partial c} = 0 \quad \text{uniformly over } ([0, T] \times \mathbf{R}^K),$$

$$\lim_{c \to \infty} \frac{\partial H_i}{\partial c} = 0 \quad \text{uniformly over } \{T\} \times \mathbf{R}^K,$$

$$\lim_{c \to 0_+} h_i(c, (t, x)) = h_i(0, (t, x)) \quad \text{uniformly over } ([0, T] \times \mathbf{R}^K),$$

$$\lim_{c \to 0_+} H_i(c, (T, x)) = H_i(0, (T, x)) \quad \text{uniformly over } \{T\} \times \mathbf{R}^K,$$

$$\frac{\partial h_i}{\partial c} > 0 \quad \text{on } \mathbf{R}_{++} \times ([0, T] \times \mathbf{R}^K),$$

$$\frac{\partial H_i}{\partial c} > 0 \quad \text{on } \mathbf{R}_{++} \times (\{T\} \times \mathbf{R}^K),$$

$$\frac{\partial^2 h_i}{\partial c^2} < 0 \quad \text{on } \mathbf{R}_{++} \times ([0, T] \times \mathbf{R}^K),$$

$$\frac{\partial^2 H_i}{\partial c^2} < 0 \quad \text{on } \mathbf{R}_{++} \times (\{T\} \times \mathbf{R}^K).$$

Note that these conditions are satisfied by all state-independent Constant Relative Risk Aversion (CRRA) utility functions. Note also that we allow quite general state-dependence of the utility function, as long as the state-dependence enters through the process $\mathcal{I}$. If the state-dependence were not measurable in the Brownian motions, there would be no hope of obtaining effective dynamic completeness with securities whose dividends *are* measurable with respect to the Brownian filtration.

The assumption that the endowments and utility functions are analytic does not impose serious economic restrictions; for example, all conventional utility functions are in fact analytic. However, the dividend paid by an option is *not* analytic because of the kink that occurs when the stock price just equals the exercise price of the option on the exercise date. Moreover, shares of a limited liability corporation should not generally be thought of as analytic because they are, in effect, options to claim the corporation's stream of earnings at an exercise price equal to the corporation's debt.[12] We require analyticity at the intermediate times $t \in (0, T)$ and not at $t = T$. Thus, our model allows us to include options and shares of limited liability corporations among our basic securities, provided that the exercise date is the terminal date T. In addition, as long as the basic securities pay dividends which are analytic functions of $\mathcal{I}$ over $(0, T)$, there is no problem in using the equilibrium prices derived from the basic securities to price options or other derivatives on those basic securities with exercise date $t \in (0, T)$.[13] The equilibrium price of any security, analytic or not, is equal to the expected value of its future dividends, evaluated at the equilibrium consumption prices. Since we prove that the basic securities are essentially dynamically complete, any option or other derivative on the basic securities can be replicated by an admissible self-financing trading strategy on the basic securities. Thus, the equilibrium price of any security is also given by the equilibrium value of the portfolio specified by the replicating strategy at that node. For more detail on equilibrium pricing of options, see Anderson and Raimondo (2005, 2006).

Note that we do *not* require G_j to be analytic, or even differentiable. Standard option payoffs are not differentiable at the strike price, and other derivatives need not be continuous. However, because derivatives represent contracts that need to be understood and enforced, they tend to have relatively simple structures, such as piecewise linearity in the underlying securities; they may or may not be continuous at the boundaries between the regions of linearity. Since our security dividends are analytic functions of the Brownian motion,

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[12]In particular, geometric Brownian motion is problematic as a model of the price of shares in a limited liability corporation.

[13]We do not consider the payout of an option at times $t < T$ as a dividend. The payout of an option is a lump, which does not lie in the consumption set at time t. Rather, if the option is in the money at its expiration date $t < T$, one exercises it by selling securities to raise the money for the exercise price, takes delivery of the underlying security, and then rebalances one's portfolio.

derivatives should be piecewise analytic functions of the Brownian motion. The assumption that the dividend at time T is continuous almost everywhere allows for the possibility that security A_j is a derivative, with exercise date T, on another security or securities, which may or may not be traded. Our formulation also allows A_j to be a stock in a limited liability corporation, since shares in limited liability corporations are in effect options to buy the earnings flow of the firm at an exercise price equal to the firm's debt.

We assume the endowments and dividends satisfy the following growth conditions (note that $|x|$ denotes the Euclidean length of the vector x):

$$(1) \qquad \exists_{r \in \mathbf{R}} \forall_{t \in [0,T], x \in \mathbf{R}^K}, \quad |f_i(t, x)| \le r + e^{r|x|},$$

$$|F_i(T, x)| \le r + e^{r|x|},$$

$$|g_j(t, x)| \le r + e^{r|x|},$$

$$|G_j(T, x)| \le r + e^{r|x|},$$

$$\left| \frac{\partial f_i(t, x)}{\partial x} \right| \le r + e^{r|x|},$$

$$\left| \frac{\partial g_j(t, x)}{\partial x} \right| \le r + e^{r|x|}.$$

These conditions are needed to show that if $\hat{\beta}_n$ is a sequence of random walks converging to Brownian motion β, the distribution of $f_i(t, \hat{\beta}_n(t, \omega))$ converges to that of $f_i(t, \beta(t, \omega))$ and satisfies a uniform L^2 integrability condition. In fact, the distribution of $f_i(t, \hat{\beta}_n(t, \omega))$ will satisfy a uniform L^p integrability condition for all $p \in [1, \infty)$.

We are grateful to the referees for pointing out Dana's (1993) treatment of consumptions that are not uniformly bounded away from zero. Let

$$c(t, x) = \begin{cases} \displaystyle\sum_{i=1}^{I} f_i(t, x) + \sum_{j=0}^{J} \eta_j g_j(t, x), & \text{if } t < T, \\ \displaystyle\sum_{i=1}^{I} F_i(t, x) + \sum_{j=0}^{J} \eta_j G_j(t, x), & \text{if } t = T, \end{cases}$$

denote the social consumption at the node (t, x). We make the following joint assumption on the utility functions and the social consumption: there exists $r \in \mathbf{R}$ such that

$$(2) \qquad \left. \frac{\partial h_i}{\partial c} \right|_{c(t,x)/I} \le r + e^{r|x|},$$

$$\left. \frac{\partial H_i}{\partial c} \right|_{c(T,x)/I} \le r + e^{r|x|}.$$

Inequality (2) says if we give each of the I agents an equal share of the social consumption, then every agent's marginal utility of consumption is bounded above by a linear exponential of the magnitude of the Brownian motion. Note that it implies that the social endowment is strictly positive at almost all nodes. The assumption would follow without any additional assumptions on the utility functions if the social consumption were uniformly bounded away from zero. However, the assumption also allows for the social consumption to approach zero, as long as the marginal utility does not grow too quickly. For example, if each agent has state-independent CRRA utility and the social endowment is a geometric Brownian motion, the condition is satisfied.

In continuous-time models, it is commonly assumed that the zeroth security is a money-market account, in other words, it is instantaneously risk-free. Since we are determining securities prices endogenously, the assumption that a security is instantaneously risk-free is an endogenous assumption. For example, if we assume that A_0 is a zero-coupon bond (i.e., $G_0(t, \omega) = 0$ for $t < T$ and $G_0(T, \omega) = 1$, so the dividends of A_0 are risk-free), the *Arrow–Debreu equilibrium price* of A_0 will *not* be instantaneously risk-free except in degenerate situations. However, as long as the equilibrium securities prices are positive Itô processes (and we shall show that they are), one is free to divide the securities price process and the consumption price process by the equilibrium price of the zeroth security. Under this renormalization, relative prices are preserved and the price of the zeroth security is identically 1, which is obviously instantaneously risk-free. When one does this, the set of admissible self-financing trading strategies is left unchanged; see Section 4.8 of Nielsen (1999) or Duffie (1996), as well as Nielsen (2007). Consequently, the consumption processes that lie in the budget set remain invariant, and the renormalized prices are equilibrium prices. This motivates the form of our exogenous nondegeneracy condition on the terminal dividends of the securities. We assume that there is an open set $V \subset \mathbf{R}^K$ such that $G_0(T, x) > 0$ for all $x \in V$ and for $j = 1, \ldots, J$ and $i = 1, \ldots, I$,

$$(3) \qquad G_j, F_i \in C^1(V) \text{ and } \exists_{x \in V} \quad \text{rank} \begin{pmatrix} \frac{\partial(G_1/G_0)}{\partial\beta}\big|_{(T,x)} \\ \vdots \\ \frac{\partial(G_J/G_0)}{\partial\beta}\big|_{(T,x)} \end{pmatrix} = K.$$

Note that if A_0 is a bond, the rank condition is equivalent to assuming that the $K \times K$ matrix

$$\begin{pmatrix} \frac{\partial G_1}{\partial\beta}\big|_{(T,x)} \\ \vdots \\ \frac{\partial G_J}{\partial\beta}\big|_{(T,x)} \end{pmatrix}$$

is nonsingular. This simply says that there is some possible terminal value of the Brownian motion so that the dividends of securities $A_1, \ldots, A_J$ are locally linearly independent.

Agent i is initially endowed with deterministic security holdings $e_{iA} = (e_{iA_0}, \ldots, e_{iA_J}) \in \mathbf{R}^{J+1}$ satisfying

$$\sum_{i=1}^{I} e_{iA_j} = \eta_j.$$

Note that the initial holdings are independent of the state ω. We require the following condition for each $i = 1, \ldots, I$:

(4) $\qquad \forall_{(t,x) \in [0,T) \times \mathbf{R}^K} \quad f_i(t, x) + e_{iA}g(t, x) \geq 0,$

$\qquad \forall_{x \in \mathbf{R}^K} \quad F_i(T, x) + e_{iA}G(T, x) \geq 0$

with strictly inequality on a set of positive measure (recall that the single time T carries measure 1). It says that if an agent never traded the securities and simply consumed his/her endowment and dividends, consumption would be nonnegative at each node and strictly positive on a set of positive measure; it guarantees that each agent has strictly positive income at every candidate equilibrium price process. We allow an agent to be endowed with a short position in securities, as long as the condition is satisfied.

To define the budget set of an agent, we need to have a way of calculating the capital gain the agent receives from a given trading strategy. In other words, we need to impose conditions on prices and strategies that ensure that the stochastic integral of a trading strategy with respect to a price process is defined. The essential requirements are that the trading strategy at time t not depend on information which has not been revealed by time t, and the trading strategy times the variation in the price yields a finite integral. Specifically, a consumption price process is an Itô process $p_C(t, \omega)$. A securities price process is an Itô process $p_A = (p_{A_0}, \ldots, p_{A_J}) : \Omega \times [0, T] \to \mathbf{R}^{J+1}$ such that the associated cumulative gains process

$$\gamma_j(t, \omega) = p_{A_j}(t, \omega) + \int_0^t p_C(s, \omega) A_j(s, \omega) \, ds$$

is a martingale. Securities are priced *cum dividend* at time T. Given a securities price process p_A, an admissible trading strategy for agent i is a row vector process z_i which is Itô integrable with respect to γ (written $z_i \in \mathcal{L}^2(\gamma)$)[14] and

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[14]Formally, $z_i \in \mathcal{L}^2(\gamma)$ if $z_i : [0, T) \times \Omega \to \mathbf{R}^{J+1}$, $z_i(t, \cdot)$ is $\mathcal{F}_t$-measurable for all $t \in [0, T)$, and z_i is measurable on the product $[0, T) \times \Omega$. If the Itô process γ is given by $d\gamma = a \, dt + \sigma \, d\beta$, then since γ is a martingale, a must be zero almost surely. Itô integrability with respect to γ

such that $\int z_i \, d\gamma$ is a martingale.[15]

Given a securities price process p_A and a consumption price process p_C, the budget set for agent i is the set of all consumption plans c_i such that there exists an admissible trading strategy so that c_i and t_i satisfy the budget constraint

$$p_A(t, \omega) \cdot z_i(t, \omega)$$

$$= p_A(0, \omega) \cdot e_{iA}(\omega) + \int_0^t z_i \, d\gamma$$

$$+ \int_0^t p_C(s, \omega)(e_i(s, \omega) - c_i(s, \omega)) \, ds$$

$$\text{for almost all } \omega \text{ and all } t \in [0, T),$$

$$0 = p_A(0, \omega) \cdot e_{iA}(0, \omega) + \int_0^T z_i \, d\gamma$$

$$+ \int_0^T p_C(s, \omega)(e_i(s, \omega) - c_i(s, \omega)) \, ds$$

$$+ p_C(T, \omega)(e_i(T, \omega) - c_i(T, \omega)) \qquad \text{for almost all } \omega.$$

Given a price process p, the demand of the agent is a consumption plan and an admissible trading strategy which satisfy the budget constraint and such that the consumption plan maximizes utility over the budget set.

An equilibrium for the economy is a securities price process p_A, a consumption price process p_C, and a profile of admissible trading strategies $z_1, \ldots, z_I$ and consumption plans $c_1, \ldots, c_I$ which lies in the demand set so that the securities and goods markets clear, that is, for almost all ω,

$$\sum_{i=1}^I z_{iA_j}(t, \omega) = \eta_j \quad \text{for } j = 0, \ldots, J \text{ and almost all } (t, \omega),$$

$$\sum_{i=1}^I c_i(t, \omega) = \sum_{i=1}^I e_i(t, \omega) + \sum_{j=0}^J \eta_j A_j(t, \omega) \quad \text{for almost all } (t, \omega).$$

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requires two conditions: that $z_i(\cdot, \omega) \cdot a(\cdot, \omega) \in L^1([0, T])$ almost surely, which is trivially satisfied, and that $z_i(\cdot, \omega)\sigma(\cdot, \omega) \in L^2([0, T])$ almost surely. A stronger condition is that $z_i \in \mathcal{H}^2(\gamma)$; the definition of $\mathcal{H}^2(\gamma)$ is the same as that of $\mathcal{L}^2(\gamma)$, except that we strengthen the condition that $z_i(\cdot, \omega)\sigma(\cdot, \omega) \in L^2([0, T])$ almost surely to $z_i\sigma \in L^2([0, T] \times \Omega)$. We also define $\mathcal{L}^2 = \mathcal{L}^2(\beta)$ and $\mathcal{H}^2 = \mathcal{H}^2(\beta)$. For more details, see Nielsen (1999).

[15]The requirement that $\int z_i \, d\gamma$ be a martingale is the standard "admissibility" condition ruling out arbitrage strategies such as the doubling strategy of Harrison and Kreps (1979).

We say that an equilibrium is effectively dynamically complete if every consumption process c which is adapted to the *Brownian* filtration and which satisfies

$$E\left(p_C(T, \omega)c(T, \omega) + \int_0^T p_C(t, \omega)c(t, \omega)\, dt \right) < \infty$$

can be financed by an admissible trading strategy.

The following theorem and proposition are proved in Appendix D.

THEOREM 2.1: *The continuous-time finance model just described has an equilibrium, which is Pareto optimal. The equilibrium pricing process is effectively dynamically complete and the admissible replicating strategies are unique.*

PROPOSITION 2.2: *Let p_A, p_c, c_i, and z_i denote the equilibrium securities prices, consumption prices, consumptions, and trading strategies. Let $\Sigma = \frac{\partial p_A}{\partial \beta}$ denote the dispersion matrix of the securities prices, let $\Sigma_i = z_i \Sigma$ denote the dispersion (row) vector of agent i's trading strategy, and let $W_i = z_i \cdot p_A$ denote agent i's securities wealth. Then*

$$(5) \qquad p_A(t, \beta) = E\bigg(p_C(T, \beta(T))A(T, \beta(T))$$

$$+ \int_t^T p_C(s, \beta(s))A(s, \beta(s))\, ds \bigg| \beta(t) = \beta \bigg),$$

$$(6) \qquad \Sigma(t, \beta) = E\bigg(\frac{p_C(T, \beta(T))A(T, \beta(T))(\beta(T) - \beta)^\top}{T - t}$$

$$+ \int_t^T \frac{p_C(s, \beta)A(s, \beta(s))(\beta(s) - \beta)^\top}{s - t} \bigg| \beta(t) = \beta \bigg).$$

There exists a unique vector $\lambda \in \mathbf{R}_+^I$ of Negishi utility weights with $\sum_{i=1}^I \lambda_i = 1$ such that for all (t, β), $c_1(t, \beta), \ldots, c_I(t, \beta)$ solve the problem

$$(7) \qquad \max\left\{ \sum_{i=1}^I \lambda_i h_i(c_i) : \sum_{i=1}^I c_i = \sum_{i=1}^I e_i(t, \beta) + \sum_{j=1}^J \eta_j A_j(t, \beta) \right\}$$

for $t \in [0, T)$ and the analogous problem with H_i substituted for h_i for $t = T$.

Each Σ_i and W_i has a continuous version (still denoted Σ_i and W_i):

$$(8) \qquad \Sigma_i(t, \beta) = E\bigg(\frac{p_C(T, \beta(T))(c_i(T, \beta(T)) - e_i(T, \beta(T)))(\beta(T) - \beta)^\top}{T - t}$$

$$+ \int_t^T \big(p_C(s, \beta(s))\big(c_i(s, \beta(s)) - e_i(s, \beta(s))\big)$$

$$\times (\beta(s) - \beta)^{\top})(s - t)^{-1} ds \Big| \beta(t) = \beta \Big),$$

$$(9) \qquad W_i(t, \beta) = E\Big(p_C(T, \beta(T))\big(c_i(T, \beta(T)) - e_i(T, \beta(T))\big)$$

$$+ \int_{s=t}^{T} p_C(s, \beta(s))\big(c_i(s, \beta(s)) - e_i(s, \beta(s))\big) ds \Big|$$

$$\beta(t) = \beta \Big).$$

There is a open set B of full measure in $(0, T) \times \mathbf{R}^K$ and analytic functions $Z_1, \ldots, Z_I$ with domain B such that $z_i(t, \omega) = Z_i(\mathcal{I}(t, \omega))$ whenever $\mathcal{I}(t, \omega) \in B$; Z_i is uniquely determined on B by the linear equations

$$\Sigma_i = Z_i \Sigma, \quad W_i = Z_i \cdot p_A.$$

$p_A, p_C, c_1, \ldots, c_I, W_1, \ldots, W_I \in L^2$; $\Sigma, \Sigma_1, \ldots, \Sigma_I \in \mathcal{H}^2$; $z_1, \ldots, z_I \in \mathcal{H}^2(\gamma)$; $p_A, p_C, W_1, \ldots, W_I$ are functions of $\mathcal{I}(t, \omega)$ which are continuous on $[0, T] \times \mathbf{R}^K$ and analytic on $(0, T) \times \mathbf{R}^K$; $\Sigma, \Sigma_1, \ldots, \Sigma_I$ are functions of $\mathcal{I}(t, \omega)$ which are continuous on $[0, T] \times V$ and analytic on $(0, T) \times \mathbf{R}^K$; p_c and c_i are given separately over $[0, T) \times \mathbf{R}^k$ and $\{T\} \times \mathbf{R}^K$ as functions of $\mathcal{I}(t, \omega)$; the functions over $[0, T) \times \mathbf{R}^K$ are analytic on $(0, T) \times \mathbf{R}^K$.

REMARK 2.3: Equation (6) says that the (j, k) entry of the dispersion matrix of the securities prices is the integral, over future times s (including $s = T$), of the regression coefficient of dividends of the jth security at time s, valued at the equilibrium consumption prices, on the change in the kth component of the Brownian motion up to time s. Equation (8) says that the kth element of Σ_i is the integral, over future times s (including $s = T$), of the regression coefficient of i's consumption minus i's endowment at time s, valued at the equilibrium consumption prices, on the change in the kth component of the Brownian motion up to time s.

3. EXAMPLE

To appreciate how our existence result compares to the previous literature, it is useful to consider the following parametric family of continuous-time securities markets. It is derived from the Merton (1973, 1990) model, except that our securities are described by their dividend processes, rather than by their price processes. Agent i is endowed with a flow rate of consumption α_i for $t \in [0, T)$ and a lump of consumption α_i at time T (in Merton, this is described

as a bequest), where $\alpha_i \in \mathbf{R}_{++}$. Agent i's utility for a stream of consumption c_i is

$$E\left(h_i(c_i(T)) + \int_0^T h_i(c_i(s))\, ds\right),$$

where h_i is a state-independent CRRA utility function with coefficient of relative risk aversion γ_i. There are $K + 1$ securities, which pay no dividends at times $t \in [0, T)$ and which pay lump dividends at time T. The zeroth security is a zero-coupon bond which pays a lump dividend of one unit of consumption at time T; the dividends of the other securities are given by $e^{\sigma \beta(T)}$, where σ is a constant $K \times K$ matrix, so the dividends are terminal values of a K-dimensional geometric Brownian motion. The zeroth security is in zero net supply, while the remaining securities are in net supply one. At time zero, agent 1 has an initial holding of $\delta_1 \in \mathbf{R}^{K+1}$ units of the securities; agent's 2's initial holding is $(0, 1, \ldots, 1) - \delta_1$. The utility functions and dividend processes are standard benchmarks in continuous-time finance.

For what set of parameters does this securities market have an equilibrium? If an equilibrium exists, is it dynamically complete? If $\alpha_1 = \alpha_2$, $\gamma_1 = \gamma_2$, and $\delta_1 = \delta_2$, this is effectively a single agent economy, and it is known that the economy has an equilibrium, but it is not known whether it is dynamically complete for $K > 1$. If $\gamma_1 \neq \gamma_2$, the candidate equilibrium is Pareto optimal, so there are Negishi weights λ_1 and λ_2 such that the candidate equilibrium consumptions maximize $h(c) = \lambda_1 h_1(c_1) + \lambda_2 h_2(c_2)$ subject to $c_1 + c_2 = c$ at each node. But the weights λ cannot be computed in closed form and h is *not* CRRA. Diasakos (2007) recently showed in the case $K = 1$ that the candidate equilibrium price is dynamically complete.[16] His proof does not require computing the candidate equilibrium price in closed form, but it is a difficult calculation. For the case $K > 1$, the only known way to show the existence of equilibrium has been to compute the candidate equilibrium price process explicitly and check that it is dynamically complete. However, the candidate equilibrium prices can be computed explicitly only for a very small set of parameter values. Thus, none of the previous results rules out the possibility that the set of parameters $(\alpha_1, \alpha_2, \gamma_1, \gamma_2, \delta_1, \sigma, T)$ for which an equilibrium *exists* has Lebesgue measure *zero*. Theorem 2.1 implies that if σ is nonsingular, then an equilibrium exists and is dynamically complete for *every* value of the other parameters, that is, every

$$(\alpha_1, \alpha_2, \gamma_1, \gamma_2, \delta_1, T)$$
$$\in \mathbf{R}_{++} \times \mathbf{R}_{++} \times \mathbf{R}_+ \times \mathbf{R}_+ \times (\{0\} \times [0, 1] \times [0, 1]) \times \mathbf{R}_{++}.$$

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[16]Diasakos considered a single agent economy, but the Negishi utility h satisfies his assumptions on the utility function.

4. DISCRETE APPROXIMATIONS

In this section, we describe our process for discretizing a continuous-time model. We then state a theorem indicating that the equilibria of the discretized economy are close to those of the continuous-time economy. This result has two important consequences. First, it provides an effective computational method for computing equilibria of the continuous-time economy. All one has to do is compute, using standard algorithms, an equilibrium of a sufficiently fine discretization. Second, actual securities markets are discrete in a number of important ways. Prices are restricted to lie in a grid, trades are carried out in integral numbers of shares, and trades take a certain amount of time to execute. Continuous-time models are useful because pricing formulas can be expressed more cleanly in continuous time than in discrete time. However, to know that the formulas obtained from continuous-time models are applicable to real markets, we need to know that the behavior of large discrete models is close to the behavior of large continuous-time models.

Here is the formal description of our discretization procedure:

Choose $n \in \mathbf{N}$. For $t \in [0, T]$, define $\hat{t} = \frac{\lfloor nt \rfloor}{n}$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x; in particular, $\hat{T} = \frac{\lfloor nT \rfloor}{n}$. Define $\Delta T = \frac{1}{n}$ and $\mathcal{T} = \{0, \Delta T, 2\Delta T, \ldots, \hat{T}\}$. Define a measure $\hat{\nu}$ on $\mathcal{T}$ by $\hat{\nu}(\{t\}) = \Delta T$ if $t < \hat{T}$ and $\hat{\nu}(\{\hat{T}\}) = 1$.

LEMMA 4.1: *We can choose $K + 1$ vectors $v_0, \ldots, v_K \in \mathbf{R}^K$ such that*

$$v_j \cdot v_k = \begin{cases} K, & \text{if } j = k, \\ -1, & \text{if } j \neq k, \end{cases}$$

$$\sum_{k=0}^{K} (v_k)_i (v_k)_j = \delta_{ij}(K + 1),$$

$$\sum_{k=0}^{K} v_k = 0,$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

See Appendix A for the proof.

To ensure that the discrete model is potentially dynamically complete, it is critical that every nonterminal node in the random walk have exactly $K + 1$ successor nodes; see the comments on this in the Introduction. Now, we describe how we construct our probability space $\hat{\Omega}$, filtration $\{\mathcal{F}_t\}$, and random walk $\hat{\beta}$. Let $\hat{\Omega} = \{\omega : \mathcal{T} \setminus \{0\} \to \{0, 1, 2, \ldots, K\}\}$. The measure $\hat{\mu}$ on $\hat{\Omega}$ is given by $\hat{\mu}(A) = \frac{|A|}{|\hat{\Omega}|}$ for every $A \in \hat{\mathcal{F}}$, the algebra of all subsets of $\hat{\Omega}$; here, $|A|$ denotes the cardinality of A. For $t \in \mathcal{T}$, $\hat{\mathcal{F}}_t$ is the algebra of all subsets of $\hat{\Omega}$ that respect the equivalence relation $\omega \sim_t \omega' \Leftrightarrow \omega(s) = \omega'(s)$ for all $s \leq t$.

For $s \in \mathcal{T}$, define the random variable $v_s(\omega) = v_{\omega(s)}$, where $v_0, \ldots, v_k$ are the vectors chosen in Lemma 4.1. If $s \neq t$, the random variables v_s and v_t are independent. Moreover, for each $\ell \in \{1, \ldots, K\}$, the random variable $(v_s)_\ell$ has mean 0 and standard deviation 1. Define $\hat{\beta} : \mathcal{T} \times \hat{\Omega} \to \mathbf{R}^K$ by

$$\hat{\beta}(t, \omega) = \sum_{0 < s \leq t, s \in \mathcal{T}} v_s(\omega)\sqrt{\Delta T}.$$

$\hat{\beta}$ is a K-dimensional random walk. Note that $\hat{\beta}(t, \cdot)$ has variance–covariance matrix tI, where I is the $K \times K$ identity matrix. Define

$$\hat{\mathcal{I}}(t, \omega) = (t, \hat{\beta}(t, \omega)).$$

Given a consumption plan $\hat{c} : \mathcal{T} \times \hat{\Omega} \to \mathbf{R}_+$, the agent's utility is

$$\hat{U}_i(\hat{c}) = E_{\hat{\mu}}\left(\left(\Delta T \sum_{s \in \mathcal{T}, s < \hat{T}} h_i(\hat{c}(t, \omega), \hat{\mathcal{I}}(t, \omega))\right)\right.$$
$$\left. + H_i(\hat{c}(\hat{T}, \omega), \hat{\beta}(\hat{T}, \omega))\right).$$

Our discrete economy is a GEI economy, the social endowment (including dividends) is strictly positive at every node, and our utility functions are completely well behaved, so the discrete economy satisfies the assumptions of the Duffie–Shafer (1985, 1986) and Magill–Shafer theorems (1990). These theorems state that there is an arbitrarily small perturbation of the endowments and security dividends such that the perturbed economy has an equilibrium. Let $\hat{e}_i(t, \omega) \geq f_i(\hat{\mathcal{I}}(t, \omega))$, $|\hat{e}_i(t, \omega) - f_i(\hat{\mathcal{I}}(t, \omega))| \leq (\Delta T)^2$ ($t < \hat{T}$), and $\hat{e}_i(\hat{T}, \omega) \geq F_i(\hat{T}, \omega)$, $|\hat{e}_i(\hat{T}, \omega) - F_i(\hat{\mathcal{I}}(\hat{T}, \omega))| \leq (\Delta T)^2$ denote the perturbed endowments. For all $\omega \in \hat{\Omega}$, let $\hat{A}(t, \omega)$ denote the perturbed dividends $\hat{A}(t, \omega) \geq g(\hat{\mathcal{I}}(t, \omega))$, $|\hat{A}(t, \omega) - g(\hat{\mathcal{I}}(t, \omega))| \leq (\Delta T)^2$ for all $t < \hat{T}$, and $\hat{A}(\hat{T}, \omega) \geq G(\hat{\mathcal{I}}(\hat{T}, \omega))$, $|\hat{A}(\hat{T}, \omega) - G(\hat{\mathcal{I}}(\hat{T}, \omega))| \leq (\Delta T)^2$. Notice that the social consumption $\hat{e}(t, \omega) + \sum_{j=0}^{J} \eta_j \hat{A}_j(t, \omega) > 0$.

Recall there are $J + 1$ securities indexed by $j = 0, \ldots, J$. A securities price process is a function $\hat{p}_A : \mathcal{T} \times \hat{\Omega} \to \mathbf{R}^{J+1}$ which is adapted with respect to $\{\hat{\mathcal{F}}_t\}_{t \in \hat{\mathcal{T}}}$. We will price securities *ex dividend* for $t < \hat{T}$; it will be convenient to price securities *cum dividend* at $t = \hat{T}$.

A consumption price process is a function $\hat{p}_C : \mathcal{T} \times \hat{\Omega} \to \mathbf{R}_+$ which is adapted with respect to $\{\hat{\mathcal{F}}_t\}_{t \in \hat{\mathcal{T}}}$.

Given a securities price process $\hat{p}_A$ and a consumption price process $\hat{p}_C$, the associated total gains process $\hat{\gamma}$ for the securities is defined to be

$$\hat{\gamma}(t, \omega) = \begin{cases} \hat{p}_A(t, \omega) + (\Delta T) \displaystyle\sum_{s \in \mathcal{T}, s \le t} \hat{p}_C(s, \omega) \hat{A}(s, \omega), & \text{if } t < \hat{T}, \\[2mm] \hat{p}_A(t, \omega) + (\Delta T) \displaystyle\sum_{s \in \mathcal{T}, s < t} \hat{p}_C(s, \omega) \hat{A}(s, \omega), & \text{if } t = \hat{T}. \end{cases}$$

Note that since securities are priced ex dividend at times $t < \hat{T}$, the dividend at time t is included in the sum and not in the securities price $\hat{p}_A$; since securities are priced cum dividend at time $\hat{T}$, the dividend at time $\hat{T}$ is included in the securities price $\hat{p}_A$ and not in the sum.

A trading strategy[17] for agent i is a row vector function $\hat{z}_i : (\mathcal{T} \cup \{-\Delta T\}) \times \hat{\Omega} \to \mathbf{R}^{J+1}$ which is adapted with respect to $\{\hat{\mathcal{F}}_t\}_{t \in \mathcal{T}}$ such that $\hat{z}_i(-\Delta T, \omega) = e_{iA}$ (this ensures that each agent arrives at time 0 holding the correct initial securities endowment; there is no trade in goods or securities at time $-\Delta T$) and $\hat{z}_i(\hat{T}, \omega) = \hat{z}_i(\hat{T} - \Delta T, \omega)$.

A consumption plan for agent i is a function $\hat{c}_i : \mathcal{T} \times \hat{\Omega} \to \mathbf{R}_+$. The budget set for agent i is the set of all consumption plans $\hat{c}_i$ such that there exists a trading strategy $\hat{z}_i$ for which $\hat{c}_i$ satisfies the budget constraint

$$\big(\hat{c}_i(s, \omega) - \hat{e}_i(s, \omega) - \hat{z}_i(s - \Delta T, \omega) \cdot \hat{A}(s, \omega) \big) \cdot \hat{p}_C(s, \omega)$$
$$= (\hat{z}_i(s, \omega) - \hat{z}_i(s - \Delta T, \omega)) \cdot \hat{p}_A(s, \omega)$$

for all $s \in \mathcal{T}$ and all $\omega \in \hat{\Omega}$. The budget equation says that the value of the agent's consumption at every node equals the value of the agent's endowment at that node plus the value of the dividends generated by the agent's portfolio at that node plus the value of the securities sold by the agent at that node.[18] Note that since $\hat{z}$ is required to be adapted to $\{\hat{\mathcal{F}}_t\}_{t \in \hat{\mathcal{T}}}$, it follows that $\hat{c}_i$ is adapted to $\{\hat{\mathcal{F}}_t\}_{t \in \hat{\mathcal{T}}}$.

Given a security price $\hat{p}$ and a consumption price $\hat{p}_C$, the demand of the agent is a consumption plan and a trading strategy which satisfy the budget constraint, and such that the consumption plan maximizes utility over the budget set.

An equilibrium for the economy is a security price process $\hat{p}$, a consumption price process $\hat{p}_C$, trading strategies $\hat{z}_i$, and consumption plans $\hat{c}_i$ which lies in

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[17]In the discrete model, stochastic integrals with zero drift are automatically martingales, so we do not need to require admissibility as a separate assumption.

[18]We take $\hat{z}_i(-\Delta T, \omega) = e_{iA}(0)$, so that the agent enters period 0 holding the securities with which s/he is endowed. Since securities are priced cum dividend at $t = \hat{T}$, we require that $\hat{z}_i(\hat{T}, \omega) = \hat{z}_i(\hat{T} - \Delta T, \omega)$.

the demand sets of the agents so that the securities and goods markets clear, that is, for all $t \in \mathcal{T}$ and all $\omega \in \hat{\Omega}$,

$$\sum_{i=1}^{I} \hat{z}_i(t, \omega) = (\eta_0, \ldots, \eta_J),$$

$$\sum_{i=1}^{I} \hat{c}_i(t, \omega) = \sum_{i=1}^{I} \hat{e}_i(t, \omega) + \sum_{j=0}^{J} \eta_j \hat{A}_j(\omega, t).$$

Since short selling is not restricted, the first-order conditions imply that the total gains process $\hat{\gamma}$ is a vector martingale (pages 230–231 of Magill and Quinzii (1996)).

Now we identify the equilibrium trading strategies and the discrete analogue $\hat{\sigma}$ of the dispersion matrix of the equilibrium securities price process and the dispersion matrix $\hat{\sigma}_i$ of agent i's securities wealth. We have the equilibrium securities prices $\hat{p}_A(t, \omega)$ and at the $K + 1$ successor nodes to (t, ω). Because the $K + 1$ possible values of $\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)$ span $\mathbf{R}^K$, and $\hat{p}_A$ and $\hat{\gamma}$ are both martingales, there is[19] a unique $(K + 1) \times K$ matrix process $\hat{\sigma}(t, \omega)$ adapted to $\hat{\mathcal{F}}_t$, satisfying

$$\hat{p}_A(t + \Delta T, \omega) + \hat{p}_C(t + \Delta T, \omega) A(t + \Delta T, \omega) - \hat{p}_A(t, \omega)$$
$$= \hat{\sigma}(t, \omega)(\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)).$$

Because the equilibrium is dynamically complete, $\hat{\sigma}(t, \omega)$ has rank K for all nodes (t, ω).

For each consumer $i = 1, \ldots, I$, define

$$\hat{D}_i(t, \omega) = E\Bigg(\Bigg(\Big(\hat{p}_C(\hat{T}, \cdot)(\hat{c}_i(\hat{T}, \cdot) - \hat{e}_i(\hat{T}, \cdot))$$
$$+ \Delta T \sum_{s=0}^{\hat{T}-\Delta T} \hat{p}_C(s, \cdot)(\hat{c}_i(s, \cdot) - \hat{e}_i(s, \cdot))\Big)\Bigg)\Big|(t, \omega)\Bigg).$$

From the definition, $\hat{D}_i$ is a martingale; it is the conditional expectation of the difference in value, at the prices of consumption, between i's consumption and i's endowment. For each i, there is[20] a unique $(K + 1) \times K$ matrix process $\hat{\sigma}_i(t, \omega)$ adapted to $\hat{\mathcal{F}}_t$ such that

$$\hat{D}_i(t + \Delta T, \omega) - \hat{D}_i(t, \omega) = \hat{\sigma}_i(t, \omega)(\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)).$$

[19]For details, see Appendix A.
[20]For details, see Appendix A.

Since the securities wealth of agent i must exactly equal the conditional expectation of the difference in value between i's future consumption and i's endowment, $\hat{z}_i(t, \omega)$ must be the unique $K \times 1$ row vector process adapted to $\hat{\mathcal{F}}_t$ which satisfies the budget constraint and the equation

$$\hat{\sigma}_i(t, \omega) = \hat{z}_i(t, \omega)\hat{\sigma}(t, \omega).$$

Thus, $\hat{z}_i$ is determined uniquely from $\hat{p}_A$ and $\hat{c}_i$ by linear algebra.

In the following theorem, whose proof is given in Appendix D, we show that equilibria of the discretized economies are close to equilibria of the continuous-time economy. If the continuous-time economy has a unique equilibrium, then p_{nA}, p_{nC}, c_{ni}, and λ_n are given by the unique equilibrium, and thus are independent of n. If the continuous-time economy has multiple equilibria, the equilibrium corresponding to the nth discretization may well cycle among the multiple equilibria. We could, for example, have the discrete-time equilibria close to one continuous-time equilibrium for n odd and close to a different continuous-time equilibrium for n even, so our continuous-time equilibrium will in general have to be chosen differently for different discretizations.

THEOREM 4.2: *Given a continuous-time economy satisfying the assumptions of Theorem* 2.1, *let* $\hat{p}_{nA}$, $\hat{p}_{nC}$, $\hat{c}_{ni}$, *and* $\hat{z}_{ni}$ *denote any equilibrium securities prices, consumption prices, consumptions, and trading strategies of the discretized sequence of economies just described. Let* $\hat{\lambda}_n$ *denote the vector of Negishi utility weights maximized at that equilibrium. Then there are equilibria of the continuous-time economy with securities prices* p_{nA}, *consumption prices* p_{nC}, *consumptions* c_{ni}, *trading strategies* z_{ni}, *and utility weights* λ_n *satisfying the conclusions of Theorem* 2.1 *and Proposition* 2.2 *such that*

(10) $|\hat{\lambda}_n - \lambda_n| \to 0,$

(11) $\|\hat{p}_{nA} - p_{nA} \circ \hat{\mathcal{I}}\|_2 \to 0 \quad and \quad \max_{t \in \mathcal{T}_n}|\hat{p}_{nA}(t, \cdot) - p_{nA}(\hat{\mathcal{I}}(t, \cdot))| \to 0$

         in probability,

(12) $\|\hat{p}_{nc} - p_{nc} \circ \hat{\mathcal{I}}\|_2 \to 0 \quad and \quad \max_{t \in \mathcal{T}_n}|\hat{p}_{nc}(t, \cdot) - p_{nc}(\hat{\mathcal{I}}(t, \cdot))| \to 0$

         in probability,

(13) $\|\hat{\sigma}_n - \Sigma_n \circ \hat{\mathcal{I}}\|_2 \to 0 \quad and \quad \max_{t \in \mathcal{T}_n}|\hat{\sigma}_n(t, \cdot) - \Sigma_n(\hat{\mathcal{I}}(t, \cdot))| \to 0$

         in probability,

(14) $\|\hat{c}_{ni} - c_{ni} \circ \hat{\mathcal{I}}\|_2 \to 0 \quad and \quad \max_{t \in \mathcal{T}_n}|\hat{c}_{ni}(t, \cdot) - c_{ni}(\hat{\mathcal{I}}(t, \cdot))| \to 0$

         in probability,

(15) $|\hat{U}_{ni}(\hat{c}_{ni}) - U_{ni}(c_{ni})| \to 0,$

(16) $|\hat{z}_{ni} - z_{ni} \circ \hat{\mathcal{I}}| \to 0$ *in probability*,

(17) $\|\hat{\sigma}_{ni} - \Sigma_{ni} \circ \hat{\mathcal{I}}\|_2 \to 0$ *and* $\max_{t \in \mathcal{T}_n} |\hat{\sigma}_{ni}(t, \cdot) - \Sigma_{ni}(\hat{\mathcal{I}}(t, \cdot))| \to 0$

        in probability,

(18) $\|\hat{z}_{ni} \cdot \hat{p}_{An} - W_{ni} \circ \hat{\mathcal{I}}\|_2 \to 0$ *and*

        $\max_{t \in \mathcal{T}_n} |\hat{z}_{ni}(t, \cdot) \cdot \hat{p}_{An} - W_{ni}(\hat{\mathcal{I}}(t, \cdot))| \to 0$ *in probability*.

5. DISCUSSION

Nonstandard analysis is a conservative extension of conventional analysis. In other words, the existence of nonstandard models follows from the conventional axioms of set theory and analysis, so the theorems presented here do not depend on any additional set theoretic or analytic axioms. In particular, the proofs presented here can be mechanically translated into standard proofs using only the usual axioms, but the resulting standard proofs would be exceedingly long and unintelligible. We believe it would be extremely difficult to provide tractable standard proofs of the convergence theorems.

We think it is likely possible to use the analytic function techniques developed here to prove that, in the settings of the previous literature,[21] the candidate equilibrium prices are dynamically complete. This would allow one to remove the assumption of dynamic completeness of the candidate equilibrium price process from the papers cited, and would rely on the techniques used in those papers and our analytic function techniques, but would not require nonstandard analysis. We have chosen not to proceed in this direction for the following three reasons: First, our ultimate goal is to extend these methods to obtain the existence of equilibria in continuous-time financial markets which are dynamically *incomplete*. The approach in the previous literature begins with an Arrow–Debreu complete markets equilibrium. Since there is no known theorem asserting the existence of equilibrium with more than one agent, an infinite-dimensional commodity space, and incomplete markets, there seems to be no hope of extending the approach in the previous literature to incomplete markets. By contrast, our approach begins with a GEI equilibrium in a discrete setting. Many steps in our argument work just as well for the dynamically incomplete case as for the dynamically complete case. While critical problems remain to be solved, our methods at least provide a way of attacking the dynamically incomplete case. Second, we feel the convergence theorem is at least as important as the existence theorem. As is often the case in nonstandard analysis, our convergence theorem is essentially an immediate corollary

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[21]See the references listed in the fifth paragraph of Section 1.

of the nonstandard existence proof; however, we doubt that a tractable stan-
dard proof of the convergence theorem can be given. Third, while our proof
makes extensive use of nonstandard analysis and stochastic processes, it is com-
pletely independent of knowledge of functional analysis. Since most who work
in continuous-time finance have a background in stochastic processes but not
in functional analysis, it is desirable to have a proof that does not depend on
functional analysis.

Anderson's construction of Brownian motion has been used to answer a
number of questions in stochastic processes, and we are able to make do in
this paper with slight extensions of it. However, we anticipate that extending
this work to the dynamically incomplete case, or the case of dividends with
jumps, will require using subsequent work in nonstandard stochastic analysis.[22]

6. MEAN REVERSION

We assumed throughout that the primitives of the model are given as func-
tions of time and the current value of the Brownian motion. It would be de-
sirable to allow the primitives to be given by more general Itô processes. Sig-
nificant technical problems would have to be overcome, so we are leaving this
problem for future work.

Most models of interest rate determination involve mean reversion, so in this
section we briefly sketch how to incorporate mean reversion into our model.

---

[22]The work of Keisler (1984) on stochastic differential equations with respect to Brownian
motion, and the work by Albeverio and Herzberg (2006), Hoover and Perkins (1983a, 1983b),
Lindstrøm (1980a, 1980b, 1980c, 1980d, 2004, 2008), and Ng (2008) on stochastic integration
with respect to more general martingales, including Lévy processes, is likely to be of particu-
lar relevance. The nonstandard theory of stochastic integration has previously been applied to
option pricing by Cutland, Kopp, and Willinger (1991a, 1991b, 1991c, 1993, 1995a, 1995b) and
Cutland, Kopp, Willinger, and Wyman (1997). Those papers, which are not in an equilibrium con-
text, primarily concern convergence of discrete versions of options to continuous-time versions,
and their methods can likely be used to establish convergence results for the equilibrium op-
tion pricing formulas developed by Anderson and Raimondo (2005, 2006). Anderson and Rashid
(1978) provided a nonstandard characterization of weak convergence. Cutland, Kopp, and Will-
inger (1993, 1995b) and Cutland (2000) proposed a convergence notion for stochastic processes,
D^2-convergence, which is stronger than the topology of weak convergence. The statement of the
convergence notion is nonstandard, and to the best of our knowledge no one has identified a stan-
dard topology on stochastic processes that corresponds to D^2-convergence. The equilibria of our
discrete approximations D^2-converge to the set of equilibria of the continuous-time economy. Ng
(2003) provided a treatment of mathematical finance based on hyperfinite random walks, and in
particular presented the Cox–Ross–Rubinstein (1979) binomial model in a nonstandard context.
Khan and Sun (1997, 2001) related the capital asset pricing model and arbitrage pricing theory in
a single-period setting. Malliavin calculus provides an approach to continuous-time finance with
asymmetric information; nonstandard treatments of Malliavin calculus were given by Cutland
and Ng (1995) and Osswald (2006). Keisler (1996) presented a nonstandard stochastic process
model of nontatonnement price adjustment; agents are randomly chosen to come to the market,
and their trades move the market price rapidly toward equilibrium.

The canonical mean-reverting process is the Ornstein–Uhlenbeck process, which is by definition a solution ϕ of the stochastic differential equation

$$(19) \qquad d\phi = a(b - \phi)\,dt + \sigma\,d\beta,$$

where a, b, and σ are real constants with $a > 0$. The Vasicek (1977) model assumes that the short-run risk-free rate of return r is an Ornstein–Uhlenbeck process; one can then derive the dynamics of bonds of various maturities.

From an equilibrium perspective, the short-run risk-free interest rate should be determined in equilibrium, not given as a primitive of the model. When markets are dynamically complete, the securities market equilibrium is the Arrow–Debreu equilibrium, in which the price of consumption is the marginal utility of consumption. In the proof of Theorem 2.1, we show that one can construct a riskless security denominated in utils, and that the risk-free interest rate in this normalization is zero. Observable interest rates are typically expressed in terms of monetary units, but ours, like most general equilibrium models, is not a monetary model. However, there are securities, such as treasury inflation-protected securities (TIPS), which are automatically adjusted for inflation, and it is reasonable to think of TIPS as being denominated in units of consumption. If we denominate prices in units of consumption, the equilibrium real risk-free interest rate is the negative of the drift term of the price of consumption:

$$r = -\frac{\frac{\partial p_C}{\partial t} + \frac{1}{2}\frac{\partial^2 p_C}{\partial \beta^2}}{p_C}.$$

r will be an Ornstein–Uhlenbeck process only for a knife-edge set of primitives of our equilibrium model.

Our model can easily generate mean reversion of the risk-free rate. For example, suppose that the social consumption is a periodic function of β such as

$$2 + \sin\beta(t, \omega),$$

and that the utility functions are state- and time-independent (in particular, there is no time discounting). When $\sin\beta(t, \omega) > 0$, social consumption has a downward drift, so $p_C(t, \omega)$ has an upward drift, so $r(t, \omega) < 0$; one can show that $r(t, \omega)$ has an upward drift. Similarly, when $\sin\beta(t, \omega) < 0$, social consumption has an upward drift, so $p_C(t, \omega)$ has a downward drift, so $r(t, \omega) > 0$ and has a downward drift. Adding impatience and expressing the interest rate in nominal rather than real terms will produce a risk-free rate which is positive and mean reverting.

Suppose ϕ is an Ornstein–Uhlenbeck process satisfying Equation (19). It is not possible to write $\phi(t, \omega)$ as a function of t and $\beta(t, \omega)$ alone, so our theorem as stated does not cover the case in which the social consumption equals

$\phi(t, \omega)$. However, our proof can be extended to the case in which the primitives are analytic functions of t and $\phi(t, \omega)$. They key is that ϕ is a Markov process, and the conditional distribution of $\phi(T)$, conditional on $(t, \phi(t))$, is well behaved in the same ways that the conditional distribution of $\beta(T)$, conditional on $(t, \beta(t))$, is well behaved.

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APPENDIX A: DETAILS ON THE DISCRETE MODEL

In this appendix, we provide more detail on the analysis of the discrete model.

PROOF OF LEMMA 4.1: Suppose first $K = 1$; let $v_0 = (1)$ and $v_1 = (-1)$. Then $v_0 \cdot v_0 = v_1 \cdot v_1 = 1$ and $v_0 \cdot v_1 = -1$, $(v_0)_1(v_0)_1 + (v_1)_1(v_1)_1 = 1 + 1 = 2$, and $(v_0)_1 + (v_1)_1 = 1 - 1 = 0$.

Now suppose we have chosen $v_0, \ldots, v_K \in \mathbf{R}^K$ such that $v_j \cdot v_k = -1$ if $j \neq k$ and $v_k \cdot v_k = K$; $\sum_{k=0}^{K}(v_k)_i(v_k)_j = \delta_{ij}(K+1)$ and $\sum_{k=0}^{K} v_k = 0$. Let

$$\tilde{v}_k = \begin{cases} \left(\sqrt{\dfrac{K+2}{K+1}}(v_k)_1, \ldots, \sqrt{\dfrac{K+2}{K+1}}(v_k)_K, -\dfrac{1}{\sqrt{K+1}}\right), \\ \quad \text{for } k = 0, \ldots, K, \\ (0, \ldots, 0, \sqrt{K+1}), \\ \quad \text{for } k = K+1. \end{cases}$$

For $j, k \in \{0, \ldots, K\}$,

$$\tilde{v}_j \cdot \tilde{v}_k = \frac{K+2}{K+1} v_j \cdot v_k + \frac{1}{K+1}$$

$$= \begin{cases} \dfrac{K+2}{K+1}(-1) + \dfrac{1}{K+1} = -1, & \text{if } j \neq k, \\ \dfrac{K+2}{K+1}K + \dfrac{1}{K+1} = \dfrac{K^2+2K+1}{K+1} = K+1, & \text{if } j = k, \end{cases}$$

$$\tilde{v}_j \cdot \tilde{v}_{K+1} = -1,$$

$$\tilde{v}_{K+1} \cdot \tilde{v}_{K+1} = K+1.$$

For $i, j \in \{1, \ldots, K\}$,

$$
\begin{aligned}
\sum_{k=0}^{K+1} (\tilde{v}_k)_i (\tilde{v}_k)_j &= \sum_{k=0}^{K} (\tilde{v}_k)_i (\tilde{v}_k)_j + 0 \\
&= \frac{K+2}{K+1} \sum_{k=2}^{K} (v_k)_i (v_k)_j \\
&= \frac{K+2}{K+1} (K+1) \delta_{ij} \\
&= \delta_{ij} (K+2).
\end{aligned}
$$

For $i \in \{1, \ldots, K\}$ and $j = K + 1$,

$$
\begin{aligned}
\sum_{k=0}^{K+1} (\tilde{v}_k)_i (\tilde{v}_k)_j &= \sum_{k=0}^{K} \sqrt{\frac{K+2}{K+1}} (v_k)_i \left(-\frac{1}{\sqrt{K+1}} \right) + 0 \cdot \sqrt{K+1} \\
&= -\frac{\sqrt{K+2}}{K+1} \sum_{k=0}^{K} (v_k)_i + 0 \\
&= 0 \\
&= \delta_{ij} (K+2).
\end{aligned}
$$

For $i = K + 1$ and $j \in \{1, \ldots, K\}$, note that the formula is invariant under switching i and j. Finally, for $i = j = K + 1$,

$$
\begin{aligned}
\sum_{k=0}^{K+1} (\tilde{v}_k)_i (\tilde{v}_k)_j &= (K+1) \frac{1}{K+1} + (K+1) \\
&= 1 + (K+1) \\
&= K + 2 \\
&= \delta_{ij} (K+2).
\end{aligned}
$$

For $j = 1, \ldots, K$,

$$
\sum_{k=0}^{K+1} (\tilde{v}_k)_j = \sqrt{\frac{K+2}{K+1}} \sum_{k=0}^{K} (v_k)_j = \sqrt{\frac{K+2}{K+1}} \cdot 0 = 0.
$$

Moreover,

$$
\sum_{k=0}^{K+1} (\tilde{v}_k)_{K+1} = -\frac{K+1}{\sqrt{K+1}} + \sqrt{K+1} = 0.
$$

This shows that $\sum_{k=0}^{K+1} \tilde{v}_k = 0$.

Thus, by induction, we can choose vectors $v_0, \ldots, v_K \in \mathbf{R}^K$ with the specified properties.   *Q.E.D.*

Determination of $\hat{\sigma}$ and $\hat{\sigma}_i$

Here, as promised in footnotes 19 and 20, we provide more detail on the determination of $\hat{\sigma}$ and $\hat{\sigma}_i$. Recall that v_k is a $K \times 1$ column vector, so $v_k^\top$ is a $1 \times K$ row vector; $\hat{\gamma}$ is a $(K+1) \times 1$ column vector, and $\hat{D}_i$ is a scalar. Let M be the $K \times K$ matrix, let Γ be the $(K+1) \times K$ matrix, and let Γ_i $(i = 1, \ldots, I)$ be the $1 \times K$ row vectors:

$$M = \sqrt{\Delta T} \begin{pmatrix} v_0^\top \\ \vdots \\ v_{K-1}^\top \end{pmatrix},$$

$$\Gamma = \begin{pmatrix} (\hat{\gamma}(t + \Delta T, \omega) - \hat{\gamma}(t, \omega)) & \cdots & (\hat{\gamma}(t + \Delta T, \omega) - \hat{\gamma}(t, \omega)) \\ \text{when } \omega(t + \Delta T) = 0 & \cdots & \text{when } \omega(t + \Delta T) = K - 1 \end{pmatrix},$$

$$\Gamma_i = \begin{pmatrix} (\hat{D}_i(t + \Delta T, \omega) - \hat{D}_i(t, \omega)) & \cdots & (\hat{D}_i(t + \Delta T, \omega) - \hat{D}_i(t, \omega)) \\ \text{when } \omega(t + \Delta T) = 0 & \cdots & \text{when } \omega(t + \Delta T) = K - 1 \end{pmatrix}.$$

Thus, the kth row $(k = 0, \ldots, K - 1)$ of M is the value of $\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)$ when $\omega(t + \Delta T) = k$, the kth column $(k = 0, \ldots, K - 1)$ of Γ is the value of $\hat{\gamma}(t + \Delta T, \omega) - \hat{\gamma}(t, \omega)$ when $\omega(t + \Delta T) = k$, and the kth entry $(k = 0, \ldots, K - 1)$ of Γ_i is the value of $\hat{D}_i(t + \Delta T, \omega) - \hat{D}_i(t, \omega)$ when $\omega(t + \Delta T) = k$; note that Γ_i is neither a row nor a column of Γ. Since $v_0, \ldots, v_{K-1}$ are linearly independent, M is nonsingular. Let

$$\hat{\sigma}(t, \omega) = \Gamma M^{-1},$$
$$\hat{\sigma}_i(t, \omega) = \Gamma_i M^{-1}.$$

$\hat{\sigma}(t, \omega)$ is the unique $(K+1) \times K$ matrix and $\hat{\sigma}_1(t, \omega), \ldots, \hat{\sigma}_I(t, \omega)$ are the unique $1 \times K$ row vectors such that

$$(20) \qquad \hat{\gamma}(t + \Delta T, \omega) - \hat{\gamma}(t, \omega) = \hat{\sigma}(t, \omega)(\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)),$$

$$\hat{D}_1(t + \Delta T, \omega) - \hat{D}_1(t, \omega) = \hat{\sigma}_1(t, \omega)(\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)),$$

$$\vdots$$

$$\hat{D}_I(t + \Delta T, \omega) - \hat{D}_I(t, \omega) = \hat{\sigma}_I(t, \omega)(\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega))$$

whenever $\omega(t + \Delta T) \in \{0, 1, \ldots, K - 1\}$. Since $\hat{\beta}, \hat{\gamma}$, and $\hat{D}_i, \ldots, \hat{D}_I$ are martingales, the sum (over the $K + 1$ successor nodes) of the change in $\hat{\beta}$, the change

in $\hat{\gamma}$, and the change in $\hat{D}_i$ is zero, so the fact that Equation (20) holds for $\omega(t + \Delta T) \in \{0, 1, \ldots, K - 1\}$ implies it holds also for $\omega(t + \Delta T) = K$. Thus, the processes $\hat{\sigma}$ and $\hat{\sigma}_1, \ldots, \hat{\sigma}_I$ are adapted.

APPENDIX B: REAL ANALYTIC FUNCTIONS OF SEVERAL VARIABLES

In this appendix, we summarize the results on real analytic functions of several variables used in our proof.

DEFINITION B.1: Let $U \subset \mathbf{R}^n$ be open. A function $F : U \to \mathbf{R}^m$ is *real analytic* if, for every $x_0 \in U$, there is a power series $G_{x_0}(x)$ centered at x_0 with a positive radius of convergence δ_{x_0} such that $F(x) = G_{x_0}(x)$ whenever $x \in U$ and $|x - x_0| < \delta_{x_0}$.

THEOREM B.2—The Analytic Implicit Function Theorem: *Suppose $X \subset \mathbf{R}^n, Y \subset \mathbf{R}^m$ are open, $F : X \times Y \to \mathbf{R}^n$ is real analytic, $x_0 \in X$, $y_0 \in Y$, $F(x_0, y_0) = 0$, and*

$$\det \frac{\partial F}{\partial x} \neq 0$$

at (x_0, y_0). Then there are neighborhoods U of x_0, and V of y_0 and a real analytic function $\phi : V \to U$ such that for all $y \in V$,

$$F(\phi(y), y) = 0.$$

Moreover, if $x \in U$, $y \in V$, and $F(x, y) = 0$, then $x = \phi(y)$.

PROOF: The conventional statement of the implicit function theorem establishes all the claims except the claim that the implicit function ϕ is real analytic. For this, see Theorem 2.3.5 of Krantz and Parks (2002). *Q.E.D.*

THEOREM B.3: *Let $U \subset \mathbf{R}^n$ be open and convex,[23] $F : U \to \mathbf{R}$ real analytic. If $\{x \in U : F(x) = 0\}$ has positive Lebesgue measure, then F is identically zero on U.*

PROOF: Lojasiewicz's structure theorem for varieties (Theorem 6.3.3 of Krantz and Parks (2002)) states that if U is an open set in $\mathbf{R}^n$ and $F : U \to \mathbf{R}$ is analytic, then for every $x_0 \in U$, there exists a neighborhood V_{x_0} of x_0 such that either $F(x) = 0$ for all $x \in V_{x_0}$ or $\{x \in V_{x_0} : F(x) = 0\}$ is a finite union of real algebraic varieties of dimension less than n. If $F(x) = 0$ for all $x \in V_{x_0}$ and $y \in U$, there is a ray that passes through V_{x_0} and through y; the restriction of F

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[23]It is sufficient to assume that U is connected and open. We prove the easier case when U is convex, since that is the case we use.

to the ray is an analytic function of a single variable, and it vanishes on an interval (the intersection of the ray with the set V_0); since it is well known that an analytic function of one variable that vanishes on an interval is identically zero, we must have $F(y) = 0$ for all $y \in U$ and we are done. On the other hand, if $\{x \in V : F(x) = 0\}$ is a finite union of algebraic varieties of dimension less than n, $\{x \in V_{x_0} : F(x) = 0\}$ has Lebesgue measure zero. There is a countable collection $\{x_n : n \in \mathbf{N}\}$ such that $\bigcup_{n \in \mathbf{N}} V_{x_n} \supset U$, so $\{x \in U : F(x) = 0\}$ has Lebesgue measure zero.   *Q.E.D.*

Finally, we show that the conditional expectation of a function of $G(\beta(T))$, conditional on $(t, \beta(t))$, is, under mild hypotheses on G, an analytic function of $(t, \beta(t))$. This fact is widely known among probabilists, but we were unable to find a specific reference that shows that the analyticity is *joint* in $(t, \beta(t))$, and we need the joint analyticity. Therefore, we provide a proof.

THEOREM B.4: *Suppose F is measurable on $\mathbf{R}^K$ and there exists $r \in \mathbf{R}$ such that*

$$|F(x)| \leq r + e^{r|x|}.$$

Let β be a standard K-dimensional Brownian motion and let

$$G(t, \beta) = E\big(F(\beta(T))|\beta(t) = \beta\big).$$

Then $G(t, \beta)$ is an analytic function of $(t, \beta) \in (0, T) \times \mathbf{R}^K$.

PROOF: Fix $t < T$. Then

$$(21) \qquad G(t, \beta) = \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} F(\beta + x) e^{-|x|^2/2(T-t)} \, dx$$

$$= \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} F(y) e^{-|y-\beta|^2/2(T-t)} \, dy$$

$$= \frac{e^{-|\beta|^2/2(T-t)}}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} e^{\beta \cdot y/(T-t)} F(y) e^{-|y|^2/2(T-t)} \, dy$$

$$= \frac{e^{-|\beta|^2/2(T-t)}}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \sum_{k=0}^{\infty} \frac{\left(\frac{\beta \cdot y}{T-t}\right)^k}{k!} F(y) e^{-|y|^2/2(T-t)} \, dy$$

$$= \frac{e^{-|\beta|^2/2(T-t)}}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \sum_{k=0}^{\infty} \frac{1}{(T-t)^k k!}$$

$$\times \sum_{k_1 + \cdots + k_K = k} \binom{k}{k_1 \ \cdots \ k_K} (\beta_1 y_1)^{k_1} \cdots (\beta_K y_K)^{k_K}$$

$$\times F(y) e^{-|y|^2/2(T-t)} \, dy$$

$$= \frac{e^{-|\beta|^2/2(T-t)}}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \sum_{k=0}^{\infty} \frac{1}{(T-t)^k}$$

$$\times \sum_{k_1+\cdots+k_K=k} \frac{(\beta_1 y_1)^{k_1}\cdots(\beta_K y_K)^{k_K}}{k_1!\cdots k_K!} F(y)e^{-|y|^2/2(T-t)}\,dy.$$

We view the two sums as an integral over

$$\{(k, k_1, \ldots, k_K): k, k_1, \ldots, k_K \in \mathbf{N}\cup\{0\}, k = k_1+\cdots+k_K\},$$

where each element has mass 1, so the expression becomes the integral of a product measure space. We want to use Fubini's theorem to interchange the order of the integral and the sums. To do this, we need to show that the integrand is L^1 on the product. In the following calculation, Equation (22) follows because the integral of $|y_1^{k_1}\cdots y_K^{k_K}|$ over any orthant equals the integral of $y_1^{k_1}\cdots y_k^{k_K}$ over the positive orthant, and there are 2^K orthants. Equation (24) follows because the component normal random variables $y_1, \ldots, y_K$ are independent, and from the formula for the k_ith moment of the absolute value of a normal random variable. In Equation (25), the integrand is greater than or equal to the integrand in Equation (23) on the set $\{y \in \mathbf{R}_+^K : |y| \leq 4r(T-t)\}$. In Equation (26), the integrand is greater than or equal to the integrand in Equation (23) on the set $\{y \in \mathbf{R}_+^K : |y| > 4r(T-t)\}$; integrating over all of $\mathbf{R}_+^K$ makes the integral still larger. In Equation (27), the second term comes from the fact that Equation (25) is the integral of a constant over $\{y \in \mathbf{R}_+^K : |y| \leq 4r(T-t)\}$ times the density of a probability distribution over $\mathbf{R}^K$; the third term comes from the formula for the k_ith moment of the absolute value of a normal random variable:

$$\left| \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \frac{y_1^{k_1}\cdots y_K^{k_K}}{k_1!\cdots k_K!} F(y)e^{-|y|^2/2(T-t)}\,dy \right|$$

$$\leq \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \frac{(r+e^{r|y|})|y_1^{k_1}\cdots y_K^{k_K}|}{k_1!\cdots k_K!} e^{-|y|^2/2(T-t)}\,dy$$

$$(22) \quad = \frac{2^K}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}_+^K} \frac{(r+e^{r|y|})y_1^{k_1}\cdots y_K^{k_K}}{k_1!\cdots k_K!} e^{-|y|^2/2(T-t)}\,dy$$

$$(23) \quad = \frac{2^K r}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}_+^K} \frac{y_1^{k_1}\cdots y_K^{k_K}}{k_1!\cdots k_K!} e^{-|y|^2/2(T-t)}\,dy$$

$$+ \frac{2^K}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}_+^K} \frac{y_1^{k_1}\cdots y_K^{k_K}e^{r|y|}}{k_1!\cdots k_K!} e^{-|y|^2/2(T-t)}\,dy$$

$$(24) \quad \leq r\frac{\sqrt{k_1!}(T-t)^{k_1/2}\cdots\sqrt{k_K!}(T-t)^{k_K/2}}{k_1!\cdots k_K!}$$

$$(25) \qquad + \frac{2^K}{(2\pi(T-t))^{K/2}}$$

$$\times \int_{\{y \in \mathbf{R}_+^K : |y| \le 4r(T-t)\}} \frac{(4(T-t))^k e^{4r(T-t)}}{k_1! \cdots k_K!} e^{-|y|^2/2(T-t)} \, dy$$

$$(26) \qquad + \frac{2^K}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}_+^K} \frac{y_1^{k_1} \cdots y_K^{k_K}}{k_1! \cdots k_K!} e^{-|y|^2/4(T-t)} \, dy$$

$$(27) \qquad \le \frac{r(T-t)^{k/2}}{\sqrt{k_1! \cdots k_K!}} + \frac{(4(T-t))^k e^{4r(T-t)}}{k_1! \cdots k_K!} + \frac{r(2(T-t))^{k/2}}{\sqrt{k_1! \cdots k_K!}}.$$

Consider the power series

$$(28) \qquad \sum_{k=0}^{\infty} \frac{1}{(T-t)^k} \sum_{k_1+\cdots+k_K=k} \beta_1^{k_1} \cdots \beta_K^{k_K} \frac{1}{(2\pi(T-t))^{K/2}}$$

$$\times \int_{\mathbf{R}^K} \frac{y_1^{k_1} \cdots y_K^{k_K}}{k_1! \cdots k_K!} F(y) e^{-|y|^2/2(T-t)} \, dy.$$

By Equation (27), the absolute value of the coefficient of $\beta_1^{k_1} \cdots \beta_K^{y_K}$ is bounded by

$$\frac{1}{(T-t)^k} \left(\frac{r(T-t)^{k/2}}{\sqrt{k_1! \cdots k_K!}} + \frac{(4(T-t))^k e^{4r(T-t)}}{k_1! \cdots k_K!} + \frac{r(2(T-t))^{k/2}}{\sqrt{k_1! \cdots k_K!}} \right)$$

$$\le \frac{3r(T-t)^{-k/2}}{\sqrt{k_1! \cdots k_K!}} + \frac{4^k e^{4r(T-t)}}{k_1! \cdots k_K!}$$

$$\le \frac{3r(T-t)^{-k/2} + 4^k e^{4r(T-t)}}{\sqrt{k_1! \cdots k_K!}}$$

$$\le \frac{(3r + e^{4r(T-t)}) \sqrt{\max\{\frac{1}{T-t}, 16\}}^k}{\sqrt{k_1! \cdots k_K!}},$$

which shows that the power series in Equation (28) converges absolutely within a positive radius of convergence, by Proposition 2.2.10 of Krantz and Parks (2002). Thus, we can apply Fubini's theorem to obtain

$$e^{-|\beta|^2/2(T-t)} \sum_{k=0}^{\infty} \frac{1}{(T-t)^k} \sum_{k_1+\cdots+k_K=k} \beta_1^{k_1} \cdots \beta_K^{k_K}$$

$$\times \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \frac{y_1^{k_1} \cdots y_K^{k_K}}{k_1! \cdots k_K!} F(y) e^{-|y|^2/2(T-t)} \, dy$$

$$= \frac{e^{-|\beta|^2/2(T-t)}}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \sum_{k=0}^{\infty} \frac{1}{(T-t)^k} \sum_{k_1+\cdots+k_K=k} \frac{(\beta_1 y_1)^{k_1}\cdots(\beta_K y_K)^{k_K}}{k_1!\cdots k_K!}$$

$$\times F(y)e^{-|y|^2/2(T-t)}\,dy$$

$$= G(t, \beta)$$

by Equation (21). Therefore, for fixed $t < T$, $G(t, \beta)$ is a product of two analytic functions, hence is an analytic function of $\beta \in \mathbf{R}^K$. For any $s \in (0, t)$, we have

$$G(s, \beta) = \frac{1}{(2\pi(t-s))^{K/2}} \int_{\mathbf{R}^K} G(t, \beta + x)e^{-|x|^2/2(t-s)}\,dx.$$

The right side is an integral with respect to x of an analytic function of (s, β, x), and hence G is analytic on $(0, t) \times \mathbf{R}^K$ by Proposition 2.2.3 of Krantz and Parks (2002). Since this is true for every $t < T$, G is analytic on $(0, T) \times \mathbf{R}^K$.   *Q.E.D.*

APPENDIX C: NONSTANDARD STOCHASTIC INTEGRATION

Up to now, all of our definitions and results have been stated without any reference to nonstandard analysis. Our proof makes extensive use of nonstandard analysis, in particular Anderson's (1976) construction of Brownian motion and the Itô integral. It is beyond the scope of this paper to develop these methods; see Anderson (2000) and Hurd and Loeb (1985) for references to nonstandard analysis.

To show that the equilibrium of the hyperfinite economy generates an equilibrium of the standard continuous-time economy, one needs to show that capital gains are the same in the two settings. Capital gains are given by Stieltjes integrals with respect to securities prices in the hyperfinite setting and by Itô integrals with respect to securities prices in the continuous-time setting.

Anderson (1976) showed that the Itô integral with respect to Brownian motion is the standard part of a Stieltjes integral with respect to a hyperfinite random walk. Anderson's theorem covers hyperfinite random walks which move independently in each component by an amount $\pm 1/\sqrt{n}$. In that random walk, each node in the tree has 2^K successor nodes. As discussed above, to obtain dynamic completeness in the hyperfinite model, we need to use a random walk in which each node has $K + 1$ successor nodes. Thus, Anderson's theorem does not cover the case considered here.

Lindstrøm (1980a, 1980b, 1980c, 1980d) showed that the stochastic integral with respect to a square integrable martingale is the standard part of a Stieltjes integral with respect to a hyperfinite SL^2 martingale. Lindstrøm's theorem is limited to one-dimensional martingales. Because the components of a vector Brownian motion are uncorrelated, a process is Itô integrable with respect to a

vector Brownian motion if and only if it is integrable with respect to each component. However, the components of a vector martingale can be correlated and, consequently, a process can be integrable with respect to a vector martingale even if it is not integrable with respect to the individual components. This fact has economic significance. If two components of the vector martingale are instantaneously nearly perfectly correlated at some point, then the equilibrium trading strategy may well require taking a nonstandard infinite long position in one security and a nonstandard infinite short position in the other. In both the hyperfinite and continuous-time models, the capital gain is well defined and finite when computed with respect to the vector martingale. However, the hyperfinite capital gain may be a positive nonstandard infinite number in one component and a negative nonstandard infinite number in the other components; they add up to a well defined finite integral when both components are considered. The continuous-time capital gain may be undefined with respect to the two components when considered separately, but well defined and finite when the integral is computed with respect to the vector martingale.

Thus, we need to extend either Anderson's theorem or Lindstrøm's theorem. The more general approach would be to extend Lindstrøm's theorem to vector martingales; such an extension is probably needed to tackle the dynamically incomplete markets case. However, it is considerably easier, and sufficient for our purposes in this paper, to extend Anderson's theorem to the particular kind of random walk considered here. This is the approach we follow.

Anderson (1976) proved that β is a standard Brownian motion provided that $\hat{\beta}$ moves up or down $\frac{1}{\sqrt{n}}$ independently in each coordinate at every node; this would require each node in the tree for $\hat{\beta}$ to have 2^K successor nodes, precluding dynamic completeness in the hyperfinite model for $K > 1$. Neither Anderson nor Keisler (1984) quite covers the random walk $\hat{\beta}$ considered here because the coordinates of $\hat{\beta}$ are uncorrelated but not independent.

For definitions of standard terms in stochastic integration (such as $\mathcal{H}^2$ and $\mathcal{L}^2$), see Nielsen (1999). For definitions of nonstandard terms such as SL^2 lifting, see Anderson (1976).

THEOREM C.1: *β is a standard Brownian motion*.

PROOF: We claim that the coordinates of β are independent. To see this, fix $x \in \mathbf{R}^K$. Then $\{x \cdot v_s : s \in \mathcal{T}\}$ is a family of independent and identically distributed (IID) random variables with standard distribution, mean zero, and finite variance σ_x, so Anderson's Theorem 21 implies that $x \cdot \beta(t, \cdot)$ is Normal mean zero variance $t\sigma_x$. Since $x \cdot \beta(t, \cdot)$ is Normal for all $x \in \mathbf{R}^K$, it is well known that $\beta(t, \cdot)$ is system Normal (see, for example, Bryc (1995, Theorem 2.2.4)). Since $\beta(t, \cdot)$ is system Normal with variance–covariance matrix tI, where I is the $K \times K$ identity matrix, the components of β are independent. Each component of the random walk $\hat{\beta}$ is a hypermartingale (martingale with respect to

the hyperfinite filtration) and so satisfies the *S*-continuity property by Keisler's continuity theorem for hypermartingales, and it follows from Anderson's proof that β is almost surely continuous. Anderson's proof that β has independent increments goes through without change in the present setting. Anderson used a slightly smaller filtration than the one considered here, while Keisler used the filtration considered here. *Q.E.D.*

THEOREM C.2: *Let $\hat{\beta}$ be the hyperfinite random walk defined above, and let $\beta = {}^{\circ}\hat{\beta}$ be the standard Brownian motion it generates. Suppose $Z \in \mathcal{H}^2$. Then there is an SL^2 lifting $\hat{Z}$ of Z. Given any SL^2 lifting $\hat{Z}$ of Z, for every $t \in \mathcal{T}$, we have*

$$^{\circ}\int_0^t \hat{Z}\,d\hat{\beta} = \int_0^{{}^{\circ}t} Z\,d\beta$$

PROOF: Lemma 31 in Anderson (1976) proves the existence of an SL^2 lifting; the hyperfinite probability space is slightly different from the one considered here, but the proof goes through without change.

Theorem 33 in Anderson (1976) shows that with respect to the hyperfinite random walk and Brownian motion considered there, the Itô integral is the standard part of the hyperfinite Stieltjes integral. The proof of Theorem 33 depends on the specific form of the hyperfinite random walk only in establishing the Itô isometry, so we show that the Itô isometry holds for the random walk $\hat{\beta}$. $\hat{Z}$ may be a 1×1 scalar process, a $1 \times K$ vector process, or a $(J+1) \times K$ matrix process. The proofs in the vector and matrix cases are virtually identical apart from notation, while the proof in the scalar case is easier, so we assume that $\hat{Z}$ is a $1 \times K$ vector process with kth component $\hat{Z}_k$,

$$\left\|\int_0^t \hat{Z}\,d\hat{\beta}\right\|_2$$

$$= \left\|\int_0^t \hat{Z}_k\,d\hat{\beta}_k\right\|_2$$

$$= \left\|\sum_{k=1}^K \sum_{s\in\mathcal{T},s<t} \hat{Z}_k(s,\cdot)(v_{s+\Delta T})_k\sqrt{\Delta T}\right\|_2$$

$$= \sum_{k=1}^K \sum_{s\in\mathcal{T},s<t} \|\hat{Z}_k(s,\cdot)(v_{s+\Delta T})_k\sqrt{\Delta T}\|_2$$

(because the terms $\hat{Z}_k(s,\omega)v_{s+\Delta T}$ are uncorrelated across s,k)

$$= \sum_{k=1}^K \sum_{s\in\mathcal{T},s<t} \Delta T\|\hat{Z}_k(s,\cdot)\|_2$$

$$= \sum_{k=1}^{K} \|\hat{Z}_k|_{\{s \in \mathcal{T}: s < t\}}\|_2$$

$$= \|\hat{Z}|_{\{s \in \mathcal{T}: s < t\}}\|_2,$$

which establishes the Itô isometry.

Since β is a standard Brownian motion by Theorem C.1, the rest of the proof goes through unchanged.   *Q.E.D.*

DEFINITION C.3: Suppose $Z \in \mathcal{L}^2$. An $\mathcal{SL}^2$ lifting of Z is an internal nonanticipating process $\hat{Z}$ such that $\hat{Z}(\cdot, \omega) \in SL^2(\mathcal{T} \setminus \{\hat{T}\})$ for almost all ω and such that $^\circ\hat{Z}(t, \omega) = Z(^\circ t, \omega)$ for almost all $(t, \omega) \in (\mathcal{T} \setminus \{\hat{T}\}) \times \Omega$.

THEOREM C.4: *Let $\hat{\beta}$ be the hyperfinite random walk defined above, and $\beta = {}^\circ\hat{\beta}$ the standard Brownian motion it generates. Suppose $Z \in \mathcal{L}^2$. Then there exists an $\mathcal{SL}^2$ lifting of Z. If $\hat{Z}$ is any $\mathcal{SL}^2$ lifting of Z, then for every $t \in \mathcal{T}$ we have*

$$^\circ \int_0^t \hat{Z} \, d\hat{\beta} = \int_0^{^\circ t} Z \, d\beta.$$

PROOF: Let $f(\omega) = \|Z(\cdot, \omega)\|_2$ and find $\bar{f}$ internal such that $^\circ\bar{f}(\omega) = f(\omega)$ $L(\hat{\mu})$-almost surely. Let $\bar{Z}$ be an internal nonanticipating process such that $^\circ\bar{Z}(t, \omega) = Z(^\circ t, \omega)$ $L(\hat{\nu}) \times L(\hat{\mu})$-almost everywhere, so for $L(\hat{\mu})$-almost all $\omega, {}^\circ\bar{Z}(\cdot, \omega) = Z(^\circ t, \omega)$ for ν-almost all $t \in \mathcal{T} \setminus \{\hat{T}\}$. For $m \in {}^*\mathbf{N}$, let

$$(\bar{Z}_m)_{ij}(t, \omega) = \max\{-m, \min\{m, \bar{Z}_{ij}(t, \omega)\}\},$$

$$\bar{f}_m(\omega) = \|\bar{Z}_m(\cdot, \omega)\|_2.$$

For all ω, we have $\bar{f}_{m+1}(\omega) \geq \bar{f}_m(\omega)$. For $L(\hat{\mu})$-almost all ω, we have $\lim_{m \in \mathbf{N}, n \to \infty} {}^\circ\bar{f}_m(\omega) = f(\omega) < \infty$. Therefore, there exists $m_0 \in \mathbf{N}$ such that for all $m \in \mathbf{N}, m \geq m_0$,

$$\hat{\mu}\left(\left\{\omega : |\bar{f}_m(\omega) - \bar{f}(\omega)| < \frac{1}{m}\right\}\right) > 1 - \frac{1}{m},$$

so we may find $m \in {}^*\mathbf{N} \setminus \mathbf{N}$ such that

$$\hat{\mu}\left(\left\{\omega : |\bar{f}_m(\omega) - \bar{f}(\omega)| < \frac{1}{m}\right\}\right) > 1 - \frac{1}{m},$$

so $^\circ\bar{f}_m(\omega) = \bar{f}(\omega)$ $L(\hat{\mu})$-almost surely; for any such ω, $\bar{Z}_m(\cdot, \omega) \in SL^2$ (Anderson (1976, Theorem 11)), so if we define $\hat{Z} = \bar{Z}_m$, $\hat{Z}$ is an $\mathcal{SL}^2$ lifting of Z.

Now, suppose $\hat{Z}$ is any $\mathcal{SL}^2$ lifting of Z. For $m \in \mathbf{N}$, define the internal stopping time

$$\hat{\tau}_m(\omega) = \max\left\{ t \in \mathcal{T} : \int_0^t \|\hat{Z}(s, \omega)\|_2^2 \, d\hat{\nu} \leq m \right\}$$

and define

$$\hat{Z}_m(t, \omega) = \begin{cases} \hat{Z}(t, \omega), & \text{if } t \leq \tau_m(\omega), \\ 0, & \text{if } t > \tau_m(\omega). \end{cases}$$

Let $Z_m(t, \omega) = {}^\circ\hat{Z}_m(\hat{t}, \omega)$. For $L(\hat{\mu})$-almost all ω, there exists $m(\omega)$ such that $\tau_{m(\omega)} = \hat{T}$, in which case $\hat{Z}_m(\cdot, \omega) = \hat{Z}(\cdot, \omega)$ and therefore

$$\int_0^t \hat{Z}_m \, d\hat{\beta} = \int_0^t \hat{Z}(\cdot, \omega) \, d\hat{\beta}.$$

By the definition of the standard stochastic integral (see, for example, page 96 of Steele (2001)) and Theorem C.2,

$$\int_0^{{}^\circ t} Z \, d\beta = \lim_{m \to \infty} \int_0^{{}^\circ t} Z_m \, d\beta$$

$$= \lim_{m \to \infty} {}^\circ \int_0^t \hat{Z}_m \, d\hat{\beta}$$

$$= {}^\circ \int_0^t \hat{Z} \, d\hat{\beta}. \qquad\qquad Q.E.D.$$

APPENDIX D: PROOFS OF THEOREMS 2.1 AND 4.2 AND PROPOSITION 2.2

We construct our probability space, filtration, and Brownian motion following Anderson's (1976) construction. Choose $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. Using this hyperfinite n, define $\mathcal{T}, \hat{t}, \hat{T}, \hat{\nu}, \hat{\mu}, \hat{\Omega}, \hat{\beta}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{U}_i$, and so on exactly as they were defined in Section 4; note that this involves perturbing endowments and dividends to ensure the existence of a Pareto optimal equilibrium, and that the measures $\hat{\nu}$ and $\hat{\mu}$ are defined by internal cardinalities. By the transfer principle, the economy has an equilibrium, and the equilibrium is Pareto optimal.

Let $(\mathcal{T}, L(\hat{\nu}))$ denote the complete Loeb measure generated by $\hat{\nu}$ on $\mathcal{T}$; note that the Loeb measure is a standard countably additive measure on the same underlying point set $\mathcal{T}$ as ν. For $B \subset [0, T]$, let $\text{st}^{-1}(B) = \{t \in \mathcal{T} : {}^\circ t \in B\}$. For any Lebesgue measurable set $B \subset [0, T]$, $\text{st}^{-1}(B)$ is Loeb measurable and $L(\hat{\nu})(\text{st}^{-1}(B)) = \nu(B)$ (Anderson (1976)).

Let $(\Omega, \mathcal{F}, L(\hat{\mu}))$ be the (complete) Loeb measure generated by $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})$ (Loeb (1975)). Although $(\Omega, \mathcal{F}, L(\hat{\mu}))$ is generated by a nonstandard construction, Loeb showed that it is a probability space in the usual standard sense.

Let

$$\mathcal{F}_t = \Big\{ B \in \mathcal{F} : L(\hat{\mu})(B \Delta C) = 0 \text{ for some } C \text{ which respects}$$

$$\text{the equivalence relation } \omega \sim \omega' \Leftrightarrow \omega \sim_s \omega' \text{ for all } s \simeq t \Big\}.$$

Let $\beta : [0, T] \times \Omega \to \mathbf{R}^K$ be defined by $\beta(t, \omega) = {}^\circ(\hat{\beta}(\hat{t}, \omega))$. β is a K-dimensional Brownian motion in the usual standard sense and $\beta(t, \cdot) = E(\beta(T, \cdot) | \mathcal{F}_t)$. Let $\mathcal{I} : [0, T] \times \Omega \to \mathbf{R}^{K+1}$ be defined by

$$\mathcal{I}(t, \omega) = (t, \beta(t, \omega)).$$

Since the equilibrium is Pareto optimal, the marginal utility of consumption is infinite at zero, the aggregate consumption is strictly positive at each node, and all agents have strictly positive income by Equation (4), the equilibrium consumptions of all agents are strictly positive at each node; they may be infinitesimal. Let Δ be the open $(I-1)$-dimensional simplex in $\mathbf{R}^I_{++}$. Pareto optimality and strict positivity of the equilibrium consumptions imply that there exists $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_I) \in {}^*\Delta$ such that at each node (t, ω), the consumptions maximize

$$\sum_{i=1}^{I} \hat{\lambda}_i {}^* h_i(\hat{c}_i(t, \omega), \hat{\mathcal{I}}(t, \omega)) \quad \text{and} \quad \sum_{i=1}^{I} \hat{\lambda}_i {}^* H_i(\hat{c}_i(\hat{T}, \omega), \hat{\mathcal{I}}(\hat{T}, \omega)).$$

Since each $\hat{c}_i(t, \omega) > 0$, each $\hat{\lambda}_i > 0$, so there is a positive constant $\hat{\iota}(t, \omega)$ such that

$$\hat{\lambda}_1 {}^* \frac{\partial h_1}{\partial c_1}\bigg|_{(\hat{c}_1(t, \omega), \hat{\mathcal{I}}(t, \omega))} = \cdots = \hat{\lambda}_I {}^* \frac{\partial h_I}{\partial c_I}\bigg|_{(\hat{c}_I(t, \omega), \hat{\mathcal{I}}(t, \omega))} = \hat{\iota}(t, \omega)$$

$$\text{for} \quad t < \hat{T},$$

$$\hat{\lambda}_1 {}^* \frac{\partial H_1}{\partial c_1}\bigg|_{(\hat{c}_1(\hat{T}, \omega), \hat{\mathcal{I}}(\hat{T}, \omega))} = \cdots = \hat{\lambda}_I {}^* \frac{\partial H_I}{\partial c_I}\bigg|_{(\hat{c}_I(\hat{T}, \omega), \hat{\mathcal{I}}(\hat{T}, \omega))} = \hat{\iota}(\hat{T}, \omega).$$

Let $\hat{c}(t, \omega) = \sum_{i=1}^{I} \hat{c}_i(t, \omega)$.

We claim that $\hat{\iota}(\cdot, \cdot) \in SL^p$ for each $p \in [1, \infty)$, and that $\hat{\iota}(t, \omega) \not\simeq 0$ for Loeb almost all (t, ω). For every node (t, ω) there is an agent $i(t, \omega)$ such that $\hat{c}_{i(t, \omega)}(t, \omega) \geq \frac{\hat{c}(t, \omega)}{I}$. The following calculation assumes $t < \hat{T}$; if $t = \hat{T}$, substitute H_i for h_i. Since $\hat{\lambda}_i \leq 1$ for each i,

$$\hat{\iota}(t, \omega) = \hat{\lambda}_{i(t, \omega)} {}^* \frac{\partial h_{i(t, \omega)}}{\partial c_{i(t, \omega)}}\bigg|_{(\hat{c}_{i(t, \omega)}(t, \omega), \hat{\mathcal{I}}(t, \omega))}$$

$$\leq {}^* \frac{\partial h_{i(t, \omega)}}{\partial c_{i(t, \omega)}}\bigg|_{((\hat{c}(t, \omega))/I, \hat{\mathcal{I}}(t, \omega))}$$

$$\leq {}^*\frac{\partial h_{i(t,\omega)}}{\partial c_{i(t,\omega)}}\Bigg|_{((\sum_{i=1}^I {}^*f_i(\hat{\mathcal{I}}(t,\omega))+\sum_{j=0}^J \eta_j {}^*g_j(\hat{\mathcal{I}}(t,\omega)))/I,\hat{\mathcal{I}}(t,\omega))}$$

$$\leq r + e^{r|\hat{\beta}(t,\omega)|}$$

by inequality (2). Therefore, $(\hat{\iota}(t,\omega))^p \leq \bar{r} + e^{\bar{r}|\hat{\beta}(t,\omega)|}$ for some $\bar{r} \in \mathbf{R}$, so $(\hat{\iota})^p \in SL^1$ by Raimondo (2002, 2005, Proposition 3.1), so $\hat{\iota} \in SL^p$. Since $\sum_{i=1}^I \hat{\lambda}_i = 1$, we can assume without loss of generality that $\lambda_1 \not\simeq 0$. Then

$$\hat{\iota}(t,\omega) = \hat{\lambda}_1 {}^*\frac{\partial h_i}{\partial c_i}\Bigg|_{(\hat{c}_i(t,\omega),\hat{\mathcal{I}}(t,\omega))}$$

$$\geq \hat{\lambda}_1 {}^*\frac{\partial h_i}{\partial c_i}\Bigg|_{(\hat{c}(t,\omega),\hat{\mathcal{I}}(t,\omega))}$$

$$\geq \hat{\lambda}_1 {}^*\frac{\partial h_i}{\partial c_i}\Bigg|_{(\sum_{i=1}^I {}^*f_i(\hat{\mathcal{I}}(t,\omega))+\sum_{j=0}^J \eta_j {}^*g_j(\mathcal{I}(t,\omega)+O((\Delta T)^2)))}$$

$$\not\simeq 0$$

whenever the social consumption is finite, that is, for Loeb almost every (t,ω). If we set $\hat{p}_C(t,\omega) = \hat{\iota}(t,\omega)$ for $t \in \mathcal{T}$, $\hat{p}_C$ gives the Arrow–Debreu prices of consumption.

We claim that $\hat{\lambda}_i \not\simeq 0$ for each $i = 1,\ldots,I$. If not, we may assume without loss of generality that $\hat{\lambda}_I \simeq 0$. For every node for which $\hat{\iota}(t,\omega) \not\simeq 0$, we have

$${}^*\frac{\partial h_I}{\partial c_I}\Bigg|_{(\hat{c}_I(t,\omega),\hat{I}(t,\omega))} = \frac{\hat{\iota}(t,\omega)}{\hat{\lambda}_I} \simeq \infty \quad \text{or}$$

$${}^*\frac{\partial H_I}{\partial c_I}\Bigg|_{(\hat{c}_I(\hat{T},\omega),\hat{I}(\hat{T},\omega))} = \frac{\hat{\iota}(\hat{T},\omega)}{\hat{\lambda}_I} \simeq \infty$$

as appropriate, so $\hat{c}_I(t,\omega) \simeq 0$, which shows that $\hat{c}_I(t,\omega) \simeq 0$ Loeb almost everywhere. Since $\hat{c}_I(t,\omega) \leq \hat{c}(t,\omega) \leq r + e^{r|\hat{\beta}(t,\omega)|} + O((\Delta T)^2)$, then as above, $\hat{c}_I \in SL^p$ for all $p \in [1,\infty)$ by Raimondo (2002, 2005, Proposition 3.1), and hence $\hat{p}_C \times \hat{c}_I \in SL^p$ for all $p \in [1,\infty)$ by the Cauchy–Schwarz inequality and Anderson (1976);

$$^{\circ}E\left(\hat{p}_C(\hat{T},\omega)\hat{c}_I(\hat{T},\omega) + \Delta T \sum_{t=0}^{\hat{T}-\Delta T} \hat{p}_c(t,\omega)\hat{c}_I(t,\omega)\right)$$

$$= E\left({}^{\circ}\hat{p}_C(\hat{T},\omega){}^{\circ}\hat{c}_I(\hat{T},\omega) + \int_0^T {}^{\circ}\hat{p}_c(t,\omega){}^{\circ}\hat{c}_I(t,\omega)\,dt\right)$$

$$= 0.$$

However, since the hyperfinite market is dynamically complete, agent I can afford any bundle satisfying the Arrow–Debreu budget constraint. In the following calculation, Equation (29) follows because $\hat{e}_I(t, \omega) \leq r + e^{r|\hat{\beta}(t,\omega)|} + O((\Delta T)^2)$ and $\hat{A}_j(t, \omega) \leq r + e^{r|\hat{\beta}(t,\omega)|} + O((\Delta T)^2)$, so $\hat{e}_I, \hat{A} \in SL^p$ for all $p \in [1, \infty)$, so $\hat{p}_c \hat{e}_I, \hat{p}_c \hat{A} \in SL^p$ for all $p \in [1, \infty)$. Agent I's income is

$$\hat{e}_{IA} \cdot \hat{p}_A(0, 0) + E\left(\hat{p}_C(\hat{T}, \cdot)\hat{e}_I(\hat{T}, \cdot) + \Delta T \sum_{t=0}^{\hat{T}-\Delta T} \hat{p}_C(t, \cdot)\hat{e}_I(t, \cdot)\right)$$

$$= E\left(\hat{p}_C(\hat{T}, \cdot)(\hat{e}_{IA}A(\hat{T}, \cdot) + \hat{e}_I(\hat{T}, \cdot))\right.$$

$$\left. + \Delta T \sum_{t=0}^{\hat{T}-\Delta T} \hat{p}_C(t, \cdot)(\hat{e}_{IA}A(t, \cdot) + \hat{e}_I(t, \cdot))\right)$$

$$(29) \qquad \simeq E\left({}^\circ\hat{p}_C(\hat{T}, \cdot)(e_{IA}A(\hat{T}, \cdot) + e_I(\hat{T}, \cdot))\right.$$

$$\left. + \int_0^T {}^\circ p_C(t, \cdot)(e_{IA}A(t, \cdot) + e_I(t, \cdot))\right)$$

$$\neq 0$$

since the expectation is of a function which is nonnegative and strictly positive on a set of positive measure by inequality (4). Thus, $\hat{c}_I$ lies strictly inside the Arrow–Debreu budget set; since preferences are strictly monotonic, it cannot be the demand of agent I, a contradiction which shows that $\hat{\lambda}_i \not\simeq 0$ for each $i = 1, \ldots, I$.

We now show that the equilibrium consumptions and consumption prices are given by standard analytic functions. Consider the standard analytic function

$$\rho : (0, \infty)^I \times \mathbf{R}_{++} \times \Delta \times \mathbf{R}_{++} \times ((0, T) \times \mathbf{R}^K) \to \mathbf{R}^{I+1}$$

defined by

$$\rho((c_1, \ldots, c_I), \iota, \lambda, c, (t, \beta))$$

$$= \left(\lambda_1 \frac{\partial h_1}{\partial c_1}\bigg|_{(c_1, t, \beta)} - \iota, \ldots, \lambda_I \frac{\partial h_I}{\partial c_I}\bigg|_{(c_I, t, \beta)} - \iota, c_1 + \cdots + c_I - c\right).$$

ρ encodes the first-order conditions for the equilibrium consumptions. The determinant of the Jacobian matrix of partial derivatives of ρ with respect to

$((c_1, \ldots, c_I), \iota)$ is given by

$$\det\left(\frac{\partial \rho}{\partial((c_1, \ldots, c_I,), \iota)}\right)$$

$$= \det\begin{pmatrix} \lambda_1 \frac{\partial^2 h_1}{\partial c_1^2} & 0 & 0 & \cdots & 0 & -1 \\ 0 & \lambda_2 \frac{\partial^2 h_2}{\partial c_2^2} & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_I \frac{\partial^2 h_I}{\partial c_I^2} & -1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}$$

$$= \det\begin{pmatrix} \lambda_1 \frac{\partial^2 h_1}{\partial c_1^2} & 0 & 0 & \cdots & 0 & -1 \\ 0 & \lambda_2 \frac{\partial^2 h_2}{\partial c_2^2} & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_I \frac{\partial^2 h_I}{\partial c_I^2} & -1 \\ 0 & 0 & 0 & \cdots & 0 & \sum_{i=1}^{I} \frac{1}{\lambda_i \frac{\partial^2 h_i}{\partial c_i^2}} \end{pmatrix}$$

$$= \lambda_1 \cdots \lambda_I \frac{\partial^2 h_1}{\partial c_1^2} \cdots \frac{\partial^2 h_I}{\partial c_I^2} \sum_{i=1}^{I} \left(1 / \left(\lambda_i \frac{\partial^2 h_i}{\partial c_i^2}\right)\right)$$

$$\neq 0$$

since $\lambda \in \Delta$ and $\frac{\partial^2 h_i}{\partial c_i^2} < 0$. By the analytic implicit function theorem (Theorem B.2 in Appendix B) and the transfer principle, there exist standard analytic functions $\hat{\pi}, \hat{\psi}_i : \Delta \times \mathbf{R}_{++} \times ((0, T) \times \mathbf{R}^K) \to \mathbf{R}$ for $i = 1, \ldots, I$ such that

$$\hat{\iota}(t, \omega) = {}^*\hat{\pi}(\hat{\lambda}, \hat{c}(t, \omega), \hat{\mathcal{I}}(t, \omega)) \quad \text{for} \quad t < \hat{T},$$
$$\hat{c}_i(t, \omega) = {}^*\hat{\psi}_i(\hat{\lambda}, \hat{c}(t, \omega), \hat{\mathcal{I}}(t, \omega)) \quad \text{for} \quad t < \hat{T}.$$

Using a similar calculation at $t = \hat{T}$ and the ordinary implicit function theorem, by Equation (3), there are standard C^1 functions $\hat{\Pi}, \hat{\Psi}_i : \Delta \times \mathbf{R}_{++} \times (\{T\} \times \mathbf{R}^K) \to \mathbf{R}$ such that

$$\hat{\iota}(\hat{T}, \omega) = {}^*\hat{\Pi}(\hat{\lambda}, \hat{c}(\hat{T}, \omega), \hat{\mathcal{I}}(\hat{T}, \omega)),$$
$$\hat{c}_i(\hat{T}, \omega) = {}^*\hat{\Psi}_i(\hat{\lambda}, \hat{c}(\hat{T}, \omega), \hat{\mathcal{I}}(\hat{T}, \omega)).$$

Let $\lambda \in \Delta$, $\pi : \mathbf{R}_{++} \times \mathbf{R}^K \times [0, T) \to \mathbf{R}$, $\Pi : \mathbf{R}_{++} \times \mathbf{R}^K \to \mathbf{R}$, $\psi_i : \mathbf{R}_{++} \times \mathbf{R}^K \times [0, T) \to \mathbf{R}$, and $\Psi_i : \mathbf{R}_{++} \times \mathbf{R}^K \to \mathbf{R}$ be defined by

$$\lambda = {}^\circ \hat{\lambda},$$

$$\pi(c, \mathcal{I}) = \hat{\pi}(\lambda, c, \mathcal{I}),$$

$$\Pi(c, \beta) = \hat{\Pi}(\lambda, c, \beta),$$

$$\psi_i(c, \mathcal{I}) = \hat{\psi}_i(\lambda, c, \mathcal{I}),$$

$$\Psi_i(c, \beta) = \hat{\Psi}_i(\lambda, c, \beta).$$

Because $\hat{\pi}$ and $\hat{\psi}$ are standard analytic functions, π and Ψ are standard analytic functions; because $\hat{\Pi}$ and $\hat{\Psi}$ are standard C^1 functions, Π and Ψ are standard C^1 functions. Let $c(t, \omega) = {}^\circ \hat{c}(t, \omega)$ and $c_i(t, \omega) = {}^\circ \hat{c}_i(t, \omega)$. Since $c(t, \omega)$ and $\psi_i(c, \beta, t)$ are analytic functions for $t \in (0, T)$, each $c_i(t, \omega)$ is an analytic function of $\mathcal{I}(t, \omega)$. Each $c_i(T, \omega)$ is a C^1 function of $c(T, \omega)$; however, since $c(T, \omega)$ is a continuous function of $\mathcal{I}(T, \omega)$ almost everywhere, $c_i(T, \omega)$ is a continuous function of $\mathcal{I}(T, \omega)$ almost everywhere.

Let

$$p_C(t, \omega) = {}^\circ \hat{p}_C(t, \omega),$$

then

$$\begin{aligned}
p_C(t, \omega) &= {}^\circ \hat{p}_C(t, \omega) \\
&= {}^\circ \big({}^* \hat{\pi}(\hat{\lambda}, \hat{c}(t, \omega), \hat{\mathcal{I}}(t, \omega)) \big) \\
&= \hat{\pi}({}^\circ \hat{\lambda}, {}^\circ \hat{c}(t, \omega), {}^\circ \hat{\mathcal{I}}(t, \omega)) \\
&= \hat{\pi}(\lambda, c(t, \omega), \mathcal{I}(t, \omega)) \\
&= \pi(c(t, \omega), \mathcal{I}(t, \omega))
\end{aligned}$$

is an analytic function of $\mathcal{I}(t, \omega)$ on $(0, T) \times \mathbf{R}^K$, since π is analytic and $c(t, \omega)$ is an analytic function of $\mathcal{I}(t, \omega)$. Since $c(T, \omega) = \Pi(c(T, \omega), \beta(T, \omega))$ is a C^1 function of $\beta(T, \omega) \in V$, $p_C(T, \cdot)$ is a C^1 function of $\beta(T, \omega) \in V$.

In the following calculation, Equation (30) holds because the first-order conditions for demand imply the price of a security is the expected value of future dividends times the weighted marginal utility of consumption, which equals the expected value of the dividend times the price of consumption. Equation (31) follows because $\hat{p}_A(\hat{T}, \cdot) = \hat{p}_C(\hat{T}, \omega) A(\hat{T}, \omega)$, $\hat{p}_C \in SL^p$ for all $p \in [1, \infty)$, and $A_j(t, \omega) \le r + e^{r|\hat{\beta}(t, \omega)|} + O((\Delta T)^2)$, so as above $\hat{p}_A(\hat{T}, \cdot) \in SL^p$ for all $p \in [1, \infty)$, by Raimondo (2002, 2005 Proposition 3.1), so $\hat{p}_C(\cdot, \cdot) \hat{A}(\cdot, \cdot) \in SL^p$ for all $p \in [1, \infty)$ (Anderson (1976)); and the internal integrands are S-continuous functions of $\hat{\mathcal{I}}(t, \omega)$ whenever ${}^\circ \hat{\mathcal{I}}(t, \omega)$ is at a point at which c and

G are continuous. Equation (32) follows because $\hat{\mathcal{I}}(\cdot, \omega)$ is almost surely S-continuous, and for any such ω and any $s \in [{}^{\circ}t, T]$, the conditional distribution of $\mathcal{I}(s, \omega)$ is the same given $({}^{\circ}t, \omega)$ as it is given (t, ω).

$$
{}^{\circ}\hat{p}_A(t, \omega)
$$

$$
= {}^{\circ}E\left(\left(\hat{p}_C(\hat{T}, \cdot)\hat{A}(\hat{T}, \cdot) + \Delta T \sum_{s=t+\Delta T}^{\hat{T}-\Delta T} \hat{p}_C(s, \cdot)\hat{A}(s, \cdot)\right)\bigg|(t, \omega)\right)
$$

$$
(30) \qquad = {}^{\circ}E\left({}^{*}\hat{\Pi}(\hat{\lambda}, \hat{c}(\hat{T}, \cdot), \hat{\mathcal{I}}(\hat{T}, \cdot))\hat{A}(\hat{T}, \cdot)|(t, \omega)\right)
$$

$$
+ {}^{\circ}E\left(\Delta T \sum_{s=t+\Delta T}^{\hat{T}-\Delta T} \left({}^{*}\hat{\pi}(\hat{\lambda}, \hat{c}(s, \cdot), \hat{\mathcal{I}}(s, \cdot))\hat{A}(s, \cdot)\right)\bigg|(t, \omega)\right)
$$

$$
= {}^{\circ}E\left({}^{*}\hat{\Pi}(\hat{\lambda}, \hat{c}(\hat{T}, \cdot), \hat{\mathcal{I}}(\hat{T}, \cdot)){}^{*}G(\hat{\mathcal{I}}(\hat{T}, \cdot)) + O((\Delta T)^2)|(t, \omega)\right)
$$

$$
+ {}^{\circ}E\left(\Delta T \sum_{s=t+\Delta T}^{\hat{T}-\Delta T} {}^{*}\hat{\pi}(\hat{\lambda}, \hat{c}(s, \cdot), \hat{\mathcal{I}}(s, \cdot))\right.
$$

$$
\left. \times \left({}^{*}g(\hat{\mathcal{I}}(s, \cdot)) + O((\Delta T)^2)\right)\bigg|(t, \omega)\right)
$$

$$
(31) \qquad = E\left(\Pi(c(T, \cdot), \mathcal{I}(T, \cdot))G(\mathcal{I}(T, \cdot))|(t, \omega)\right)
$$

$$
+ E\left(\int_{{}^{\circ}t}^{T} \pi(c(s, \cdot), \mathcal{I}(s, \cdot))g(\mathcal{I}(s, \cdot))\, ds\bigg|(t, \omega)\right)
$$

$$
(32) \qquad = E\left(\Pi(c(T, \cdot), \mathcal{I}(T, \cdot))G(\mathcal{I}(T, \cdot))\right.
$$

$$
\left. + \int_{{}^{\circ}t}^{T} \pi(c(s, \cdot), \mathcal{I}(s, \cdot))\left(g(\mathcal{I}(s, \cdot))\, ds\right)\bigg|({}^{\circ}t, \omega)\right)
$$

Let

$$
p_A(t, \omega) = E\left(\Pi(c(T, \cdot), \mathcal{I}(T, \cdot))G(\mathcal{I}(T, \cdot))\right.
$$

$$
\left. + \int_{{}^{\circ}t}^{T} \pi(c(s, \cdot), \mathcal{I}(s, \cdot))\left(g(\mathcal{I}(s, \cdot))\, ds\right)\bigg|(t, \omega)\right)
$$

We have shown that ${}^{\circ}\hat{p}_A(t, \omega)$ is an S-continuous function of $(t, \hat{\beta}(t, \omega)) \in \text{st}^{-1}([0, T) \times \mathbf{R}^K)$ and that ${}^{\circ}\hat{p}_A(t, \omega) = p_A(t, \omega)$ whenever the path $\hat{\beta}(\cdot, \omega)$ is S-continuous, that is, Loeb-almost everywhere. For almost all ω, $\hat{p}_A(\cdot, \omega)$ is an S-continuous function of $t \in \mathcal{T}$. Moreover, p_A is an analytic function of

$(t, \beta(t, \omega)) \in (0, T) \times \mathbf{R}^K$ by Theorem B.4 and Proposition 2.2.3 of Krantz and Parks (2002).

In the following calculation of $\hat{\gamma}(t + \Delta T, \omega) - \hat{\gamma}(t, \omega)$, note that we do not calculate how the future dividend *values* are affected by the information revealed by $\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)$. Instead, we calculate how the *probability* of each possible future dividend is affected by the information revealed by $\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega) = \sqrt{\Delta T} v_{\omega(t+\Delta T)}$. The calculation uses the fact that the distribution of the random walk is multinomial. In order to know the value of $\hat{\beta}(s, \omega) - \hat{\beta}(t, \omega)$, it is enough to count how many of the $\frac{s-t}{\Delta T}$ steps of the random walk in this time interval lie in each of the possible directions $v_0, \ldots, v_K$; it does not depend on the order in which the steps occur. In Equation (33), we write $\hat{p}_C \hat{A}(s, \hat{\beta}(t, \omega) + \sum_{i=0}^{K} k_i v_i \sqrt{\Delta T})$ as an abbreviation for

$$\hat{p}_C\left(s, \hat{\beta}(t, \omega) + \sum_{i=0}^{K} k_i v_i \sqrt{\Delta T}\right) \hat{A}\left(s, \hat{\beta}(t, \omega) + \sum_{i=0}^{K} k_i v_i \sqrt{\Delta T}\right).$$

Equation (34) holds because for each $s > t, s \in \mathcal{T}$, including $s = \hat{T}$, and for $k_0, \ldots, k_K \in {}^*(\mathbf{N} \cup \{0\})$ satisfying $k_0 + \cdots + k_K = (s - t)/\Delta T$,

$$\frac{\sqrt{\Delta T}\left(\sum_{i=0}^{K} k_i v_i^\top \sqrt{\Delta T}\right) v_{\omega(t+\Delta T)}}{s - t}$$

$$= \frac{\Delta T}{s - t} \sum_{i=0}^{K} k_i v_i \cdot v_{\omega(t+\Delta T)}$$

$$= \frac{\Delta T}{s - t}\left(K k_{\omega(t+\Delta T)} + (-1) \sum_{i \neq \omega(t+\Delta T)} k_i\right)$$

$$= \frac{\Delta T}{s - t}\left(K k_{\omega(t+\Delta T)} - \left(\frac{s - t}{\Delta T} - k_{\omega(t+\Delta T)}\right)\right)$$

$$= \frac{\Delta T}{s - t}\left((K + 1) k_{\omega(t+\Delta T)} - \frac{s - t}{\Delta T}\right)$$

$$= \frac{(K + 1) k_{\omega(t+\Delta T)}}{(s - t)/\Delta T} - 1.$$

Equation (36) follows from the inequalities $\hat{A}(t, \omega) \leq r + e^{r|\hat{\beta}(t, \omega)|} + O((\Delta T)^2)$ and $|\hat{A}\hat{\beta}| \leq |\hat{\beta}|(r + e^{r|\hat{\beta}|}) \leq (r|\hat{\beta}| + e^{(r+1)|\hat{\beta}|}) \leq 2e^{(r+1)|\hat{\beta}|}$, so as above $\hat{A}, \hat{A}\hat{\beta} \in SL^p$ for all $p \in [1, \infty)$ by Raimondo (2002, 2005 Proposition 3.1), and we previously showed $\hat{p}_C \in SL^p$ for all $p \in [0, \infty)$; it follows that $\hat{p}_A(\hat{T}, \cdot) = \hat{p}_C(\hat{T}, \cdot)\hat{A}(\hat{T}, \cdot) \in SL^p$ and $\hat{p}_C \hat{A}\hat{\beta} \in SL^p$ for all $p \in [1, \infty)$ (Anderson (1976)); and the internal integrands are S-continuous functions of $\hat{\mathcal{I}}(t, \omega)$ whenever

$^{\circ}\hat{\mathcal{I}}(t, \omega)$ is at a point at which c and G are continuous. Equation (37) follows because $\hat{\mathcal{I}}(\cdot, \omega)$ is almost surely S-continuous, and for any such ω and any $s \in [^{\circ}t, T]$, the conditional distribution of $\mathcal{I}(s, \omega)$ is the same given $(^{\circ}t, \omega)$ as it is given (t, ω):

$$
\begin{aligned}
&\hat{\gamma}(t + \Delta T, \omega) - \hat{\gamma}(t, \omega) \\
&= \hat{p}_A(t + \Delta T, \omega) + (\Delta T) \sum_{s \in \mathcal{T}, s \le t + \Delta T} \hat{p}_C(s, \omega) \hat{A}(s, \omega) \\
&\quad - \left(\hat{p}_A(t, \omega) + (\Delta T) \sum_{s \in \mathcal{T}, s \le t} \hat{p}_C(s, \omega) \hat{A}(s, \omega) \right) \\
&= \hat{p}_A(t + \Delta T, \omega) + \hat{p}_c(t + \Delta T, \omega) \hat{A}(t + \Delta T, \omega) - \hat{p}_A(t, \omega) \\
&= E\big((\hat{p}_A(\hat{T}, \cdot))|(t + \Delta T, \omega)\big) - E\big((\hat{p}_A(\hat{T}, \cdot))|(t, \omega)\big) \\
&\quad + \Delta T \sum_{s = t + \Delta T}^{\hat{T} - \Delta T} \big(E\big((\hat{p}_C(s, \cdot) \hat{A}(s, \cdot))|(t + \Delta T, \omega)\big) \\
&\quad - E\big((\hat{p}_C(s, \cdot) \hat{A}(s, \cdot))|(t, \omega)\big)\big) \\
&= \frac{1}{(K+1)^{(\hat{T}-t)/\Delta T}} \sum_{k_0 + \cdots + k_K = (\hat{T}-t)/\Delta T} \hat{p}_A\left(\hat{T}, \hat{\beta}(t, \omega) + \sum_{i=0}^{K} k_i v_i \sqrt{\Delta T} \right) \\
&\quad \times \left(\frac{(K+1)(((\hat{T}-t)/\Delta T) - 1)!}{k_0! k_1! (k_{\omega(t+\Delta T)} - 1)! \cdots k_K!} - \frac{((\hat{T}-t)/\Delta T)!}{k_0! k_1! \cdots k_K!} \right) \\
&\quad + \Delta T \sum_{s = t + \Delta T}^{\hat{T} - \Delta T} \frac{1}{(K+1)^{(s-t)/\Delta T}} \\
(33) \quad &\quad \times \sum_{k_0 + \cdots + k_K = (s-t)/\Delta T} \hat{p}_C \hat{A}\left(s, \hat{\beta}(t, \omega) + \sum_{i=0}^{K} k_i v_i \sqrt{\Delta T} \right) \\
&\quad \times \left(\frac{(K+1)(((s-t)/\Delta T) - 1)!}{k_0! k_1! (k_{\omega(t+\Delta T)} - 1)! \cdots k_K!} - \frac{((s-t)/\Delta T)!}{k_0! k_1! \cdots k_K!} \right) \\
&= \frac{1}{(K+1)^{(\hat{T}-t)/\Delta T}} \sum_{k_0 + \cdots + k_K = (\hat{T}-t)/\Delta T} \hat{p}_A\left(\hat{T}, \hat{\beta}(t, \omega) + \sum_{i=0}^{K} k_i v_i \sqrt{\Delta T} \right) \\
&\quad \times \left(\frac{((\hat{T}-t)/\Delta T)!}{k_0! k_1! \cdots k_K!} \right) \left(\frac{(K+1) k_{\omega(t+\Delta T)}}{((\hat{T}-t)/\Delta T)} - 1 \right)
\end{aligned}
$$

$$+ \Delta T \sum_{s=t+\Delta T}^{\hat{T}-\Delta T} \frac{1}{(K+1)^{(s-t)/\Delta T}}$$

$$\times \sum_{k_0+\cdots+k_K=(s-t)/\Delta T} \hat{p}_C \hat{A}\left(s, \hat{\beta}(t,\omega) + \sum_{i=0}^{K} k_i v_i \sqrt{\Delta T}\right)$$

$$\times \left(\frac{((s-t)/\Delta T)!}{k_0! k_1! \cdots k_K!}\right)\left(\frac{(K+1)k_{\omega(t+\Delta T)}}{((s-t)/\Delta T)} - 1\right)$$

$$= \frac{1}{(K+1)^{(\hat{T}-t)/\Delta T}}$$

$$\times \sum_{k_0+\cdots+k_K=(\hat{T}-t)/\Delta T} \hat{p}_A\left(\hat{T}, \hat{\beta}(t,\omega) + \sum_{i=0}^{K} k_i v_i \sqrt{\Delta T}\right)$$

$$\times \left(\frac{((\hat{T}-t)/\Delta T)!}{k_0! k_1! \cdots k_K!}\right)\left(\frac{\sqrt{\Delta T}\left(\sum_{i=0}^{K} k_i v_i \sqrt{\Delta T}\right) \cdot v_{\omega(t+\Delta T)}}{\hat{T}-t}\right)$$

$$+ \Delta T \sum_{s=t+\Delta T}^{\hat{T}-\Delta T} \frac{1}{(K+1)^{(s-t)/\Delta T}}$$

$$\times \sum_{k_0+\cdots+k_K=(s-t)/\Delta T} \left(\hat{p}_C \hat{A}\left(s, \hat{\beta}(t,\omega) + \sum_{i=0}^{K} k_i v_i \sqrt{\Delta T}\right)\right.$$

$$(34) \qquad \left.\times \left(\frac{((s-t)/\Delta T)!}{k_0! k_1! \cdots k_K!}\right)\left(\frac{\sqrt{\Delta T}\left(\sum_{i=0}^{K} k_i v_i \sqrt{\Delta T}\right) \cdot v_{\omega(t+\Delta T)}}{s-t}\right)\right)$$

$$(35) \qquad = E\left(\frac{\hat{p}_A(\hat{T}, \hat{\beta}(\hat{T},\cdot))(\hat{\beta}(\hat{T},\cdot) - \hat{\beta}(t,\cdot))^{\top}}{\hat{T}-t}\right.$$

$$\left.+ \frac{\Delta T \sum_{s=t+\Delta T}^{\hat{T}-\Delta T} \hat{p}_C \hat{A}(s, \hat{\beta}(s,\cdot))(\hat{\beta}(s,\cdot) - \hat{\beta}(t,\cdot))^{\top}}{s-t}\right|(t,\omega)\right)$$

$$\times v_{\omega(t+\Delta T)}\sqrt{\Delta T}$$

$$(36) \qquad = E\left(\frac{p_A(T, \beta(T,\cdot))(\beta(T,\cdot) - \beta(t,\cdot))^{\top}}{T-t}\right.$$

$$\left.+ \int_{\circ t}^{T} \frac{p_C(s, \beta(s,\cdot)) A(s, \beta(s,\cdot))(\beta(s,\cdot) - \beta(t,\cdot))^{\top}}{s - {}^{\circ}t} \, ds\right|(t,\omega)\right)$$

$$\times (\hat{\beta}(t+\Delta T, \omega) - \hat{\beta}(t,\omega)) + o(\sqrt{\Delta T}),$$

$$\hat{\sigma}(t, \omega)$$

$$\simeq E\left(\frac{p_A(T, \beta(T, \cdot))(\beta(T, \cdot) - \beta(t, \cdot))^\top}{T - t}\right.$$

$$\left. + \int_{\circ t}^{T} \frac{p_C(s, \beta(s, \cdot)) A(s, \beta(s, \cdot))(\beta(s, \cdot) - \beta(t, \cdot))^\top}{s - {}^\circ t} \, ds \,\middle|\, (t, \omega)\right)$$

$$(37) \qquad = E\left(\frac{p_C(T, \beta(T, \cdot)) A(T, \beta(T, \cdot))(\beta(T, \cdot) - \beta(t, \cdot))^\top}{T - t}\right.$$

$$\left. + \int_{\circ t}^{T} \frac{p_C(s, \beta(s, \cdot)) A(s, \beta(s, \cdot))(\beta(s, \cdot) - \beta(t, \cdot))^\top}{s - {}^\circ t} \, ds \,\middle|\, ({}^\circ t, \omega)\right).$$

This shows that $\hat{\sigma}$ is an S-continuous function of $(t, \hat{\beta}) \in \mathrm{st}^{-1}([0, T) \times \mathbf{R}^K)$. Note that the integrand is the product of the $(K + 1) \times 1$ column vector A and the $1 \times K$ row vector $\beta^\top$, so it is a $(K + 1) \times K$ matrix. Letting $\Sigma : [0, T) \times \mathbf{R}^K \to \mathbf{R}^{(K+1) \times K}$ be given by

$$\Sigma(t, \beta) = E\left(\frac{p_C(T, \beta(T)) A(T, \beta(T))(\beta(T) - \beta(t))^\top}{T - t}\right.$$

$$\left. + \int_{t}^{T} \frac{p_C(s, \beta(s)) A(s, \beta(s))(\beta(s) - \beta(t))^\top}{s - t} \, ds \,\middle|\, \beta(t) = \beta\right)$$

we have

$${}^\circ \hat{\sigma}(t, \omega) = \Sigma(\mathcal{I}(t, \omega))$$

for every (t, ω) such that $\hat{\mathcal{I}}(t, \omega)$ is finite. Since $\hat{\sigma}$ is SL^2 and adapted to $\{\hat{\mathcal{F}}_t\}$, and is an SL^2 lifting of $\Sigma \circ \mathcal{I}$, $\sigma \circ \mathcal{I}$ is L^2 and adapted to $\{\mathcal{F}_t\}$ (Anderson (1976)). Therefore, by Theorem C.2, we have

$$p_A(t, \omega) = {}^\circ \hat{p}_A(\hat{t}, \omega)$$

$$= {}^\circ \hat{p}_A(0, \omega) + {}^\circ \sum_{s=0}^{\hat{t} - \Delta T} \left(\hat{\sigma}(s, \omega)(\hat{\beta}(s + \Delta T, \omega) - \hat{\beta}(s, \omega))\right.$$

$$\left. - (\Delta T) \hat{p}_C(s, \omega) \hat{A}(s, \omega)\right)$$

$$= p_A(0, \omega) + \int_0^t \Sigma(\mathcal{I}(s, \omega)) \, d\beta(s, \omega)$$

$$- \int_0^t p_C(s, \omega) A(s, \omega) \, ds,$$

$$\gamma(t, \omega) = {}^\circ \hat{\gamma}(\hat{t}, \omega)$$

$$= {}^\circ \hat{p}_A(0, \omega) + {}^\circ \sum_{s=0}^{\hat{t}-\Delta T} \hat{\sigma}(s, \omega)(\hat{\beta}(s + \Delta T, \omega) - \hat{\beta}(s, \omega))$$

$$= p_A(0, \omega) + \int_0^t \Sigma(\mathcal{I}(s, \omega)) \, d\beta(s, \omega).$$

Thus, γ is the total gains process of p_A and is a vector martingale. Since p_A is an analytic (hence C^2) function of (t, β), by Itô's lemma and the uniqueness of Itô coefficients,

$$\Sigma(t, \beta) = \frac{\partial p_A(t, \beta)}{\partial \beta}$$

is a partial derivative of an analytic function, so Σ is analytic on $(0, T) \times \mathbf{R}^K$.
  Extend Σ to $\{T\} \times V$ by

$$\Sigma(T, \beta) = \frac{\partial p_A(T, \beta)}{\partial \beta}$$

for $\beta \in V$; note that here, we are using $\beta \in \mathbf{R}^K$ to denote a particular value that the Brownian motion might take. We claim that Σ is a continuous function of $(t, \beta) \in [0, T] \times V$. Notice that the second term in the definition of Σ tends to zero as $t \to T$, uniformly over β ranging over compact subsets of $\mathbf{R}^K$, so we can restrict attention to the first term. Suppose $\beta_0 \in V$. Fix $\varepsilon > 0$. Since $A(T, \cdot)$ is C^1 on V and $p_C(T, \cdot)$ is C^1 on V, $p_A(T, \cdot) = A(T, \cdot) p_C(T, \cdot)$ is C^1 on $\{T\} \times V$. Since V is open, we may find $\delta > 0$ such that $B(\beta_0, 2\delta) \subset V$ and

$$\beta, y \in B(\beta_0, 2\delta) \quad \Rightarrow \quad \left\| \frac{\partial p_A}{\partial \beta} \bigg|_{(T, y)} - \frac{\partial p_A}{\partial \beta} \bigg|_{(T, \beta)} \right\| < \frac{\varepsilon}{3}.$$

For any $\beta \in B(\beta_0, \delta)$ and $y \in B(\beta, \delta)$, we have $y \in B(\beta_0, 2\delta)$. Using the mean value theorem one component at a time, we find that

$$\left| p_A(T, y) - p_A(T, \beta) - \frac{\partial p_A}{\partial \beta} \bigg|_{(T, \beta)} (y - \beta) \right| < \frac{\varepsilon |y - \beta|}{3},$$

so

$$\left\| \left(p_A(T, y) - p_A(T, \beta) - \frac{\partial p_A}{\partial \beta} \bigg|_{(T, \beta)} (y - \beta) \right) (y - \beta)^\top \right\| < \frac{\varepsilon |y - \beta|^2}{3}.$$

Since $p_C \le r + e^{r|\beta|}$ and $|A_j(T, y)| \le r + e^{r|y|}$, and $\|\partial p_A / \partial \beta|_{(T, \beta)}\|$ is uniformly bounded over $\beta \in B(\beta_0, \delta)$, there is a constant $\bar{r} \in \mathbf{R}$ such that for every $\beta \in B(\beta_0, \delta)$, for all $y \in \mathbf{R}^K$,

$$\left\| \left(p_A(T, y) - p_A(T, \beta) - \frac{\partial p_A}{\partial \beta} \bigg|_{(T, \beta)} (y - \beta) \right) (y - \beta)^\top \right\| \le \bar{r} + e^{\bar{r}|y|}.$$

Find $t_0 < T$ such that for all $t \in [t_0, T)$,

$$\frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K \backslash B(\beta, \delta)} \frac{\bar{r} + e^{\bar{r}|y|}}{T-t} e^{-|y-\beta|^2/2(T-t)} \, dy < \frac{\varepsilon}{3}.$$

If $t \in [t_0, T)$ and $\beta \in B(\beta_0, \delta)$,

$$\left\| \frac{E(p_A(T, \beta(T, \cdot))(\beta(T, \cdot) - \beta(t, \cdot))^\top | \beta(t, \omega) = \beta)}{T-t} - \left. \frac{\partial p_A}{\partial \beta} \right|_{(T, \beta_0)} \right\|$$

$$= \left\| \frac{1}{(2\pi)^{K/2}} \int_{\mathbf{R}^K} \frac{p_A(T, \beta + \sqrt{T-t}\,x)\sqrt{T-t}\,x^\top}{T-t} e^{-|x|^2/2} \, dx \right.$$

$$\left. - \left. \frac{\partial p_A}{\partial \beta} \right|_{(T, \beta_0)} \right\|$$

$$= \left\| \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \frac{p_A(T, y)(y - \beta)^\top}{T-t} e^{-|y-\beta|^2/(2(T-t))} \, dy \right.$$

$$\left. - \left. \frac{\partial p_A}{\partial \beta} \right|_{(T, \beta_0)} \right\|$$

$$\leq \left\| \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \frac{\left. \frac{\partial p_A}{\partial \beta} \right|_{(T, \beta)}(y - \beta)(y - \beta)^\top}{T-t} e^{-|y-\beta|^2/(2(T-t))} \, dy \right.$$

$$\left. - \left. \frac{\partial p_A}{\partial \beta} \right|_{(T, \beta_0)} \right\|$$

$$+ \left\| \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K} \frac{p_A(t, \beta)(y - \beta)^\top}{T-t} e^{-|y-\beta|^2/(2(T-t))} \, dy \right\|$$

$$+ \left\| \frac{1}{(2\pi(T-t))^{K/2}} \int_{B(\beta, \delta)} \frac{\varepsilon(y - \beta)(y - \beta)^\top}{3(T-t)} e^{-|y-\beta|^2/(2(T-t))} \, dy \right\|$$

$$+ \left| \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbf{R}^K \backslash B(\beta, \delta)} \frac{\bar{r} + e^{\bar{r}|y|}}{T-t} e^{-|y-\beta|^2/(2(T-t))} \, dy \right|$$

$$= \left\| \left. \frac{\partial p_A}{\partial \beta} \right|_{(T, \beta)} - \left. \frac{\partial p_A}{\partial \beta} \right|_{(T, \beta_0)} \right\| + 0 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$< \frac{\varepsilon}{3} + 0 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which shows that Σ is continuous on $[0, T] \times V$.

Now, we show that the pricing process is dynamically complete. Let Σ_j ($j = 0, \ldots, J$) denote the jth row of Σ and let $\bar{\Sigma}(t, \beta)$ be the $K \times K$ matrix whose

jth row ($j = 1, \ldots, J$) is

$$\bar{\Sigma}_j(t, \beta) = \frac{p_{A_0}(t, \beta)\Sigma_j(t, \beta) - p_{A_j}(t, \beta)\Sigma_0(t, \beta)}{(p_{A_0}(t, \beta))^2}$$

$$= \frac{\partial(p_{A_j}/p_{A_0})}{\partial\beta}.$$

Notice that

(38) $\mathrm{rank}\,\bar{\Sigma}(t, \beta) \le \mathrm{rank}\,\Sigma(t, \beta).$

Let

$$B = \{\mathcal{I} \in (0, T) \times \mathbf{R}^K : \det\bar{\Sigma}(\mathcal{I}) = 0\}.$$

Suppose that B has positive Lebesgue measure. Then $\det\bar{\Sigma}(\mathcal{I}) = 0$ for every $\mathcal{I} \in B$. The determinant is a polynomial function of the entries of the matrix, hence is an analytic function of $\mathcal{I} \in \mathbf{R}^K \times (0, T)$, so $\det\bar{\Sigma}(\mathcal{I})$ must be identically zero on $(0, T)$;[24] since it is continuous on $[0, T] \times V$, it is identically zero on $\{T\} \times V$. Using the nondegeneracy assumption (Equation (3)), choose ω_0 such that $\hat{\beta}(\cdot, \omega_0)$ is finite, $^\circ\hat{\beta}(\hat{T}, \omega_0) \in V$, and

$$\det\begin{pmatrix} \frac{\partial(G_1/G_0)}{\partial\beta}\big|_{\mathcal{I}(T,\omega_0)} \\ \vdots \\ \frac{\partial(G_J/G_0)}{\partial\beta}\big|_{\mathcal{I}(T,\omega_0)} \end{pmatrix} \neq 0.$$

Since the securities prices are equilibrium prices, they must be arbitrage-free, so we have for $j = 1, \ldots, J$ and any ω such that $^\circ\hat{\beta}(\hat{T}, \omega) \in V$,

$$\frac{p_{A_j}(T, \omega)}{p_{A_0}(T, \omega)} \simeq \frac{\hat{p}_{A_j}(\hat{T}, \omega)}{\hat{p}_{A_0}(\hat{T}, \omega)}$$

$$= \frac{\hat{A}_j(\hat{T}, \omega)}{\hat{A}_0(\hat{T}, \omega)}$$

$$= \frac{{}^*G_j(\hat{T}, \hat{\beta}(\hat{T}, \omega)) + O((\Delta T)^2)}{{}^*G_0(\hat{T}, \hat{\beta}(\hat{T}, \omega)) + O((\Delta T)^2)}$$

$$\simeq \frac{{}^*G_j(\hat{T}, \hat{\beta}(\hat{T}, \omega))}{{}^*G_0(\hat{T}, \hat{\beta}(\hat{T}, \omega))}$$

$$\simeq \frac{G_j(T, \beta(T, \omega))}{G_0(T, \beta(T, \omega))},$$

[24]See Theorem B.3 in Appendix B.

so

$$\det \bar{\Sigma}(T, \beta(T, \omega_0))$$

$$= \det \begin{pmatrix} \frac{\partial(p_{A_1}/p_{A_0})}{\partial \beta}\Big|_{\mathcal{I}(T,\omega_0)} \\ \vdots \\ \frac{\partial(p_{A_J}/p_{A_0})}{\partial \beta}\Big|_{\mathcal{I}(T,\omega_0)} \end{pmatrix} = \det \begin{pmatrix} \frac{\partial(G_1/G_0)}{\partial \beta}\Big|_{\mathcal{I}(T,\omega_0)} \\ \vdots \\ \frac{\partial(G_J/G_0)}{\partial \beta}\Big|_{\mathcal{I}(T,\omega_0)} \end{pmatrix} \neq 0,$$

a contradiction which proves that B is a set of measure zero.

If we let $B_t = \{\beta \in \mathbf{R}^K : (t, \beta) \in B\}$ denote the t section of B, then by Fubini's theorem, $\lambda(\{t : B_t \text{ has positive Lebesgue measure}\}) = 0$. Since the distribution of $\beta(t, \cdot)$ is absolutely continuous with respect to Lebesgue measure,

$$(\lambda \times L(\hat{\mu}))\big(\{(t, \omega) : (\mathcal{I}(t, \omega)) \in B\}\big)$$

$$= \int_{[0,T]} L(\hat{\mu})\big(\{\omega : (\mathcal{I}(t, \omega)) \in B\}\big) d\lambda$$

$$= \int_{[0,T]} L(\hat{\mu})\big(\{\omega : \beta(t, \omega) \in B_t\}\big) d\lambda$$

$$= 0.$$

Thus, we have shown that rank $\bar{\Sigma}(t, \beta(t, \omega)) = K$, and hence rank $\Sigma(t, \beta(t, \omega)) = K$, except on a set of measure zero.

Now, we construct a money-market account: an admissible self-financing trading strategy which is instantaneously riskless. We first construct a money-market account in the hyperfinite model. Because the hyperfinite economy is internally dynamically complete, for every node (t, ω), $\hat{\sigma}(t, \omega)$ has rank K. We claim that $\hat{p}_A(t, \omega)$ does not lie in the span of the columns of $\hat{\sigma}(t, \omega)$. If it did, note that dynamic completeness and the absence of arbitrage imply that there is a security holding whose value (ex dividend) at (t, ω) is nonzero and whose value (cum dividend) is nonzero and the same at each of the successor nodes, so $z \cdot \hat{\sigma}(t, \omega) = 0$. Since $\hat{p}_A(t, \omega)$ lies in the span of the columns of $\hat{\sigma}(t, \omega)$, then $z \cdot \hat{p}_A(t, \omega) = 0$, a contradiction which establishes the claim. Let $\hat{\sigma}'(t, \omega)$ be the $(K + 1) \times (K + 1)$ matrix whose first column is $\hat{p}_A(t, \omega)$ and whose remaining columns are the columns of $\hat{\sigma}(t, \omega)$; we have just shown that $\hat{\sigma}'(t, \omega)$ is nonsingular for every (t, ω), so there is a unique $1 \times (K + 1)$ row vector $\hat{z}(t, \omega)$ satisfying the equation

$$\hat{z}(t, \omega)\hat{\Sigma}'(t, \omega) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

so $\hat{z}(t, \omega)\hat{p}_A(t, \beta(t, \omega)) = 1$ and $\hat{z}(t, \omega)\hat{\sigma}(t, \beta(t, \omega)) = 0$. Since $\hat{\gamma}$ is an internal martingale, $\int \hat{z}\,d\hat{\gamma}$ is an internal martingale; since $\hat{z}\hat{\sigma} = 0$, $\int \hat{z}\,d\hat{\gamma}$ is identically zero, so $\hat{z}$ is self-financing.

Fix (t, ω) such that rank $\bar{\Sigma}(t, \beta(t, \omega)) = K$. If $p_A(t, \omega)$ lies in the span of the columns of $\Sigma(t, \beta(t, \omega))$, then there is a $K \times 1$ column vector x such that $p_A(t, \omega) = \Sigma(t, \beta(t, \omega))x$, so the directional derivative of p_A in the direction $x/|x|$ is parallel to p_A, so the directional derivative of the normalized prices p_A/p_{A_0} in the direction $x/|x|$ is zero, so $x \in \ker \bar{\Sigma}$, so rank $\bar{\Sigma} \leq K - 1$, contradiction. Thus, $^\circ\hat{\sigma}'$ is nonsingular, so $\hat{z}(t, \omega)$ is finite. Thus, $\hat{z}(t, \omega)$ is finite almost everywhere, so we define a trading strategy almost everywhere by $z(t, \omega) = {}^\circ\hat{z}(t, \omega)$. $z(t, \omega)\Sigma(t, \beta(t, \omega)) = {}^\circ(\hat{z}(t, \omega)\hat{\sigma}(t, \omega)) = 0$ and $z(t, \omega)p_A(t, \omega) = {}^\circ(\hat{z}(t, \omega)\hat{p}_A(t, \omega)) = 1$. Trivially, $\hat{z}\hat{\sigma}$ is an $\mathcal{SL}^2$ lifting of $z\Sigma$, so $\int z\,d\gamma = {}^\circ\int \hat{z}\,d\hat{\gamma}$ by Theorem C.4, so $\int z\,d\gamma$ is identically zero. Thus, z is self-financing, admissible, and instantaneously risk-free, so it is a money-market account. Note that the risk-free rate of interest, expressed in the equilibrium prices, is zero.[25]

Now let c be any consumption bundle adapted to the Brownian filtration with finite value at the consumption price p_C. Recall that γ is the total gains process associated with the equilibrium securities prices p_A. Let

$$V(\omega) = \left(p_C(T, \omega)c(T, \omega) + \int_0^T p_C(t, \omega)c(t, \omega)\,dt \right).$$

Since γ is a martingale, the process which is identically one is a state price process[26] for γ. The dispersion matrix of γ is Σ, the dispersion matrix of p_A; we have seen it has rank K almost surely. Thus, the assumptions of Theorem 5.6 of Nielsen (1999) are satisfied. (Alternatively, since γ is a martingale, the true probability is a martingale measure for γ, so the assumptions of Duffie (1996, Proposition 6.I) are satisfied.) Since $E(V) < \infty$, there is an admissible self-financing trading strategy z_V which replicates $V(\omega)$ at time T.[27] Let $z_c(t, \omega) = z_V(t, \omega) - ((\int_0^t p_C(s, \omega)c(s, \omega)\,ds)/\bar{z}(t, \omega)p_A(t, \omega))\bar{z}(t, \omega)$ for $t \in [0, T)$. Then z' is an admissible trading strategy which finances the consumption c with respect to the total gains process γ; it is, of course, not self-financing unless c is identically zero for $t < T$. z' is unique by Nielsen (1999, Proposition 5.5). Therefore, p_A is effectively dynamically complete.

We now derive the formulas for the hyperfinite equilibrium trading strategies and show that they are sufficiently regular to extract a candidate trading

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[25]Recall that the prices of consumption are marginal utility, so the units are utils. Equilibrium requires that a riskless security that costs 1 util today be worth 1 util tomorrow.

[26]See Nielsen (1999, Sec. 4.3, p. 130) for the definition of a state price process.

[27]Nielsen established admissibility, while Duffie did not require it in the definition of dynamic completeness and did not prove it here.

strategy in continuous time. Let

$$
\hat{W}_i(t, \omega) = E\Bigg(\hat{p}_C(\hat{T}, \hat{\beta}(\hat{T}))(\hat{c}_i(\hat{T}, \hat{\beta}(\hat{T})) - \hat{e}_i(\hat{T}, \hat{\beta}(\hat{T})))
$$

$$
+ \Delta T \sum_{s=t}^{\hat{T} - \Delta T} \hat{p}_C(s, \hat{\beta}(s))\big(\hat{c}_i(s, \hat{\beta}(s)) - \hat{e}_i(s, \hat{\beta}(s))\big)\Big|(t, \omega)\Bigg).
$$

Since $\hat{z}_i$ finances $\hat{c}_i$, we must have

$$
\hat{z}_i(t, \omega) \cdot \hat{p}_A(t, \omega) = \hat{W}_i(t, \omega).
$$

By the same arguments we used to derive the formulas for $\hat{p}_A$ and $\hat{\sigma}$, we have for every ω such that $\hat{\beta}(\cdot, \omega)$ is S-continuous, for all $t \in \mathcal{T}$,

$$
(39) \qquad \hat{W}_i(t, \omega)
$$

$$
\simeq E\Bigg(p_C(T, \beta(T))\big(c_i(T, \beta(T)) - e_i(T, \beta(T))\big)
$$

$$
+ \int_{\circ t}^{T} p_C(s, \beta(s))\big(c_i(s, \beta(s)) - e_i(s, \beta(s))\big)\, ds\Big|(\circ t, \omega)\Bigg),
$$

$$
\hat{D}_i(t + \Delta T, \omega) - \hat{D}_i(t, \omega)
$$

$$
= E\Bigg(\frac{p_C(T, \beta(T))(c_i(T, \beta(T)) - e_i(T, \beta(T)))(\beta(T) - \beta(t))^{\top}}{T - t}
$$

$$
+ \int_{\circ t}^{T} \big(p_C(s, \beta(s))\big(c_i(s, \beta(s)) - e_i(s, \beta(s))\big)(\beta(s) - \beta(t))^{\top}\big)
$$

$$
\times (s - \circ t)^{-1}\, ds\Big|(\circ t, \omega)\Bigg)
$$

$$
\times (\hat{\beta}(t + \Delta T, \omega) - \hat{\beta}(t, \omega)) + o(\sqrt{\Delta T}),
$$

$$
(40) \qquad \hat{\sigma}_i(t, \omega)
$$

$$
\simeq E\Bigg(\frac{p_C(T, \beta(T))(c_i(T, \beta(T)) - e_i(T, \beta(T)))(\beta(T) - \beta(t))^{\top}}{T - \circ t}
$$

$$
+ \int_{\circ t}^{T} \big(p_C(s, \beta(s))\big(c_i(s, \beta(s)) - e_i(s, \beta(s))\big)(\beta(s) - \beta(t))^{\top}\big)
$$

$$
\times (s - \circ t)^{-1}\, ds\Big|(\circ t, \omega)\Bigg).
$$

Define standard functions $\Sigma_1, \ldots, \Sigma_I : [0, T) \times \mathbf{R}^K \to \mathbf{R}^K$ and $W_1, \ldots, W_I : [0, T] \times \mathbf{R}^K \to \mathbf{R}^K$ by

$$
\Sigma_i(t, \beta)
$$
$$
= E\Bigg(\frac{p_C(T, \beta(T))(c_i(T, \beta(T)) - e_i(T, \beta(T)))(\beta(T) - \beta(t))^\top}{T - t}
$$
$$
+ \int_t^T \big(p_C(s, \beta(s))\big(c_i(s, \beta(s)) - e_i(s, \beta(s))\big)(\beta(s) - \beta(t))^\top\big)
$$
$$
\times (s - t)^{-1}\, ds \Big| \beta(t) = \beta \Bigg),
$$

$$
W_i(t, \beta) = E\Bigg(p_C(T, \beta(T))\big(c_i(T, \beta(T)) - e_i(T, \beta(T))\big)
$$
$$
+ \int_t^T p_C(s, \beta(s))\big(c_i(s, \beta(s)) - e_i(s, \beta(s))\big)\, ds \Big| \beta(t) = \beta \Bigg).
$$

$\Sigma_1, \ldots, \Sigma_I$ are continuous on $[0, T)$ and analytic on $(0, T)$; $W_1, \ldots, W_I$ are continuous on $[0, T]$ and analytic on $(0, T)$. Equations (39) and (40) show that

$$
\hat{\sigma}_i(t, \omega) \simeq \Sigma_i(\mathcal{I}(^\circ t, \omega)),
$$
$$
\hat{z}_i(t, \omega) \cdot \hat{p}_A(t, \omega) \simeq W_i(\mathcal{I}(^\circ t, \omega))
$$

for every ω such that $\hat{\beta}(\cdot, \omega)$ is S-continuous. Summing up, we have

(41) $\hat{p}_A(t, \omega) \simeq p_A(^\circ t, \omega),$

$\hat{p}_c(t, \omega) \simeq p_c(^\circ t, \omega),$

$\hat{\sigma}(t, \omega) \simeq \Sigma(\mathcal{I}(^\circ t, \omega)),$

$\hat{c}_i(t, \omega) \simeq c_i(^\circ t, \omega),$

$\hat{z}_i(t, \omega) \cdot \hat{\sigma}(t, \omega) = \hat{\sigma}_i(t, \omega) \simeq \Sigma_i(\mathcal{I}(^\circ t, \omega)),$

$\hat{z}_i(t, \omega) \cdot \hat{p}_A(t, \omega) \simeq W_i(\mathcal{I}(^\circ t, \omega))$

for all $t \in \mathcal{T}$, for every ω such that $\hat{\beta}(\cdot, \omega)$ is S-continuous. Thus, for every ω such that $\hat{\beta}(\cdot, \omega)$ is S-continuous, we have for *every* $t \in \mathcal{T}$,[28]

(42) $\hat{p}_A(t, \omega) \simeq p_A(^\circ t, \omega) \simeq {}^* p_A(\hat{\mathcal{I}}(t, \omega)),$

---

[28]In Equation (42), when we write ${}^* p_A(\hat{\mathcal{I}}(t, \omega))$, we mean p_A is a standard function of $\mathcal{I}(t, \omega) \in [0, T] \times \mathbf{R}^K$; take the nonstandard extension of this standard function defined on $[0, T] \times \mathbf{R}^K$ and evaluate it at $\hat{\mathcal{I}}(t, \omega)$. The analogous definition is used for ${}^* p_c, {}^* \Sigma, {}^* \Sigma_1, \ldots, {}^* \Sigma_I, {}^* W_1, \ldots, {}^* W_I$.

$$\hat{p}_c(t, \omega) \simeq p_c({}^\circ t, \omega) \simeq {}^*p_c(\hat{\mathcal{I}}(t, \omega)),$$

$$\hat{\sigma}(t, \omega) \simeq \Sigma(\mathcal{I}({}^\circ t, \omega)) \simeq {}^*\Sigma(\hat{\mathcal{I}}(t, \omega)),$$

$$\hat{c}_i(t, \omega) \simeq c_i(\mathcal{I}({}^\circ t, \omega)) \simeq {}^*c_i(\hat{\mathcal{I}}(t, \omega)),$$

$$\hat{z}_i(t, \omega) \cdot \hat{\sigma}(t, \omega) = \hat{\sigma}_i(t, \omega) \simeq \Sigma_i(\mathcal{I}({}^\circ t, \omega)) \simeq {}^*\Sigma_i(\hat{\mathcal{I}}(t, \omega)),$$

$$\hat{z}_i(t, \omega) \cdot \hat{p}_A(t, \omega) \simeq W_i(\mathcal{I}({}^\circ t, \omega)) \simeq {}^*W_i(\hat{\mathcal{I}}(t, \omega)).$$

Given ω such that $\hat{\beta}(\cdot, \omega)$ is S-continuous,

$$\max_{t \in \mathcal{T}} |\hat{p}_A(t, \omega) - {}^*p_A(\hat{\mathcal{I}}(t, \omega))| \simeq 0$$

since $\mathcal{T}$ is hyperfinite, and every hyperfinite set contains its maximum by the transfer principle. But for $L(\hat{\mu})$-almost all ω, $\hat{\beta}(t, \omega)$ is S-continuous (see the proof of Theorem C.1), so for $L(\hat{\mu})$-almost all ω,

$$\max_{t \in \mathcal{T}} |\hat{p}_A(t, \omega) - {}^*p_A(\hat{\mathcal{I}}(t, \omega))| \simeq 0.$$

Fix $\varepsilon \in \mathbf{R}_{++}$.

$$\left\{ \omega : \max_{t \in \mathcal{T}} |\hat{p}_A(t, \omega) - {}^*p_A(\hat{\mathcal{I}}(t, \omega))| < \varepsilon \right\}$$

is an internal set which contains a set of Loeb measure 1, so

$$(43) \qquad \hat{\mu}\left(\left\{ \omega : \max_{t \in \mathcal{T}} |\hat{p}_A(t, \omega) - {}^*p_A(\hat{\mathcal{I}}(t, \omega))| < \varepsilon \right\} \right) > 1 - \varepsilon.$$

We have already noted that $\hat{p}_C, \hat{p}_A \in SL^p$ for all $p \in [1, \infty)$. Equation (35) and the growth conditions (Equation (1)) imply that, regardless of whether $\hat{\mathcal{I}}(t, \omega)$ is finite, $|\hat{\sigma}(t, \omega)| \le r + e^{r|\hat{\beta}(t,\omega)|}$ and $|\hat{\sigma}_i(t, \omega)| \le r + e^{r|\hat{\beta}(t,\omega)|}$, which implies as above by Raimondo (2002, 2005, Proposition 3.1) that $\hat{\sigma}, \hat{\sigma}_1, \ldots, \hat{\sigma}_I \in SL^p$ for all $p \in [1, \infty)$, in particular, for $p = 2$. $\hat{z}_i \cdot \hat{p}_A$ is bounded below by minus the value (at $\hat{p}_c$) of i's future endowment and bounded above by the value of total market future consumption; hence there exists $\bar{\bar{r}} \in \mathbf{R}$ such that $\hat{z}_i \cdot \hat{p}_A(t, \omega) \le \bar{\bar{r}} + e^{\bar{\bar{r}}(|\hat{\beta}(t,\omega)|+T)}$, so $\hat{z}_i \cdot \hat{p}_A$ is SL^p for all $p \in [1, \infty)$. $\hat{c}_i(t, \omega) \le r + e^{r|\hat{\beta}(t,\omega)|}$, so $\hat{c}_i$ is SL^p for all $p \in [1, \infty)$:

$$\hat{p}_{Aj}(t, \omega) \le r + e^{r(|\hat{\beta}(t,\omega)|+T)} \quad \Rightarrow \quad p_{Aj}(\mathcal{I}(t, \omega)) \le r + e^{r(|\beta(t,\omega)|+T)}$$

$$\Rightarrow \quad {}^*p_{Aj}(\hat{\mathcal{I}}(t, \omega)) \le r + e^{r(|\hat{\beta}(t,\omega)|+T)}$$

$$\Rightarrow \quad {}^*p_{Aj} \circ \hat{\mathcal{I}} \in SL^p \quad (1 \le p < \infty).$$

Similarly, $^*p_c \circ \hat{\mathcal{I}}, {}^*\Sigma \circ \hat{\mathcal{I}}, {}^*c_i \circ \hat{\mathcal{I}}, {}^*\Sigma_i \circ \hat{\mathcal{I}}, {}^*W_i \circ \hat{\mathcal{I}} \in SL^p$ for all $p \in [1, \infty)$. Since $\hat{p}_A$ and $^*p_A \circ \mathcal{I}$ are SL^p for all $p \in [1, \infty)$ and $^\circ(\hat{p}_A(\cdot, \cdot) - {}^*p_A(\hat{\mathcal{I}}(\cdot, \cdot)))$ is zero on a set of full Loeb measure,

$$(44) \qquad \|\hat{p}_A - {}^*p_A \circ \hat{\mathcal{I}}\|_2 \simeq \left\|^\circ(\hat{p}_A - {}^*p_A \circ \hat{\mathcal{I}})\right\|_2 = 0.$$

Now we consider the form of the equilibrium trading strategies $\hat{z}_i$. $\hat{\beta}(t, \omega)$ is finite and $\det \bar{\Sigma}(t, \beta(t, \omega)) \neq 0$ at (Loeb) almost every node; fix such a node (t_0, ω_0). Then $\hat{\beta}(t_0 + \Delta T, \omega_0)$ is also finite. From the growth condition on dividends, it follows that $\hat{A}(t_0 + \Delta T, \omega_0)$ is finite. Let $\omega_1, \ldots, \omega_K$ be elements of Ω such that $\omega_k(s) = \omega_0(s)$ for $s \leq t_0$ and $\omega_k(t_0 + \Delta T) = k$; we can assume without loss of generality that $\omega_0(t_0 + \Delta T) = 0$ (recall that $\omega(t_0 + \Delta T) = k$ means that $\hat{\beta}(t_0 + \Delta T, \omega) - \hat{\beta}(t_0, \omega) = v_k \sqrt{\Delta T}$). Since $\hat{z}_i$ finances $\hat{c}_i$, we have, for $k = 0, \ldots, K$,

$$\hat{\sigma}_i(t_0, \omega_0)(\hat{\beta}(t_0 + \Delta T, \omega_k) - \hat{\beta}(t_0, \omega_0))$$
$$= \hat{D}_i(t_0 + \Delta T, \omega_k) - \hat{D}_i(t_0, \omega_0)$$
$$= \hat{z}_i(t_0, \omega_0) \cdot (\hat{\gamma}(t_0 + \Delta T, \omega_k) - \hat{\gamma}(t_0, \omega_0))$$
$$= \hat{z}_i(t_0, \omega_0) \cdot \left(\hat{\sigma}(t_0, \omega_0)(\hat{\beta}(t_0 + \Delta T, \omega_k) - \hat{\beta}(t_0, \omega_0))\right)$$
$$= \left(\hat{z}_i(t_0, \omega_0)(\hat{\sigma}(t_0, \omega_0))\right)(\hat{\beta}(t_0 + \Delta T, \omega_k) - \hat{\beta}(t_0, \omega_0)).$$

Since $\{v_0, \ldots, v_K\}$ spans $\mathbf{R}^K$, we have

$$\hat{z}_i(t_0, \omega_0)\hat{\sigma}(t_0, \omega_0) = \hat{\sigma}_i(t_0, \omega_0).$$

In addition, we have

$$\hat{z}_i(t_0, \omega_0) \cdot \hat{p}_A(t_0, \omega_0) = \hat{W}_i(t_0, \omega_0).$$

Let

$$\tilde{\sigma}(t_0, \omega_0) = (\hat{\sigma}(t_0, \omega_0) | \hat{p}_A(t_0, \omega_0)),$$
$$\tilde{\sigma}_i(t_0, \omega_0) = (\hat{\sigma}_i(t_0, \omega_0) | \hat{W}_i(t_0, \omega_0))$$

denote the matrix whose first K columns are the columns of $\hat{\sigma}(t_0, \omega_0)$ and whose $(K+1)$st column is $\hat{p}_A(t_0, \omega_0)$, and the row vector whose first K entries are the entries of $\hat{\sigma}_i(t_0, \omega_0)$ and whose $(K+1)$st entry is $\hat{W}_i(t_0, \omega_0)$. Then

$$\hat{z}_i(t_0, \omega_0)\tilde{\sigma}(t_0, \omega_0) = \tilde{\sigma}_i(t_0, \omega_0)$$

and $\tilde{\sigma}(t_0, \omega_0)$ is $(K+1) \times (K+1)$.

We claim that $°\tilde{\sigma}(t_0, \omega_0)$ is nonsingular. $°\hat{\sigma}(t_0, \omega_0) = \Sigma(\mathcal{I}(°t, \omega))$ is $(K + 1) \times K$ and has rank K by Equation (38). If $p_A(°t_0, \omega_0) = °\hat{p}_A(t_0, \omega_0)$ lies in the span of the columns of $\Sigma(\mathcal{I}(°t_0, \omega_0))$, there is a vector $x \in \mathbf{R}^K$ such that $\Sigma(\mathcal{I}(°t_0, \omega_0))x = p_A(°t_0, \omega_0)$. Since $\Sigma(\mathcal{I}(°t_0, \omega_0)) = \partial p_A / \partial \beta$, we have for $\alpha \in \mathbf{R}$,

$$\begin{aligned}
p_A(°t_0, \beta(°t_0, \omega_0) + \alpha x) &= p_A(\mathcal{I}(°t_0, \omega_0)) + \alpha \Sigma(\mathcal{I}(°t_0, \omega_0))x + o(\alpha) \\
&= p_A(\mathcal{I}(°t_0, \omega_0)) + \alpha p_A(\mathcal{I}(°t_0, \omega_0)) + o(\alpha) \\
&= (1 + \alpha) p_A(\mathcal{I}(°t_0, \omega_0)) + o(\alpha)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\alpha}&\left(\frac{p_{A_j}(°t_0, \beta(°t_0, \omega_0) + \alpha x)}{p_{A_0}(°t_0, \beta(°t_0, \omega_0) + \alpha x)} - \frac{p_{A_j}(\mathcal{I}(°t_0, \omega_0))}{p_{A_0}(\mathcal{I}(°t_0, \omega_0))} \right) \\
&= \frac{1}{\alpha}\left(\frac{(1 + \alpha) p_{A_j}(\mathcal{I}(°t_0, \omega_0)) + o(\alpha)}{(1 + \alpha) p_{A_0}(\mathcal{I}(°t_0, \omega_0)) + o(\alpha)} - \frac{p_{A_j}(\mathcal{I}(°t_0, \omega_0))}{p_{A_0}(\mathcal{I}(°t_0, \omega_0))} \right) \\
&= \frac{1}{\alpha}\left(\frac{p_{A_j}(\mathcal{I}(°t_0, \omega_0))}{p_{A_0}(\mathcal{I}(°t_0, \omega_0))} + o(\alpha) - \frac{p_{A_j}(\mathcal{I}(°t_0, \omega_0))}{p_{A_0}(\mathcal{I}(°t_0, \omega_0))} \right) \\
&= o(1) \quad \text{as} \quad \alpha \to 0,
\end{aligned}$$

so the directional derivative of p_{A_j}/p_{A_0} in the direction x is zero. Since $\bar{\Sigma}(\mathcal{I}(°t_0, \omega_0))$ is the Jacobian of $(p_{A_1}, \ldots, p_{A_J})/p_{A_0}$,

$$\bar{\Sigma}(\mathcal{I}(°t_0, \omega_0))x = 0,$$

contradicting the fact that $\det \bar{\Sigma}(\mathcal{I}(°t_0, \omega_0)) \neq 0$. Accordingly, $°\tilde{\sigma}(t_0, \omega_0)$ has rank $K + 1$ and thus is invertible. Since the formula for the inverse of a matrix is a polynomial function of the matrix components, the coefficients of the inverse matrix are given by a standard analytic function of $\mathcal{I}(°t_0, \omega_0)$. $\tilde{\sigma}(t_0, \omega_0)$ is also invertible and

$$°(\tilde{\sigma}(t_0, \omega_0)^{-1}) = (°\tilde{\sigma}(t_0, \omega_0))^{-1}.$$

Then

$$\hat{z}_i(t_0, \omega_0) = \tilde{\sigma}_i(t_0, \omega_0)(\tilde{\sigma}(t_0, \omega_0))^{-1},$$

so there are standard analytic functions[29] $Z_1, \ldots, Z_I : (((0, T) \times \mathbf{R}^K) \setminus B) \to \mathbf{R}^{K+1}$ such that

$$\hat{z}_i(t_0, \omega) \simeq Z_i(\mathcal{I}(°t_0, \omega)) \simeq {}^* Z_i(\hat{\mathcal{I}}(t_0, \omega))$$

---

[29]Recall that $B = \{\mathcal{I} \in (0, T) \times \mathbf{R}^K : \det \bar{\Sigma}(\mathcal{I}) = 0\}$ is a set of measure zero.

whenever

$$\mathcal{I}({}^{\circ}t_0, \omega_0) \in ((0, T) \times \mathbf{R}^K) \setminus B.$$

Define $z_i(t, \omega) = {}^{\circ}\hat{z}_i(\hat{t}, \omega)$; $\hat{z}_i\hat{\sigma} = \hat{\sigma}_i$ is an SL^2 lifting of $z_i\sigma$, which belongs to $\mathcal{H}^2$.

We now show that $(p_A, p_C, (c_1, \ldots, c_I), (Z_1, \ldots, Z_I))$ is an equilibrium of the Loeb continuous-time economy with utility weights λ and induces an equilibrium with utility weights λ for the original continuous-time economy. For Loeb almost all (t, ω),

$$\sum_{i=1}^{I} c_i(t, \omega) = \sum_{i=1}^{I} {}^{\circ}\hat{c}_i(\hat{t}, \omega)$$

$$= {}^{\circ}\left(\sum_{i=1}^{I} \hat{c}_i(\hat{t}, \omega) \right)$$

$$= {}^{\circ}\left(\sum_{i=1}^{I} \hat{e}_i(\hat{t}, \omega) + \sum_{j=0}^{J} \eta_j \hat{A}_j(\hat{t}, \omega) \right)$$

$$= \sum_{i=1}^{I} {}^{\circ}\hat{e}_i(\hat{t}, \omega) + \sum_{j=0}^{J} \eta_j {}^{\circ}\hat{A}_j(\hat{t}, \omega)$$

$$= \sum_{i=1}^{I} e_i(t, \omega) + \sum_{j=0}^{J} \eta_j A_j(t, \omega),$$

$$\sum_{i=1}^{I} z_i(t, \omega) = \sum_{i=1}^{I} {}^{\circ}\hat{z}_i(\hat{t}, \omega)$$

$$= {}^{\circ}\left(\sum_{i=1}^{I} \hat{z}_i(\hat{t}, \omega) \right)$$

$$= {}^{\circ}(\eta_0, \ldots, \eta_J).$$

Thus, the consumptions $c_1, \ldots, c_I$ clear the goods market and the trading strategies $z_1, \ldots, z_I$ clear the securities market.

By Theorem C.4, for all $t \in [0, T]$,

$$p_A(t, \omega) = {}^{\circ}\hat{p}_A(\hat{t}, \omega)$$

$$= {}^{\circ}\left(\hat{\gamma}(\hat{t}, \omega) - \Delta T \sum_{s \leq \hat{t}, s < \hat{T}} \hat{p}_C(s, \omega)\hat{A}(s, \omega) \right)$$

$$= {}^{\circ}\left(\hat{\gamma}(0, \omega) + \int_0^{\hat{t}} \hat{\sigma}\, d\hat{\beta} - \Delta T \sum_{s \leq \hat{t}, s < \hat{T}} \hat{p}_C(s, \omega)\hat{A}(s, \omega) \right)$$

$$= \gamma(0, \omega) + \int_0^t \sigma\, d\beta - \int_0^t p_C(s, \omega)A(s, \omega)\, ds.$$

Given any hyperfinite trading strategy $\hat{z}_i$ and $t \in \mathcal{T}$, we have

$$\int_0^t \hat{z}_i\, d\hat{\gamma} = \sum_{s \in \mathcal{T}, s < t} \hat{z}_i(s, \omega) \cdot (\hat{\gamma}(s + \Delta T, \omega) - \hat{\gamma}(s, \omega))$$

$$= \sum_{s \in \mathcal{T}, s < t} (\hat{z}_i(s, \omega)\hat{\sigma}(s, \omega)) \cdot (\hat{\beta}(s + \Delta T, \omega) - \hat{\beta}(s, \omega))$$

$$= \int_0^t (\hat{z}_i\hat{\sigma})\, d\hat{\beta}.$$

Since $\hat{c}_1, \ldots, \hat{c}_I, z_1, \ldots, z_I$ are adapted to the hyperfinite filtration, $\hat{c}_1, \ldots, \hat{c}_I$, $z_1, \ldots, z_I$ are adapted to the Loeb filtration; since $\hat{c}_1, \ldots, \hat{c}_I \in SL^p$ for all $p \in [1, \infty)$, $c_1, \ldots, c_I \in L^p$ for all $p \in [1, \infty)$ (Anderson (1976)). For all $t \in [0, T)$,

$$p_A(t, \omega) \cdot z_i(t, \omega) - p_A(0, \omega) \cdot e_{iA}(\omega)$$

$$- \int_0^t z_i\, d\gamma - \int_0^t p_C(s, \omega)(e_i(s, \omega) - c_i(s, \omega))\, ds$$

$$= p_A(t, \omega) \cdot z_i(t, \omega) - p_A(0, \omega) \cdot e_{iA}(\omega)$$

$$- \int_0^t z_i\sigma\, d\beta - \int_0^t p_C(s, \omega)(e_i(s, \omega) - c_i(s, \omega))\, ds$$

$$\simeq \hat{p}_A(\hat{t}, \omega) \cdot \hat{z}_i(t, \omega) - \hat{p}_A(0, \omega) \cdot \hat{e}_{iA}(\omega)$$

$$- \int_0^{\hat{t}} \hat{z}_i\hat{\sigma}\, d\hat{\beta} - \int_0^{\hat{t}} \hat{p}_C(s, \omega)(\hat{e}_i(s, \omega) - \hat{c}_i(s, \omega))\, ds$$

$$= \hat{p}_A(\hat{t}, \omega) \cdot \hat{z}_i(\hat{t}, \omega) - \hat{p}_A(0, \omega) \cdot \hat{e}_{iA}(\omega)$$

$$- \int_0^{\hat{t}} \hat{z}_i\, d\hat{\gamma} - \int_0^{\hat{t}} \hat{p}_C(s, \omega)(\hat{e}_i(s, \omega) - \hat{c}_i(s, \omega))\, ds$$

$$= 0,$$

since the trading strategy $\hat{z}_i$ finances the consumption $\hat{c}_i$. Thus, the trading strategy z_i finances the consumption c_i.

For all nodes (t, ω) in the hyperfinite economy,

$$\hat{\lambda}_i{}^* h_i'(c_i(t, \omega), \mathcal{I}(t, \omega)) = \hat{p}_C(t, \omega)$$

for each i. Recalling that $\hat{\lambda}_i \not\approx 0$, so $\lambda_i \neq 0$, and $^\circ \hat{p}_C \in (0, \infty)$ Loeb almost everywhere,

$$
\begin{aligned}
\lambda_i \frac{\partial h_i}{\partial c_i}\bigg|_{(c_i(t,\omega),\mathcal{I}(t,\omega))} &= {}^\circ(\hat{\lambda}_i){}^\circ \left({}^* \frac{\partial h_i}{\partial c_i}\bigg|_{(\hat{c}_i(t,\omega),\hat{\mathcal{I}}(t,\omega))} \right) \\
&= {}^\circ \left(\hat{\lambda}_i{}^* \frac{\partial h_i}{\partial c_i}\bigg|_{(\hat{c}_i(t,\omega),\hat{\mathcal{I}}(t,\omega))} \right) \\
&= {}^\circ \hat{p}_C(\hat{\mathcal{I}}(t, \omega)) \\
&= p_C(\mathcal{I}(t, \omega))
\end{aligned}
$$

holds for almost all nodes in the Loeb continuous-time economy. This shows that for almost all nodes, the consumptions $c_1(t, \omega), \ldots, c_I(t, \omega)$ satisfy the necessary conditions for the Negishi weight problem (maximize $\sum_{i=1}^{I} \lambda_i h_i(c_i(t, \omega), \mathcal{I}(t, \omega))$ subject to $\sum_{i=1}^{I} c_i(t, \omega) = c(t, \omega)$), as well as the necessary conditions for demand.

Since preferences are strictly concave, the necessary conditions are in fact sufficient. Here, we present the details for the demand; the argument for the Negishi weight problem is similar. Suppose that c_i' lies in the budget set. Then

$$
\begin{aligned}
E\left(p_C(T)c_i'(T) + \int_0^T p_C(t)c_i'(t)\,dt \right) & \\
\leq e_{iA} \cdot p_A(0) + E\left(\int_0^T p_C(t)e_i(t)\,dt \right) & \\
= E\left(p_C(T)c_i(T) + \int_0^T p_C(t)c_i(t)\,dt \right). &
\end{aligned}
$$

Since h_i and H_i are concave,

$$h_i(c_i'(t, \beta), (t, \beta))$$
$$\leq h_i(c_i(t, \beta), (t, \beta)) + \frac{\partial h_i}{\partial c_i}\bigg|_{(c_i(t,\beta),(t,\beta))} (c_i'(t, \beta) - c_i(t, \beta)),$$
$$H_i(c_i'(T, \beta), (T, \beta))$$
$$\leq H_i(c_i(T, \beta), (T, \beta)) + \frac{\partial H_i}{\partial c_i}\bigg|_{(c_i(T,\beta),(T,\beta))} (c_i'(T, \beta) - c_i(T, \beta)),$$

so

$$U(c_i') - U(c_i)$$

$$= E\left(H_i(c_i'(T)) - H_i(c_i(T)) + \int_0^t h_i(c_i'(t)) - h_i(c_i(t))\, dt\right)$$

$$\leq E\left(\frac{\partial H_i}{\partial c_i}\bigg|_{(c_i(T),T,(\beta))} (c_i'(T) - c_i(T))\right.$$

$$\left. + \int_0^t \frac{\partial h_i}{\partial c_i}\bigg|_{(c_i(t),(t,\beta))} (c_i'(t) - c_i(t))\, dt\right)$$

$$= \frac{1}{\lambda_i} E\left(p_C(T)(c_i'(T) - c_i(T)) + \int_0^t p_C(t)(c_i'(t) - c_i(t))\, dt\right)$$

$$= \frac{1}{\lambda_i}\left(E\left(p_C(T)c_i'(T) + \int_0^T p_C(t)c_i'(t)\, dt\right)\right.$$

$$\left. - E\left(p_C(T)c_i(T) + \int_0^T p_C(t)c_i(t)\, dt\right)\right)$$

$$\leq 0,$$

so c_i lies in agent i's demand set. This shows that we have an equilibrium for the Loeb continuous-time economy and that λ is a vector of Negishi utility weights for the Loeb continuous-time economy.

Now, consider the original continuous-time economy specified in the statement of Theorem 2.1. To distinguish the original economy from the Loeb economy, we will denote the Brownian motion in the original economy by $\bar{\beta}$ and let $\bar{\mathcal{I}}(t, \omega) = (t, \bar{\beta}(t, \omega))$. The equilibrium prices, consumptions, and (except for a set of measure zero) trading strategies of the Loeb economy are given by functions of $\mathcal{I}(t, \omega)$:

$$p_A(t, \omega) = P_A(\mathcal{I}(t, \omega)),$$
$$p_C(t, \omega) = P_C(\mathcal{I}(t, \omega)),$$
$$c_i(t, \omega) = C_i(\mathcal{I}(t, \omega)),$$
$$z_i(t, \omega) = Z_i(\mathcal{I}(t, \omega)).$$

Using these functions, define a candidate equilibrium for the original continuous-time economy:

$$\bar{p}_A(t, \omega) = P_A(\bar{\mathcal{I}}(t, \omega)),$$
$$\bar{p}_C(t, \omega) = P_C(\bar{\mathcal{I}}(t, \omega)),$$

$$\bar{c}_i(t, \omega) = C_i(\bar{\mathcal{I}}(t, \omega)),$$

$$\bar{z}_i(t, \omega) = Z_i(\bar{\mathcal{I}}(t, \omega)).$$

Fix $\bar{\omega}_0 \in \bar{\Omega}$ and suppose that $\omega_0 \in \Omega$ satisfies $\beta(t, \omega_0) = \bar{\beta}(t, \bar{\omega}_0)$:

$$\bar{z}_i(t, \bar{\omega}_0)\bar{p}_A(t, \bar{\omega}_0) = Z_i(t, \bar{\beta}(t, \bar{\omega}_0))P_A(t, \bar{\beta}(t, \bar{\omega}_0))$$

$$= Z_i(t, \beta(t, \omega_0))P_A(t, \beta(t, \omega_0))$$

$$= W_i(t, \beta(t, \omega_0))$$

$$= W_i(t, \bar{\beta}(t, \omega_0)),$$

where W_i is an analytic function of $(t, \beta) \in (0, T) \times \mathbf{R}^K$. By Itô's lemma, $W_i(t, \bar{\beta}(t))$ must satisfy the same stochastic differential equation with respect to $\bar{\beta}$ as $W_i(t, \beta(t))$ satisfies with respect to β, and the coefficients of the stochastic differential equation are functions of $(t, \bar{\beta})$ and (t, β), respectively:

$$d\big(W_i(t, \beta(t))\big) = a(t, \beta(t)) \, dt + b(t, \beta(t)) \, d\beta,$$

$$d\big(W_i(t, \bar{\beta}(t))\big) = a(t, \bar{\beta}(t)) \, dt + b(t, \bar{\beta}(t)) \, d\bar{\beta}.$$

Since z_i finances c_i, we have

$$a(t, \beta) = (C_i(t, \beta) - f_i(t, \beta)),$$

$$b(t, \beta) = \frac{\partial D_i(t, \beta)}{\partial \beta}$$

for all $(t, \beta) \in (0, T) \times \mathbf{R}^K$, so

$$a(t, \bar{\beta}) = (C_i(t, \bar{\beta}) - f_i(t, \bar{\beta})),$$

$$b(t, \bar{\beta}) = \frac{\partial D_i(t, \bar{\beta})}{\partial \bar{\beta}},$$

so $\bar{z}_i$ finances $\bar{c}_i$. Since z_i is admissible, $\bar{z}_i$ is admissible and

$$E\bigg(\bar{p}_C(T)\bar{c}_i(T) + \int_0^T \bar{p}_C(t)\bar{c}_i(t) \, dt \bigg)$$

$$= \bar{e}_{iA} \cdot \bar{p}_A(t) + E\bigg(\int_0^T \bar{p}_C(t)\bar{e}_i(t) \, dt \bigg),$$

$$\sum_{i=1}^I \bar{c}_i(t, \beta) = \sum_{i=1}^I c_i(t, \beta)$$

$$= \sum_{i=1}^{I} e_i(t, \beta) + \sum_{j=0}^{J} \eta_j A_j(t, \beta)$$

$$= \sum_{i=1}^{I} \bar{e}_i(t, \beta) + \sum_{j=0}^{J} \eta_j \bar{A}_j(t, \beta),$$

$$\sum_{i=1}^{I} \bar{z}_i(t, \beta) = \sum_{i=1}^{I} z_i(t, \beta)$$

$$= \sum_{j=0}^{J} \eta_j.$$

Thus, the markets for consumption and securities clear at almost all nodes in the original continuous-time economy.

For almost all nodes in the original continuous time economy,

$$\lambda_i \frac{\partial h_i}{\partial c_i}\bigg|_{(\bar{c}_i(t, \bar{\beta}), (t, \bar{\beta}))} = \lambda_i \frac{\partial h_i}{\partial c_i}\bigg|_{(c_i(t, \beta), (t, \beta))}$$

$$= p_C(\mathcal{I}(t, \omega))$$

$$= \bar{p}_C(\bar{\mathcal{I}}(t, \omega)).$$

This shows that for almost all nodes, the consumptions $c_1(t, \omega), \ldots, c_I(t, \omega)$ satisfy the necessary conditions for the Negishi weight problem maximize $\sum_{i=1}^{I} \lambda_i h_i(c_i(t, \omega), \mathcal{I}(t, \omega))$ subject to $\sum_{i=1}^{I} c_i(t, \omega) = c(t, \omega)$, as well as the necessary conditions to be the demand.

Since preferences are strictly concave, the necessary conditions are in fact sufficient; the argument is exactly the same as the argument above for the Loeb continuous-time economy. This shows that we have an equilibrium for the original continuous-time economy, and that λ is a vector of Negishi utility weights for the original continuous-time economy.

This completes the proofs of Theorem 2.1 and Proposition 2.2.

Let $\mathcal{E}$ denote the set of all equilibria of the original continuous-time economy. Recall that we have been working in a discretization indexed by a particular $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, but suppressing the index n. Putting the index back into the notation, we have shown that for $n \in {}^*\mathbf{N} \setminus \mathbf{N}$,

$$(\lambda_n, p_{nA}, p_{nC}, (c_{n1}, \ldots, c_{nI}), (Z_{n1}, \ldots, Z_{nI})) \in \mathcal{E},$$

so[30]

$$(\lambda_n, {}^*p_{nA}, {}^*p_{nC}, ({}^*c_{n1}, \ldots, {}^*c_{nI}), ({}^*Z_{n1}, \ldots, {}^*Z_{nI})) \in {}^*\mathcal{E}.$$

---

[30]Since $\lambda_n \in \mathbf{R}$, ${}^*\lambda_n = \lambda_n$.

Fix $\varepsilon \in \mathbf{R}_{++}$. Equation (43) tells us that

$$\exists_{m \in {}^*\mathbf{N}} \forall_{n \geq m} \exists_{(\lambda_n, p_{nA}, p_{nC}, (Z_{n1}, \ldots, Z_{nI}), (c_{n1}, \ldots, c_{nI})) \in {}^*\mathcal{E}},$$

$$\hat{\mu}\left(\left\{\omega : \max_{t \in \mathcal{T}_n} \left| \hat{p}_{nA}(t, \omega) - {}^*p_{nA}(\hat{\mathcal{I}}(t, \omega)) \right| < \varepsilon \right\}\right) > 1 - \varepsilon$$

(any $m \in {}^*\mathbf{N} \setminus \mathbf{N}$ will do). Since the construction of the hyperfinite economy and the price process $\hat{p}_{nA}$ is just the transfer of the construction for finite n, the transfer principle tells us that

$$\exists_{m \in \mathbf{N}} \forall_{n \geq m} \exists_{(\lambda_n, p_{nA}, p_{nC}, (Z_{n1}, \ldots, Z_{nI}), (c_{n1}, \ldots, c_{nI})) \in \mathcal{E}},$$

$$\hat{\mu}\left(\left\{\omega : \max_{t \in \mathcal{T}_n} \left| \hat{p}_{nA}(t, \omega) - p_{nA}(\hat{\mathcal{I}}(t, \omega)) \right| < \varepsilon \right\}\right) > 1 - \varepsilon,$$

which is the convergence in probability part of Equation (11); Equation (16) and the convergence in probability parts of Equations (12), (13), (14), (17), and (18) in the statement of Theorem 4.2 follow in exactly the same way. Equation (44) tells us that

$$\exists_{m \in {}^*\mathbf{N}} \forall_{n \geq m} \exists_{(\lambda_n, p_{nA}, p_{nC}, (Z_{n1}, \ldots, Z_{nI}), (c_{n1}, \ldots, c_{nI})) \in {}^*\mathcal{E}}, \quad \| \hat{p}_{nA} - {}^*p_{nA} \circ \hat{\mathcal{I}} \|_2 < \varepsilon.$$

As before, the transfer principle tells us that

$$\exists_{m \in \mathbf{N}} \forall_{n \geq m} \exists_{(\lambda_n, p_{nA}, p_{nC}, (Z_{n1}, \ldots, Z_{nI}), (c_{n1}, \ldots, c_{nI})) \in \mathcal{E}}, \quad \| \hat{p}_{nA} - p_{nA} \circ \hat{\mathcal{I}} \|_2 < \varepsilon.$$

This is the convergence in norm part of Equations (11); Equations (10) and (15), and the convergence in norm parts of Equations (12), (13), (14), (17), and (18) in the statement of Theorem 4.2 follow in exactly the same way. This completes the proof of Theorem 4.2.   *Q.E.D.*

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