1. (100 points) Define or state and briefly discuss the importance of each of the following within or for economic theory:

(a) Kakutani’s Fixed Point Theorem
(b) Lebesgue measure zero
(c) Core of an exchange economy
(d) First Welfare Theorem in an Arrow-Debreu economy
(e) Index Theorem

2. (80 points) Consider an Edgeworth Box economy, where

\[
\omega_1 = (2, 1) \quad \omega_2 = (1, 2)
\]

\[
u_1(x_{11}, x_{21}) = \sqrt{x_{11}x_{21}} \quad u_2(x_{12}, x_{22}) = \sqrt{x_{12}x_{22}}
\]

(a) Find a Walrasian equilibrium.
(b) Show that the allocation \( x_1 = (1, 1), x_2 = (2, 2) \) is Pareto optimal. Without using the Second Welfare Theorem, show that this allocation is a Walrasian equilibrium with transfers.
3. (120 points) Consider the function $z : \Delta^0 \times \mathbb{R} \to \mathbb{R}^2$ defined by

$$z(p, \alpha) = \left(\frac{1}{p_1} + \alpha \cos(2\pi p_1), -\frac{1 + \alpha p_1 \cos(2\pi p_1)}{p_2}\right)$$

Note that $\cos(0) = 1$ and $\frac{d}{dx} \cos x = -\sin x$.

(a) For what values of $\alpha$ does the function $z_\alpha(p) = z(p, \alpha)$ satisfy the conditions of the Debreu-Gale-Kuhn-Nikaido Lemma?

(b) For what values of $\alpha$ does there exist $p \in \Delta$ such that $z(p, \alpha) = 0$?

(c) Show that for every $\alpha \in \mathbb{R}$ and every $\varepsilon > 0$, there is an exchange economy with two agents whose excess demand function agrees with $z_\alpha$ on $\{p \in \Delta : p_1 \in [\varepsilon, 1 - \varepsilon]\}$.

(d) Show that there is a set $A \subset \mathbb{R}$ of Lebesgue measure zero such that for every $\alpha \not\in A$, the economy with excess demand $z_\alpha$ is regular.