Justifying (or Undermining) the Price-Taking Assumption

- Many formulations: Core, Ostroy’s No Surplus Condition, Bargaining Set, Shapley-Shubik Market Games (noncooperative), other noncooperative games

- Core is the most commonly used. The core is the set of all allocations such that no coalition (set of agents) can improve on or block the allocation (make all of its members better off) by seceding from the economy and only trading among its members.

- Core is institution-free; no mention of prices.

- “Core convergence” means roughly that

  For economies with a large number of agents, core allocations are “approximately Walrasian.”

- “Approximately Walrasian” means different things in different contexts, depending on what we are willing to assume.

- Three motivations for the study of the core:
  
  - Walrasian allocations lie in the core: Important strengthening of First Welfare Theorem, under same minimal assumptions as First Welfare Theorem.

    * (Positive): Strong stability property of Walrasian equilibrium: no group of individuals would choose to upset the equilibrium by recontracting among themselves.
* (Normative): If distribution of initial endowments is equitable, no group is treated unfairly at a core allocation. Since Walrasian allocations lie in the core, this is a Group Fairness Property of Walrasian Equilibrium.

- **Core Convergence strengthens Second Welfare Theorem**

  * Second Welfare Theorem says every Pareto Optimum is a Walrasian Equilibria with Transfers.
  
  * Core convergence asserts that core allocations of large economies are *nearly* Walrasian *without* transfers.
  
  * One version states that core allocations can be realized as *exact* Walrasian equilibrium with *small* income transfers.

* **Strong “unbiasedness” property of Walrasian equilibrium**

  - Restricting to Walrasian outcomes does not narrow possible outcomes beyond narrowing occurring in the core.
  
  - (Normative) No hidden implications for welfare of different groups beyond equity issues in the initial endowment distribution.
  
  - (Normative) Assuming distribution of endowments is equitable, any allocation that is far from Walrasian will not be in the core, and hence will treat some group unfairly.

- **Core Convergence justifies Price-Taking, Core Nonconvergence suggests Price-Taking is Implausible:**

  * The definition of Walrasian equilibrium contains (hidden in plain sight) assumption that economic agents act as price-takers.
* In real markets, we see prices used to equate supply and demand, but this does not guarantee Walrasian outcome.

* Agents possessing market power may choose to supply quantities different from the competitive supply for the prevailing price, thereby altering that price and leading to a non-Pareto Optimal outcome.

* If outcome is not Walrasian, Welfare Theorems, Existence, Determinacy would have limited implications for real economies.

* (Positive) Core convergence and nonconvergence allows us to identify situations in which price-taking is more or less reasonable.


* Edgeworth criticized Walras, thought the core, not the set of Walrasian equilibria, was best positive description of outcomes from market mechanism.

  · In particular, the definition of the core does not impose the assumption of price-taking behavior made by Walras.

  · Furthermore, if any allocation not in the core arose, some group would find it in its interests to recontract. Edgeworth thus argues that the core is the significant positive equilibrium concept.

  · If core is correct positive concept, core convergence justifies price-taking. Core convergence says all trade takes place at almost a single price. Agent who tries to bargain
cannot influence prices much, and cannot change outcome much (argument more compelling with stronger convergence notions).

- If core is correct positive concept, core nonconvergence undermines price-taking. Edgeworth himself argued that in real life, the presence of large firms leads to failure of price-taking.

- **Definition 1** In an exchange economy, a *coalition* is a set

  \[ S \subseteq \{1, \ldots, I\} \]

  A coalition \( S \) *blocks* or *improves on* an exact allocation \( x \) by \( x' \) if

  \[ \sum_{i \in S} x'_i = \sum_{i \in S} \omega_i \]

  and

  \[ \forall i \in S \ x'_i \succ_i x_i \]

  The *core* is the set of all exact allocations which cannot be improved on by any nonempty coalition.

- Notice we follow MWG and require \( x'_i \succ_i x_i \) for all \( i \in S \); this is analogous to the definition of weakly Pareto Optimal. *Natural:* status quo should be focal, need strict improvement to join a coalition to upset the status quo.

- Notice that the definition of blocking by a coalition does not specify what happens to the individuals outside the coalition. One might imagine individuals not in the blocking coalition making a counter-proposal to some of those in the blocking coalition; the Bargaining Set takes these counterproposals into account.
Core in Edgeworth Box
Core in 2-fold replica
(Two agents of each type)
It is a common mistake to ask, at a core allocation, what coalition(s) are active. A core allocation is defined by the fact that no coalition can defeat it.

**Theorem 2** *In an exchange economy, every core allocation is weakly Pareto Optimal.*

**Proof:** If $x$ is not weakly Pareto Optimal, then there exists $x'$,

$$\sum_{i=1}^{I} x'_i = \bar{\omega}, \ x'_i >_i x_i$$

Then $S = \{1, \ldots, I\}$ improves on $x$ by $x'$, so $x$ is not in the core.\[\]

**Theorem 3 (Strong First Welfare Theorem)** *In an exchange economy, every Walrasian Equilibrium lies in the core.*

**Proof:** Suppose $(p^*, x^*)$ is a Walrasian Equilibrium. If $x^*$ is not in the core, there exists $S \subseteq I$, $S \neq \emptyset$ and $x'_i (i \in S)$ such that

$$\sum_{i \in S} x'_i = \sum_{i \in S} \omega_i, \ \forall i \in S \ x'_i >_i x^*_i$$

Since $x^*_i \in D_i(p^*)$,

$$p^* \cdot x'_i > p^* \cdot \omega_i$$

so

$$p^* \cdot \sum_{i \in S} x'_i = \sum_{i \in S} p^* \cdot x'_i$$

$$> \sum_{i \in S} p^* \cdot \omega_i$$

$$= p^* \cdot \sum_{i \in S} \omega_i$$

but

$$\sum_{i \in S} x'_i = \sum_{i \in S} \omega_i$$

contradiction. Therefore, $x^*$ is in the core.\[\]
Theorem 4 Suppose we are given an exchange economy with \(L\) commodities, \(I\) agents, and preferences \(\succ_1, \ldots, \succ_I\) satisfying weak monotonicity (if \(x \gg y\), then \(x \succ_i y\)) and the following free disposal condition:

\[ x \gg y, \ y \succ_i z \Rightarrow x \succ_i z. \]

If \(x\) is in the core, then there exists \(p \in \Delta\) such that

\begin{align*}
\frac{1}{I} \sum_{i=1}^{I} |p \cdot (x_i - \omega_i)| & \leq \frac{2L}{I} \max\{\|\omega_1\|_{\infty}, \ldots, \|\omega_I\|_{\infty}\} \\
\frac{1}{I} \sum_{i=1}^{I} \inf \{p \cdot (y - x_i) : y \succ_i x_i\} & \leq \frac{4L}{I} \max\{\|\omega_1\|_{\infty}, \ldots, \|\omega_I\|_{\infty}\}
\end{align*}

(1) (2)

where \(\|x\|_{\infty} = \max\{|x_1|, \ldots, |x_L|\}\).

- Equation (1) says that trade occurs almost at the price \(p\), and that each \(x_i\) is almost in the budget set.

- Equation (2) says that the price \(p\) almost supports \(\succ_i\) at \(x_i\).

- If we knew the left sides of Equations (1) and (2) were zero, then

\[ p \cdot (x_i - \omega_i) = 0 \Rightarrow x_i \in B_i(p) \]

\[ y \succ_i x_i \Rightarrow p \cdot y \geq p \cdot \omega_i \]

so \(x\) is a Walrasian quasiequilibrium! Thus, every core allocation satisfies a perturbation of the definition of Walrasian Equilibrium: agent \(i\)'s consumption need not lie in his/her budget set, but it can’t be far outside; anything strictly preferred need not be outside the budget set, but it can’t be far below the budget frontier.
\textbf{Equation (1)}

\[ \max_{i} \frac{c_i}{\lambda_i} \text{ s.t. } i = 1, \ldots, l \]

\textbf{Equation (2)}

\[ \max_{i} \sum_{l} \text{ s.t. } i = 1, \ldots, l \]
Outline of Proof: Follow the proof of the Second Welfare Theorem.

- Suppose $x$ is in the core. Define

$$B_i = \{ y - \omega_i : y \succ_i x \} \cup \{0\}$$

$$= (\{ y : y \succ_i x \} \cup \{ \omega_i \}) - \omega_i$$

$$B = \sum_{i=1}^{I} B_i$$

The first term in the definition of $B_i$ corresponds to members of a potential improving coalition; for accounting purposes, we assign members outside the coalition their endowments. Note that $B_i$ is not convex, even if $\succ_i$ is a convex preference.

- Claim: If $x$ is in the core, then

$$B \cap \mathbb{R}^L_{-\infty} = \emptyset$$

Suppose $z \in B \cap \mathbb{R}^L_{-\infty}$. Then

$$\exists z_i \in B_i, z = \sum_{i=1}^{I} z_i$$

Let

$$S = \{ i : z_i \neq 0 \}$$

Since $z \ll 0$, $S \neq \emptyset$. For $i \in S$, let

$$x'_i = \omega_i + z_i - \frac{z}{|S|}$$

$$x'_i \gg \omega_i + z_i \succ_i x_i \text{ (definition of } B_i)$$

$$x'_i \succ_i x_i \text{ (free disposal)}$$

$$\sum_{i \in S} x'_i = \sum_{i \in S} \omega_i + \sum_{i \in S} z_i - z$$

$$= \sum_{i \in S} \omega_i + z - z$$

$$= \sum_{i \in S} \omega_i$$

so $S$ can improve on $x$ by $x'$, so $x$ is not in the core.
• Let

\[ v = -L \left( \max_{i=1,\ldots,I} ||\omega_i||_\infty, \ldots, \max_{i=1,\ldots,I} ||\omega_i||_\infty \right) \]

Claim:

\((\text{con } B) \cap (v + R^L_-) = \emptyset\)

If \( z \in \text{con } B \), by the Shapley-Folkman Theorem, and relabelling the agents, we may write

\[ z = \sum_{i=1}^I z_i \]

\[ z_i \in \text{con } B_i \ (i = 1, \ldots, I), \]

\[ z_i \in B_i \ (i \notin \{1, \ldots, L\}) \]

Choose

\[ \hat{z}_i = \begin{cases} 0 & \text{if } i = 1, \ldots, L \\ z_i & \text{if } i = L + 1, \ldots, I \end{cases} \]

Then \( \sum_{i=1}^I \hat{z}_i \in B \) so

\[ \sum_{i=1}^I \hat{z}_i \not\ll 0 \]

If \( z \ll v \), then

\[ \sum_{i=1}^I \hat{z}_i = \sum_{i=1}^L 0 + \sum_{i=L+1}^I z_i \]

\[ \leq \sum_{i=1}^L (\omega_i + z_i) + \sum_{i=L+1}^I z_i \]

(since \( z_i \in \text{con } B_i \), \( \omega_i + z_i \in \text{con } (\omega_i + B_i) \)

\[ \subset \text{con } R^L_+ = R^L_+ \]

\[ = \sum_{i=1}^L \omega_i + \sum_{i=1}^I z_i \]

\[ = \sum_{i=1}^L \omega_i + z \]
\[ \sum_{i=1}^{L} \omega_i + v \leq 0 \]

so

\[ B \cap \mathbb{R}_-^L \neq \emptyset \]

a contradiction which proves the claim.

- By Minkowski’s Theorem, there exists \( p \neq 0 \) such that

\[
\sup p \cdot \left( v + \mathbb{R}_-^L \right) \leq \inf p \cdot (\text{con } B)
\]

If \( p_\ell < 0 \) for some \( \ell \), then

\[
\sup p \cdot \left( z + \mathbb{R}_-^L \right) = +\infty
\]

\[
\inf p \cdot (\text{con } B) \leq 0
\]

contradiction, so \( p > 0 \) and we can normalize \( p \in \Delta \).

\[
\inf p \cdot B \geq \inf p \cdot (\text{con } B)
\]

\[
\geq p \cdot v
\]

\[
= -L \max \{ \|\omega_1\|_\infty, \ldots, \|\omega_I\|_\infty \}
\]

- Adapt the remainder of the proof of the Second Welfare Theorem (requires a few tricks).