The Second Welfare Theorem in the Arrow Debreu Economy

Theorem 1 (Second Welfare Theorem) (Pure Exchange Case) If \( x^* \) is Pareto Optimal in a pure exchange economy, with strongly monotone, continuous, convex preferences, there exists a price vector \( p^* \) and an income transfer \( T \) such that \((p^*, x^*, T)\) is a Walrasian Equilibrium with Transfers.

Outline of Proof:

• Let

\[
A_i = \{ x'_i - x^*_i : x'_i \succeq_i x^*_i \}
\]

\[
A = \sum_{i=1}^{I} A_i = \{a_1 + \cdots + a_I : a_i \in A_i\}
\]

Then \(0 \notin A\) (if it were, we’d have a Pareto improvement).

• By Minkowski’s Theorem, find \( p^* \neq 0 \) such that

\[
\inf p^* \cdot A \geq 0
\]

• Show \((\mathbb{R}_+^L \setminus \{0\}) \subset A_i\) and hence \( p^* \geq 0 \).

• Show \( \inf p^* \cdot A_i = 0 \) for each \( i \).

• Define \( T \) to make \( x^*_i \) affordable at \( p^* \):

\[
T_i = p^* \cdot x^*_i - p^* \cdot \omega_i
\]

Show \( \sum_{i=1}^{I} T_i = 0 \) and

\[
x^*_i \in Q_i(p^*, T)
\]
• Use strong monotonicity to show that $p^* \gg 0$.

• Show

\[ p^* \gg 0 \Rightarrow Q_i(p^*, T) = D_i(p^*, T) \]

Now, for the details:

• Let

\[ A_i = \{ x'_i - x^*_i : x'_i \succ_i x^*_i \} \]

\[ A = \sum_{i=1}^I A_i = \{ a_1 + \cdots + a_I : a_i \in A_i \} \]

Claim:

\[ 0 \not\in A \]

If $0 \in A$, there exists $a_i \in A_i$ such that

\[ \sum_{i=1}^I a_i = 0 \]

Let

\[ x'_i = x^*_i + a_i \]

Since $x'_i - x^*_i = a_i \in A_i$, we have

\[ x'_i \succ_i x^*_i \]

\[ \sum_{i=1}^I x'_i = \sum_{i=1}^I (x^*_i + a_i) \]

\[ = \sum_{i=1}^I x^*_i + \sum_{i=1}^I a_i \]

\[ = \sum_{i=1}^I x^*_i \]

\[ = \bar{\omega} \]

Therefore, $x'$ is an exact allocation, $x'$ Pareto improves $x^*$, so $x^*$ is not Pareto Optimal, contradiction. Therefore, $0 \not\in A$. 

\[ \Box \]
\[ \exists p^* \neq 0 \quad \inf p^* \cdot A \geq 0 \]

\( A_i \) is convex, so \( A \) is convex (easy exercise). By Minkowski’s Theorem, there exists \( p^* \neq 0 \) such that

\[
0 = p^* \cdot 0 \leq \inf p^* \cdot A = \sum_{i=1}^{I} \inf p^* \cdot A_i
\]

The fact that \( \inf p^* \cdot A = \sum_{i=1}^{I} \inf p^* \cdot A_i \) is an exercise; once you figure out what you have to prove, it is obvious.

- We claim that \( p^* \geq 0 \).
Suppose not, so \( p^*_\ell < 0 \) for some \( \ell \), WLOG \( p^*_1 < 0 \). Let

\[
x'_i = x^*_i + \left( \frac{1}{p^*_1}, 0, \ldots, 0 \right)
\]

By strong monotonicity, \( x'_i \succ_i x^*_i \), so

\[
\left( \frac{1}{p^*_1}, 0, \ldots, 0 \right) \in A_i
\]

So

\[
\inf p^* \cdot A_i \leq p^* \cdot \left( \frac{1}{p^*_1}, 0, \ldots, 0 \right) = -1 < 0
\]

\[
\inf p^* \cdot A = \sum_{i=1}^{I} \inf p^* \cdot A_i \leq -I < 0
\]

a contradiction that shows \( p^* \geq 0 \).

- We claim that \( \inf p^* \cdot A_i = 0 \) for each \( i \):
Suppose \( \varepsilon > 0 \). By strong monotonicity,

\[
x^*_i + (\varepsilon, \ldots, \varepsilon) \succ_i x^*_i
\]
so

\[(\varepsilon, \ldots, \varepsilon) \in A_i\]

so

\[\inf p^* \cdot A_i \leq p^* \cdot (\varepsilon, \ldots, \varepsilon)\]

Since \(\varepsilon\) is an arbitrary positive number, \(\inf p^* \cdot A_i\) is less than every positive number, so

\[\inf p^* \cdot A_i \leq 0\]

Since \(\sum_{i=1}^{I} \inf p^* \cdot A_i \geq 0\),

\[\inf p^* \cdot A_i = 0 \ (i = 1, \ldots, I)\]

- Define \(T\) to make \(x_i^*\) affordable at \(p^*\). We claim that \(T\) is an income transfer and

\[x_i^* \in Q_i(p^*, T)\]

Let

\[T_i = p^* \cdot x_i^* - p^* \cdot \omega_i\]

\[\sum_{i=1}^{I} T_i = \sum_{i=1}^{I} (p^* \cdot x_i^* - p^* \cdot \omega_i)\]

\[= p^* \cdot \left(\sum_{i=1}^{I} x_i^* - \sum_{i=1}^{I} \omega_i\right)\]

\[= p^* \cdot (\bar{\omega} - \bar{\omega})\]

\[= 0\]

so \(T\) is an income transfer.

\[p^* \cdot x_i^* = p^* \cdot (\omega_i + (x_i^* - \omega_i))\]

\[= p^* \cdot \omega_i + p^* \cdot (x_i^* - \omega_i)\]

\[= p^* \cdot \omega_i + T_i\]
so

\[ x_i^* \in B_i(p^*, T) \]

If \( x'_i \succ_i x_i^* \), then \( x'_i - x_i^* \in A_i \), so

\[
\begin{align*}
p^* \cdot x'_i &= p^* \cdot (x_i^* + (x'_i - x_i^*)) \\
&= p^* \cdot x_i^* + p^* \cdot (x'_i - x_i^*) \\
&\geq p^* \cdot x_i^* + \inf p^* \cdot A_i \\
&= p^* \cdot x_i^* \\
&= p^* \cdot \omega_i + T_i
\end{align*}
\]

so

\[ x_i^* \in Q_i(p^*, T) \]

- Use strong monotonicity to show that \( p^* \gg 0 \).

**Lemma 2** If \( \succeq_i \) is continuous and complete, and \( x \succ_i y \), then there exists \( \varepsilon > 0 \) such that

\[
(B(x, \varepsilon) \cap X_i) \succ_i y
\]

**Proof:** If \( (B(x, \varepsilon) \cap X_i) = \{x\} \) for some \( \varepsilon > 0 \), i.e. \( x \) is an isolated point in \( X_i \), then the lemma is true, since \( x \succ_i y \).

If \( x \) is not an isolated point in \( X_i \), then we can find \( x_n \to x, x_n \in X_i, x_n \npre_i y \); by completeness, we have \( y \succeq_i x_n \) for each \( n \). Since \( \succeq_i \) is continuous, \( y \succeq_i x \), so \( x \npre_i y \), a contradiction which proves the lemma.

Since \( p^* \geq 0 \) and \( p^* \neq 0 \), \( p^* > 0 \); since in addition \( \bar{\omega} \gg 0 \), \( p^* \cdot \bar{\omega} > 0 \), so

\[
p^* \cdot \omega_i + T_i > 0 \text{ for some } i
\]

If \( p^*_\ell = 0 \) for some \( \ell \) (WLOG \( \ell = 1 \)), let

\[
x'_i = x_i^* + (1, 0, \ldots, 0)
\]
By strong monotonicity, \( x'_i \succ_i x_i^* \).

\[
p^* \cdot x'_i = p^* \cdot x_i^* = p^* \cdot \omega_i + T_i > 0
\]

Find \( \ell \) (WLOG \( \ell = 2 \)) such that

\[
p^*_\ell > 0, \ x'_2 > 0
\]

Since \( x'_i \succ_i x_i^* \), let \( \varepsilon > 0 \) be chosen to satisfy the conclusion of the Lemma. If necessary, we may make \( \varepsilon \) smaller to ensure that \( \varepsilon \leq 2x'_2 \). Let

\[
x''_i = x'_i - (0, \varepsilon/2, 0, \ldots, 0)
\]

Since \( X_i = \mathbb{R}^L_+ \), \( x''_i \in X_i \), so by the Lemma, \( x''_i \succ_i x_i^* \). But \( p^* \cdot x''_i < p \cdot x'_i = p^* \cdot \omega_i + T_i \), which shows that \( x_i^* \not\in Q_i(p^*, T) \), a contradiction which proves that \( p^* \gg 0 \).

\( \bullet \) Show

\[
p^* \gg 0 \Rightarrow Q_i(p^*, T) = D_i(p^*, T)
\]

– Case 1: \( p^* \cdot \omega_i + T_i = 0 \). Since \( p^* \gg 0 \), \( B_i(p^*, T) = \{0\} \), so

\[
Q_i(p^*, T) = D_i(p^*, T) = \{0\}
\]

– Case 2: \( p^* \cdot \omega_i + T_i > 0 \)

Suppose \( x \in Q_i(p^*, T) \) but \( x \not\in D_i(p^*, T) \). Then there exists \( z \succ_i x \) such that \( z \in B_i(p^*, T) \), hence \( p^* \cdot z \leq p^* \cdot \omega_i + T_i \). Since \( x \in Q_i(p^*, T) \), \( p^* \cdot z \geq p^* \cdot \omega_i + T_i \), so

\[
p^* \cdot z = p^* \cdot \omega_i + T_i > 0
\]

By Lemma 2, there exists \( \varepsilon > 0 \) such that

\[
|z' - z| < \varepsilon, \ z' \in \mathbb{R}^L_+ \Rightarrow z' \succ x
\]
Let 
\[ z' = z \left(1 - \frac{\varepsilon}{2|z|}\right) \]
Since \( z \in \mathbb{R}_L^L, z' \in \mathbb{R}_L^L \).

\[ |z' - z| = \frac{|\varepsilon z|}{2|z|} = \frac{\varepsilon}{2} < \varepsilon \]
so \( z' \succ x \).

\[ p^* \cdot z' = p^* \cdot z \left(1 - \frac{\varepsilon}{2|z|}\right) \]
\[ = (p^* \cdot \omega_i + T_i) \left(1 - \frac{\varepsilon}{2|z|}\right) \]
\[ < p^* \cdot \omega_i + T_i \]

which contradicts the assumption that \( x \in Q_i(p^*, T) \). This shows \( Q_i(p^*, T) \subset D_i(p^*, T) \); since clearly \( D_i(p^*, T) \subset Q_i(p^*, T) \), \( Q_i(p^*, T) = D_i(p^*, T) \).

**What if preferences are not convex?**

- Second Welfare Theorem may fail if preferences are nonconvex.
- Diagram gives an economy with two goods and two agents, and a Pareto optimum \( x^* \) so that so that the utility levels of \( x^* \) cannot be approximated by a Walrasian Equilibrium with Transfers.
- If \( p^* \) is the price which locally supports \( x^* \), and \( T \) is the income transfer which makes \( x \) affordable with respect to the prices \( p^* \), there is a unique Walrasian equilibrium with transfers \( (z^*, q^*, T) \); \( z^* \) is much more favorable to agent I and much less favorable to agent II than \( x^* \) is.
• This is the worst that can happen under standard assumptions on preferences. Given a Pareto optimum $x^*$, there is a Walrasian quasiequilibrium with transfers $(z^*, p^*, T)$ such that all but $L$ people are indifferent between $x^*$ and $z^*$. Those $L$ people are treated quite harshly (they get zero consumption). One could be less harsh and give these $L$ people carefully chosen consumption bundles in the convex hull of their quasidemand sets, but one would then have to forbid them from trading, a prohibition that would in practice be difficult to enforce.