Economics 201B

Nonconvex Preferences and Approximate Equilibria

1 The Shapley-Folkman Theorem

The Shapley-Folkman Theorem is an elementary result in linear algebra, but it is apparently unknown outside the mathematical economics literature. It is closely related to Caratheodory’s Theorem, a linear algebra result which is well known to mathematicians. The Shapley-Folkman Theorem was first published in Starr [3], an important early paper on existence of approximate equilibria with nonconvex preferences.

Theorem 1.1 (Caratheodory) Suppose \( x \in \text{con} \ A \), where \( A \subset \mathbb{R}^L \). Then there are points \( a_1, \ldots, a_{L+1} \in A \) such that \( x \in \text{con} \{a_1, \ldots, a_{L+1}\} \).

Theorem 1.2 (Shapley-Folkman) Suppose \( x \in \text{con} \ (A_1 + \cdots + A_I) \), where \( A_i \subset \mathbb{R}^L \). Then we may write \( x = a_1 + \cdots + a_I \), where \( a_i \in \text{con} \ A_i \) for all \( i \) and \( a_i \in A_i \) for all but \( L \) values of \( i \).

We derive both Caratheodory’s Theorem and the Shapley-Folkman Theorem from the following lemma:

Lemma 1.3 Suppose \( x \in \text{con} \ (A_1 + \cdots + A_I) \) where \( A_i \subset \mathbb{R}^L \). Then we may write

\[
x = \sum_{i=1}^{I} \sum_{j=0}^{m_i} \lambda_{ij} a_{ij}
\]

with \( \sum_{i=1}^{I} m_i \leq L \); \( a_{ij} \in A_i \) and \( \lambda_{ij} > 0 \) for each \( i, j \); and \( \sum_{j=0}^{m_i} \lambda_{ij} = 1 \) for each \( i \).

Proof:
1. Suppose $x \in \text{con} (A_1 + \cdots + A_I)$. Then we may write

$$x = \sum_{j=0}^{m} \lambda_j \sum_{i=1}^{I} a_{ij} = \sum_{i=1}^{I} \sum_{j=0}^{m} \lambda_j a_{ij}$$  \hspace{1cm} (2)$$

with $\lambda_j > 0$, $\sum_{j=0}^{m} \lambda_j = 1$. Letting $\lambda_{ij} = \lambda_j$ and $m_i = m$ for each $i$, we have an expression for $x$ in the form of equation 1.

2. Suppose we have any expression for $x$ in the form of equation 1 with $\sum_{i=1}^{I} m_i > L$. Then the set

$$\{a_{ij} - a_{i0} : 1 \leq i \leq I, 1 \leq j \leq m_i\}$$  \hspace{1cm} (3)$$

contains $\sum_{i=1}^{I} m_i > L$ vectors in $\mathbb{R}^L$, and hence is linearly dependent. Therefore, we can find $\beta_{ij}$ not all zero such that

$$\sum_{i=1}^{I} \sum_{j=1}^{m_i} \beta_{ij}(a_{ij} - a_{i0}) = 0.$$  \hspace{1cm} (4)$$

3. Given any $t \geq 0$, we have

$$x = \sum_{i=1}^{I} \sum_{j=0}^{m_i} \lambda_{ij} a_{ij} + t \sum_{i=1}^{I} \sum_{j=1}^{m_i} \beta_{ij}(a_{ij} - a_{i0})$$

$$= \sum_{i=1}^{I} \left[ \sum_{j=1}^{m_i} (\lambda_{ij} + t\beta_{ij}) a_{ij} + \left(\lambda_{i0} - t \sum_{j=1}^{m_i} \beta_{ij}\right) a_{i0} \right].$$  \hspace{1cm} (5)$$

Fix $i$. Observe that the sum of the coefficients of the terms $a_{i0}, \ldots, a_{im_i}$ in equation 5 is

$$\sum_{j=1}^{m_i} (\lambda_{ij} + t\beta_{ij}) + \lambda_{i0} - t \sum_{j=1}^{m_i} \beta_{ij} = \sum_{j=0}^{m_i} \lambda_{ij} = 1,$$  \hspace{1cm} (6)$$

so the expression in equation 5 is in the form of equation 1 provided that each of the coefficients is strictly positive. For $t = 0$, all coefficients are strictly positive. $\beta_{ij} \neq 0$ for some $i, j$ with $j \geq 1$; thus for $t$ sufficiently large, the coefficient of $a_{ij}$ will be either negative or will exceed 1, in which case the coefficient of some other term will be negative. Thus,
there is some $t > 0$ such that at least one of the $a_{ij}$ has a zero coefficient; let $t$ be the smallest such value. By deleting any $a_{ij}$ whose coefficients are zero, and renumbering if necessary, equation 5 becomes

$$x = \sum_{i=1}^{I} \sum_{j=0}^{\hat{m}_i} \hat{\lambda}_{ij} a_{ij}$$

(7)

with $\sum_{i=1}^{I} \hat{m}_i < \sum_{i=1}^{I} m_i$. Thus, we have an expression for $x$ in the form of equation 1, but with a smaller value of $\sum_{i=1}^{I} m_i$. Repeat this process until we obtain an expression in the form of equation 1 with $\sum_{i=1}^{I} m_i \leq L$.

Proof of Caratheodory’s Theorem: In Lemma 1.3, take $I = 1$. Then we have $x = \sum_{j=1}^{m_1} \lambda_{1j} a_{1j}$ with $m_1 - 1 \leq L$; hence, we have $x = \sum_{j=1}^{m} \lambda_{j} a_{j}$ with $m \leq L + 1$.

Proof of the Shapley-Folkman Theorem: Because $\sum_{i=1}^{I} (m_i - 1) \leq L$, we have $m_i = 1$ except for at most $L$ values of $i$. Let $a_i = \sum_{j=1}^{m_i} \lambda_{ij} a_{ij} \in \text{con } A_i$. If $m_i = 1$, $a_i = \sum_{j=1}^{1} \lambda_{ij} a_{ij} = a_{i1} \in A_i$, so equation 1 gives an expression for $x$ in the form required.

2 Existence of Approximate Walrasian Equilibrium

The material in this section is taken from Anderson, Khan and Rashid [1] and Geller [2]. The assumptions in those papers are stated in terms of strict preference relations, $\succ$, rather than weak preference relations, $\succeq$; we will follow the same formulation here.

Theorem 2.1 Suppose we are given a pure exchange economy, where for each $i = 1, \ldots, I$, $\succ_i$ satisfies

1. continuity: $\{(x, y) \in \mathbb{R}_+^L \times \mathbb{R}_+^L : x \succ_i y\}$ is relatively open in $\mathbb{R}_+^L \times \mathbb{R}_+^L$;

2. for each individual $i$, the consumption set is $\mathbb{R}_+^L$, i.e. each good is perfectly divisible, and each agent is capable of surviving on zero consumption;
3. acyclicity: there is no collection $x_1, x_2, \ldots, x_m$ such that $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_m \succ_i x_1$;

Then there exists $p^* \gg 0$ and $z^*_i \in D_i(p^*)$ such that

$$\frac{1}{I} \sum_{\ell=1}^{L} \max \left\{ \left( \sum_{i=1}^{I} z^*_i - \sum_{i=1}^{I} \omega_i \right)_\ell, 0 \right\} \leq 2 \sqrt{\frac{L}{I}} \max \{ \| \omega_i \|_1 : i = 1, \ldots, I \}$$

(8)

where $\| x \|_1 = \sum_{\ell=1}^{L} | x_\ell |$.

The proof has much in common with the proof of the Debreu-Gale-Kuhn-Nikaido Lemma. One works on a compact subset of the interior of the price simplex.\footnote{To be more precise, it is convenient to work on $\Delta' = \{ p \in \mathbb{R}_+^L : \sqrt{\frac{L}{I}} \leq p_\ell \leq 1 (1 \leq \ell \leq L) \}$.} One considers the same correspondence as in the Debreu-Gale-Kuhn-Nikaido Lemma, except that one uses the convex hull of the demand sets instead of the demand function. One finds a fixed point $(p^*, x^*)$. Use the definition of the correspondence to show that

$$\left( \sum_{i=1}^{I} x^*_i \right)_\ell \leq \sqrt{\frac{L}{I}} \max \{ \| \omega_i \|_1 : i = 1, \ldots, I \}$$

for all $\ell = 1, \ldots, L$.\footnote{This bound is related to the lower bound on prices in the definition of $\Delta'$.} From the definition of the correspondence, $x^* = \sum_{i=1}^{I} x^*_i$, where $x^*_i \in \text{con } D_i(p^*)$ for all $i = 1, \ldots, I$. Use the Shapley-Folkman Theorem to find $y^*_i$ with $\sum_{i=1}^{I} y^*_i = \sum_{i=1}^{I} x^*_i$ and $y^*_i \in D_i(p^*)$ for all but $L$ of the individuals. Let $y^*_i = y^*_i$ for all but the $L$ exceptional individuals, and let $z^*_i$ be an arbitrary element of $D_i(p^*)$ for the remaining individuals; this establishes a bound on the difference between $\sum_{i=1}^{I} z^*_i$ and $\sum_{i=1}^{I} x^*_i$, which proves the desired result.

The result can also be applied in the case of indivisibilities (i.e. nonconvexities in the consumption set). In that case, one obtains a bound on the excess quasidemand rather than demand. Even with nonconvex preferences, demand has closed graph, so Kakutani’s Theorem applies; with indivisibilities, however, demand need not have closed graph, so one needs to consider quasidemand, which does have closed graph.
References

