Part I

1. Prove that if $X$ and $Y$ are vector spaces over the same field $F$ and $\dim X = \dim Y$, then $X$ and $Y$ are isomorphic.

2. Consider the function
   \[ f(x, y) = 3x^2 + 3y^2 - 2xy + x^4 + y^5 \]

   (a) Show that \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) is a critical point of $f$.

   (b) Determine whether $f$ has a local max, a local min, or neither at \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \).

   (c) Does $f$ have a global max, a global min, or neither at \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \)?

Part II

3. Consider the Initial Value Problem
   \[ y'' = -y, \quad y(0) = y'(0) = 1 \quad (1) \]

   (a) Write this as a first order linear Initial Value Problem using the variables $y_1 = y$ and $y_2 = y'$.

   (b) Find the eigenvalues of the matrix obtained in part (a).

   (c) Find the unique solution of the Initial Value Problem in Equation (1). \textit{Hint:} you can use the product of three complex matrices if you wish, but there is a simpler approach.

   (d) Now consider the Initial Value Problem
   \[
   \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 - y_1^2/100 \\ -y_1 - y_2^2/100 \end{pmatrix}, \quad y_1(0) = y_2(0) = 1 \quad (2)
   \]

   Show that the unique stationary point for Equation (2) is \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \). Show that the linearized equation corresponding to Equation (2) is the Initial Value Problem you found in part (a). Find a function $G : \mathbb{R}^2 \to \mathbb{R}_+$ such that every solution of the linearized differential equation follows a level set of $G$. 
(e) Now suppose \( \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \) is a solution of the nonlinear Initial Value Problem in Equation (2). Compute \( \frac{dG(y(t))}{dt} \). What does this tell you about the behavior of \( y(t) \) as \( t \to \infty \)?

Part III

4. Suppose \( F : X \times \Omega \to \mathbb{R} \) is continuous, where \( X \) is a compact subset of \( E^m \) and \( \Omega \subseteq E^n \). Define

\[
\psi(\omega) = \{ x \in X : F(x, \omega) = \sup \{ F(z, \omega) : z \in X \} \}
\]

(a) Show that for all \( \omega \in \Omega \), \( \psi(\omega) \neq \emptyset \).

(b) Show that \( \psi \) is upper hemicontinuous. Hint: Suppose not. Negate the definition of upper hemicontinuity and show that this implies the existence of a sequence \( \{x_n\} \) with certain properties. Take a convergent subsequence \( \{x_{n_k}\} \), and derive a contradiction.