Economics 204
Lecture 3–Wednesday, July 29, 2009
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Section 2.1, Metric Spaces and Normed Spaces

Generalization of distance notion in $\mathbf{R}^n$

Definition 1 A *metric space* is a pair (X, d), where X is a set and $d : X \times X \to \mathbf{R}_+$, satisfying

1. $\forall_{x,y \in X}\ d(x, y) \geq 0,\ d(x, y) = 0 \Leftrightarrow x = y$
2. $\forall_{x,y \in X}\ d(x, y) = d(y, x)$
3. (*triangle inequality*)

$$\forall_{x,y,z \in X}\ d(x, y) + d(y, z) \geq d(x, z)$$

$$\begin{array}{ccccc} & & y & & \\ & \nearrow & & \searrow & \\ x & & \longrightarrow & & z \end{array}$$

Definition 2 Let V be a vector space over $\mathbf{R}$. A *norm* on V is a function $\| \cdot \| : V \to \mathbf{R}_+$ satisfying

1. $\forall_{x \in V}\ \|x\| \geq 0$
2. $\forall_{x \in V}\ \|x\| = 0 \Leftrightarrow x = 0$
3. (*triangle inequality*)

$$\forall_{x,y \in V}\ \|x + y\| \leq \|x\| + \|y\|$$

$$\begin{array}{ccccc} & & x & & \\ x & \nearrow & & \searrow & y \\ 0 & & \longrightarrow & & x + y \\ y & \searrow & & \nearrow & x \\ & & y & & \end{array}$$

4. $\forall_{\alpha \in \mathbf{R}, x \in V} \; \|\alpha x\| = |\alpha| \|x\|$

A *normed vector space* is a vector space over $\mathbf{R}$ equipped with a norm.

Theorem 3 *Let $(V, \| \cdot \|)$ be a normed vector space. Let $d : V \times V \Rightarrow \mathbf{R}_+$ be defined by*

$$d(v, w) = \|v - w\|$$

Then (V, d) is a metric space.

Proof: We must verify that d satisfies all the properties of a metric.

1.

$$\begin{aligned} d(v, w) = \|v - w\| &\geq 0 \\ d(v, w) = 0 &\Leftrightarrow \|v - w\| = 0 \\ &\Leftrightarrow v - w = 0 \\ &\Leftrightarrow (v + (-w)) + w = w \\ &\Leftrightarrow v + ((-w) + w) = w \\ &\Leftrightarrow v + 0 = w \\ &\Leftrightarrow v = w \end{aligned}$$

2. First, note that for any $x \in V$, $0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$, so $0 \cdot x = 0$. Then $0 = 0 \cdot x = (1 - 1) \cdot x = 1 \cdot x + (-1) \cdot x = x + (-1) \cdot x$, so we have $(-1) \cdot x = (-x)$.

$$\begin{aligned} d(v, w) &= \|v - w\| \\ &= |-1| \|v - w\| \\ &= \|(-1)(v + (-w))\| \\ &= \|(-1)v + (-1)(-w)\| \end{aligned}$$

$$
\begin{aligned}
&= \|-v+w\| \\
&= \|w+(-v)\| \\
&= \|w-v\| \\
&= d(w,v)
\end{aligned}
$$

3.

$$
\begin{aligned}
d(u,w) &= \|u-w\| \\
&= \|u+(-v+v)-w\| \\
&= \|u-v+v-w\| \\
&\leq \|u-v\|+\|v-w\| \\
&= d(u,v)+d(v,w)
\end{aligned}
$$

Examples of Normed Vector Spaces

- E^n: n-dimensional Euclidean space.

$$V = \mathbf{R}^n,\ \ \|x\|_2 = |x| = \sqrt{\sum_{i=1}^{n} (x_i)^2}$$

- $$V = \mathbf{R}^n,\ \ \|x\|_1 = \sum_{i=1}^{n} |x_i|$$

- $$V = \mathbf{R}^n,\ \ \|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$$

- $$C([0,1]),\ \ \|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}$$

- $$C([0,1]),\ \ \|f\|_2 = \sqrt{\int_0^1 (f(t))^2\, dt}$$

- $$C([0,1]),\ \ \|f\|_1 = \int_0^1 |f(t)|\, dt$$

Theorem 4 (Cauchy-Schwarz Inequality)
If $v, w \in \mathbf{R}^n$, then

$$\begin{aligned} \left(\sum_{i=1}^{n} v_i w_i\right)^2 &\leq \left(\sum_{i=1}^{n} v_i^2\right)\left(\sum_{i=1}^{n} w_i^2\right) \\ |v \cdot w|^2 &\leq |v|^2 |w|^2 \\ |v \cdot w| &\leq |v||w| \end{aligned}$$

Read the proof in De La Fuente. The Cauchy-Schwarz Inequality is essential in proving the triangle inequality in E^n. Note that $v \cdot w = |v||w| \cos\theta$ where θ is the angle between v and w:

$$\begin{array}{ccc} v & & w \\ \nwarrow & \theta & \nearrow \\ & 0 & \end{array}$$

Definition 5 Two norms $\|\cdot\|$ and $\|\cdot\|'$ on the same vector space V are said to be *Lipschitz-equivalent* if

$$\exists_{m,M} > 0 \; \forall_{x \in V} \; m\|x\| \leq \|x\|' \leq M\|x\|$$

Equivalently,

$$\exists_{m,M} > 0 \; \forall_{x \in V, x \neq 0} \; m \leq \frac{\|x\|'}{\|x\|} \leq M$$

Theorem 6 (10.8 on page 107 of de la Fuente)** *All norms on $\mathbf{R}^n$ are Lipschitz-equivalent.*

**The Theorem is correct, but the proof in de la Fuente has a problem.

However, infinite-dimensional spaces support norms which are not Lipschitz-equivalent. For example, on $C([0,1])$, let f_n be the function

$$f_n(t) = \begin{cases} 1 - nt & \text{if } t \in \left[0, \frac{1}{n}\right] \\ 0 & \text{if } t \in \left(\frac{1}{n}, 1\right] \end{cases}$$

Then

$$\frac{\|f_n\|_1}{\|f_n\|_\infty} = \frac{\frac{1}{2n}}{1} = \frac{1}{2n} \to 0$$

Definition 7 In a metric space (X, d), define

$$\begin{aligned} B_\varepsilon(x) &= \textit{open ball with center } x \textit{ and radius } \varepsilon \\ &= \{y \in X : d(y, x) < \varepsilon\} \\ B_\varepsilon[x] &= \textit{closed ball with center } x \textit{ and radius } \varepsilon \\ &= \{y \in X : d(y, x) \le \varepsilon\} \\ S \subseteq X & \quad \text{is } \textit{bounded} \text{ if} \\ & \quad \exists_{x \in X, \beta \in \mathbf{R}} \forall_{s \in S}\ d(s, x) \le \beta \\ \operatorname{diam}(S) &= \sup\{d(s, s') : s, s' \in S\} \\ d(A, x) &= \inf_{a \in A} d(a, x) \\ d(A, B) &= \inf_{a \in A} d(B, a) \\ &= \inf\{d(a, b) : a \in A, b \in B\} \end{aligned}$$

Note that $d(A, x)$ cannot be a metric (since a metric is a function on $X \times X$, the first and second arguments must be objects of the same type); in addition, $d(A, B)$ does not define a metric on the space of subsets of X. *Another, more useful notion of the distance between sets is the Hausdorff distance, will probably see it in 201B*

Section 2.2: Convergence of sequences in metric spaces

Definition 8 Let (X, d) be a metric space. A sequence $\{x_n\}$

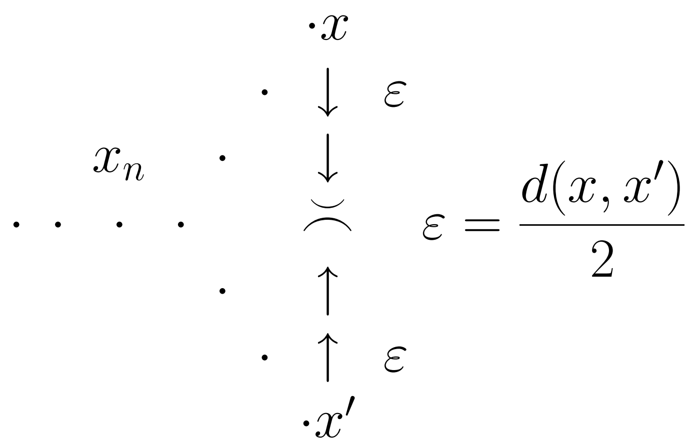
converges to x (written $x_n \longrightarrow x$ or $\lim_{n\to\infty} x_n = x$) if

$$\forall_{\varepsilon>0}\exists_{N(\varepsilon)\in\mathbf{N}}\ n > N(\varepsilon) \Rightarrow d(x_n, x) < \varepsilon$$

This is exactly the same as the definition of convergence of a sequence of real numbers, except we replace $|\cdot|$ in $\mathbf{R}$ by the metric d.

Theorem 9 (Uniqueness of Limits) *In a metric space (X, d), if $x_n \longrightarrow x$ and $x_n \longrightarrow x'$, then $x = x'$.*

Proof:



Suppose $\{x_n\}$ is a sequence in X, $x_n \longrightarrow x$, $x_n \longrightarrow x'$, $x \neq x'$. Since $x \neq x'$, $d(x, x') > 0$. Let

$$\varepsilon = \frac{d(x, x')}{2}$$

Then there exist $N(\varepsilon)$ and $N'(\varepsilon)$ such that

$$\begin{aligned} n > N(\varepsilon) &\Rightarrow d(x_n, x) < \varepsilon \\ n > N'(\varepsilon) &\Rightarrow d(x_n, x') < \varepsilon \end{aligned}$$

Choose

$$n > \max\{N(\varepsilon), N'(\varepsilon)\}$$

Then

$$d(x, x') \leq d(x, x_n) + d(x_n, x')$$

$$
\begin{aligned}
&< \varepsilon + \varepsilon \\
&= 2\varepsilon \\
&= d(x, x') \\
d(x, x') &< d(x, x')
\end{aligned}
$$

a contradiction.■

c is a *cluster point* of a sequence $\{x_n\}$ in a metric space (X, d) if

$$\forall_{\varepsilon>0}\ \{n : x_n \in B_\varepsilon(c)\} \text{ is an infinite set.}$$

Equivalently,

$$\forall_{\varepsilon>0, N\in\mathbf{N}} \exists_{n>N}\ x_n \in B_\varepsilon(c)$$

Example:

$$x_n = \begin{cases} 1 - \frac{1}{n} & \text{if } n \text{ even} \\ \frac{1}{n} & \text{if } n \text{ odd} \end{cases}$$

For n large and odd, x_n is close to zero; for n large and even, x_n is close to one. The sequence does not converge; the set of cluster points is $\{0, 1\}$.

If $\{x_n\}$ is a sequence and $n_1 < n_2 < n_3 < \cdots$, then $\{x_{n_k}\}$ is called a *subsequence.*

Note that we take some of the elements of the parent sequence, *in the same order.*

Example: $x_n = \frac{1}{n}$, so $\{x_n\} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$. If $n_k = 2k$, then $\{x_{n_k}\} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\right)$.

Theorem 10 (2.4 in De La Fuente, plus ...) *Let* (X, d) *be a metric space,* $c \in X$, *and* $\{x_n\}$ *a sequence in* X. *Then* c *is a cluster point of* $\{x_n\}$ *if and only if there is a subsequence* $\{x_{n_k}\}$ *such that* $\lim_{k\to\infty} x_{n_k} = c$.

Proof: Suppose c is a cluster point of $\{x_n\}$. We inductively construct a subsequence that converges to c. For $k = 1$, $\{n : x_n \in$

$B_1(c)\}$ is infinite, so nonempty; let

$$n_1 = \min\{n : x_n \in B_1(c)\}$$

Now, suppose we have chosen $n_1 < n_2 < \cdots < n_k$ such that

$$x_{n_j} \in B_{\frac{1}{j}}(c) \text{ for } j = 1, \ldots, k$$

$\{n : x_n \in B_{\frac{1}{k+1}}(c)\}$ is infinite, so it contains at least one element bigger than n_k, so let

$$n_{k+1} = \min\left\{n : n > n_k,\ x_n \in B_{\frac{1}{k+1}}(c)\right\}$$

Thus, we have chosen $n_1 < n_2 < \cdots < n_k < n_{k+1}$ such that

$$x_{n_j} \in B_{\frac{1}{j}}(c) \text{ for } j = 1, \ldots, k, k+1$$

Thus, by induction, we obtain a subsequence $\{x_{n_k}\}$ such that

$$x_{n_k} \in B_{\frac{1}{k}}(c)$$

Given any $\varepsilon > 0$, by the Archimedean property, there exists $N(\varepsilon) > 1/\varepsilon$.

$$\begin{aligned} k > N(\varepsilon) \ &\Rightarrow\ x_{n_k} \in B_{\frac{1}{k}}(c) \\ &\Rightarrow\ x_{n_k} \in B_\varepsilon(c) \end{aligned}$$

so

$$x_{n_k} \to c \text{ as } k \to \infty$$

Conversely, suppose that there is a subsequence $\{x_{n_k}\}$ converging to c. Given any $\varepsilon > 0$, there exists $K \in \mathbf{N}$ such that

$$k > K \Rightarrow d(x_{n_k}, c) < \varepsilon \Rightarrow x_{n_k} \in B_\varepsilon(c)$$

Therefore,

$$\{n : x_n \in B_\varepsilon(c)\} \supseteq \{n_{K+1}, n_{K+2}, n_{K+3}, \ldots\}$$

Since $n_{K+1} < n_{K+2} < n_{K+3} < \cdots$, this set is infinite, so c is a cluster point of $\{x_n\}$. ■

Section 2.3: Sequences in R and $\mathbf{R}^m$

Definition 11 A sequence of real number $\{x_n\}$ is *increasing* (*decreasing*) if $x_{n+1} \geq x_n$ ($x_{n+1} \leq x_n$) for all n.

Definition 12 If $\{x_n\}$ is a sequence of real numbers, $\{x_n\}$ *tends to infinity* (written $x_n \to \infty$ or $\lim x_n = \infty$) if

$$\forall_{K \in \mathbf{R}} \exists_{N(K)} \; n > N(K) \Rightarrow x_n > K$$

Similarly define $\lim x_n = -\infty$.

We don't say the sequence *converges* to infinity; the term "converge" is limited to the case of finite limts.

Theorem 13 (Theorem 3.1') *Let* $\{x_n\}$ *be an increasing (decreasing) sequence of real numbers. Then* $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbf{N}\}$ *(*$\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbf{N}\}$*). In particular, the limit exists.*

Proof: Read the proof in the book, and figure out how to handle the unbounded case. ■

Lim Sups and Lim Infs Handout:

Consider a sequence $\{x_n\}$ of real numbers. Let

$$\begin{aligned} \alpha_n &= \sup\{x_k : k \geq n\} \\ &= \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} \\ \beta_n &= \inf\{x_k : k \geq n\} \end{aligned}$$

Either $\alpha_n = +\infty$ for all n, or $\alpha_n \in \mathbf{R}$ and $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$.
Either $\beta_n = -\infty$ for all n, or $\beta_n \in \mathbf{R}$ and $\beta_1 \leq \beta_2 \leq \beta_3 \leq \cdots$.

Definition 14

$$\limsup_{n\to\infty} x_n = \begin{cases} +\infty & \text{if } \alpha_n = +\infty \text{ for all } n \\ \lim \alpha_n & \text{otherwise.} \end{cases}$$

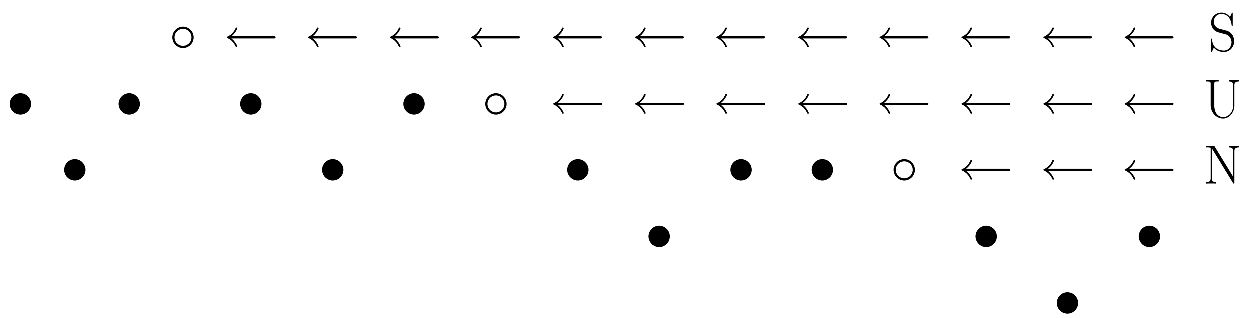
$$\liminf_{n\to\infty} x_n = \begin{cases} -\infty & \text{if } \beta_n = -\infty \text{ for all } n \\ \lim \beta_n & \text{otherwise.} \end{cases}$$

Theorem 15 *Let $\{x_n\}$ be a sequence of real numbers. Then*

$$\lim_{n\to\infty} x_n = \gamma \in \mathbf{R} \cup \{-\infty, \infty\}$$
$$\Leftrightarrow \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \gamma$$

Return to Section 2.3:

Theorem 16 (Theorem 3.2, Rising Sun Lemma) *Every sequence of real numbers contains an increasing subsequence or a decreasing subsequence or both.*



Proof: Let

$$S = \{s \in \mathbf{N} : \forall_{n>s}\ x_s > x_n\}$$

Either S is infinite, or S is finite.

If S is infinite, let

$$\begin{aligned} n_1 &= \min S \\ n_2 &= \min (S \setminus \{n_1\}) \\ n_3 &= \min (S \setminus \{n_1, n_2\}) \\ &\vdots \\ n_{k+1} &= \min (S \setminus \{n_1, n_2, \ldots, n_k\}) \end{aligned}$$

Then $n_1 < n_2 < n_3 < \cdots$.

$$\begin{aligned} x_{n_1} > x_{n_2} &\quad \text{since } n_1 \in S \text{ and } n_2 > n_1 \\ x_{n_2} > x_{n_3} &\quad \text{since } n_2 \in S \text{ and } n_3 > n_2 \\ &\quad \vdots \\ x_{n_k} > x_{n_{k+1}} &\quad \text{since } n_k \in S \text{ and } n_{k+1} > n_k \\ &\quad \vdots \end{aligned}$$

so $\{x_{n_k}\}$ is a strictly decreasing subsequence of $\{x_n\}$.

If S is finite and nonempty, let $n_1 = (\max S) + 1$; if $S = \emptyset$, let $n_1 = 1$. Then

$$\begin{aligned} n_1 \notin S &\text{ so } \exists_{n_2 > n_1}\ x_{n_2} \geq x_{n_1} \\ n_2 \notin S &\text{ so } \exists_{n_3 > n_2}\ x_{n_3} \geq x_{n_2} \\ &\vdots \\ n_k \notin S &\text{ so } \exists_{n_{k+1} > n_k}\ x_{n_{k+1}} \geq x_{n_k} \\ &\vdots \end{aligned}$$

so $\{x_{n_k}\}$ is a (weakly) increasing subsequence of $\{x_n\}$. ■

Theorem 17 (Thm. 3.3, Bolzano-Weierstrass) *Every bounded sequence of real numbers contains a convergent subsequence.*

Proof: Let $\{x_n\}$ be a bounded sequence of real numbers. By the Rising Sun Lemma, find an increasing or decreasing subsequence $\{x_{n_k}\}$. If $\{x_{n_k}\}$ is increasing, then by Theorem 3.1', $\lim x_{n_k} = \sup\{x_{n_k} : k \in \mathbf{N}\} \leq \sup\{x_n : n \in \mathbf{N}\} < \infty$, since the sequence is bounded; since the limit is finite, the subsequence converges. Similarly, if the subsequence is decreasing, it converges. ■