Section 2.1, Metric Spaces and Normed Spaces

Generalization of distance notion in $\mathbb{R}^n$

**Definition 1** A *metric space* is a pair $(X, d)$, where $X$ is a set and $d : X \times X \to \mathbb{R}_+$, satisfying

1. $\forall x, y \in X \ d(x, y) \geq 0, \ d(x, y) = 0 \iff x = y$
2. $\forall x, y \in X \ d(x, y) = d(y, x)$
3. (triangle inequality) $\forall x, y, z \in X \ d(x, y) + d(y, z) \geq d(x, z)$

**Definition 2** Let $V$ be a vector space over $\mathbb{R}$. A *norm* on $V$ is a function $\| \cdot \| : V \to \mathbb{R}_+$ satisfying

1. $\forall x \in V \ \|x\| \geq 0$
2. $\forall x \in V \ \|x\| = 0 \iff x = 0$
3. (triangle inequality) $\forall x, y \in V \ \|x + y\| \leq \|x\| + \|y\|$
4. \( \forall \alpha \in \mathbb{R}, x \in V \) \( \|\alpha x\| = |\alpha|\|x\| \)

A **normed vector space** is a vector space over \( \mathbb{R} \) equipped with a norm.

**Theorem 3** Let \((V, \| \cdot \|)\) be a normed vector space. Let \( d : V \times V \Rightarrow \mathbb{R}_+ \) be defined by

\[
d(v, w) = \|v - w\|
\]

Then \((V, d)\) is a metric space.

**Proof:** We must verify that \( d \) satisfies all the properties of a metric.

1. \[
d(v, w) = \|v - w\| \geq 0
\]

\[
d(v, w) = 0 \iff \|v - w\| = 0
\]
\[
\iff v - w = 0
\]
\[
\iff (v + (-w)) + w = w
\]
\[
\iff v + ((-w) + w) = w
\]
\[
\iff v + 0 = w
\]
\[
\iff v = w
\]

2. First, note that for any \( x \in V \), \( 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x \), so \( 0 \cdot x = 0 \). Then \( 0 = 0 \cdot x = (1 - 1) \cdot x = 1 \cdot x + (-1) \cdot x = x + (-1) \cdot x \), so we have \( (-1) \cdot x = (-x) \).

\[
d(v, w) = \|v - w\|
\]
\[
= | -1|\|v - w\|
\]
\[
= \|( -1)(v + (-w))\|
\]
\[
= \|( -1)v + (-1)(-w)\|
\]
\[ = \| - v + w \| \]
\[ = \| w + (-v) \| \]
\[ = \| w - v \| \]
\[ = d(w, v) \]

3.

\[ d(u, w) = \| u - w \| \]
\[ = \| u + (-v + v) - w \| \]
\[ = \| u - v + v - w \| \]
\[ \leq \| u - v \| + \| v - w \| \]
\[ = d(u, v) + d(v, w) \]

**Examples of Normed Vector Spaces**

- \( E^n \): \( n \)-dimensional Euclidean space.

\[ V = \mathbb{R}^n, \quad \| x \|_2 = |x| = \sqrt{\sum_{i=1}^{n} (x_i)^2} \]

- \( V = \mathbb{R}^n, \quad \| x \|_1 = \sum_{i=1}^{n} |x_i| \)

- \( V = \mathbb{R}^n, \quad \| x \|_\infty = \max\{|x_1|, \ldots, |x_n|\} \)

- \( C([0, 1]), \quad \| f \|_\infty = \sup\{|f(t)| : t \in [0, 1]\} \)

- \( C([0, 1]), \quad \| f \|_2 = \sqrt{\int_{0}^{1} (f(t))^2 \, dt} \)

- \( C([0, 1]), \quad \| f \|_1 = \int_{0}^{1} |f(t)| \, dt \)
Theorem 4 (Cauchy-Schwarz Inequality)
If \(v, w \in \mathbb{R}^n\), then
\[
\left( \sum_{i=1}^{n} v_i w_i \right)^2 \leq \left( \sum_{i=1}^{n} v_i^2 \right) \left( \sum_{i=1}^{n} w_i^2 \right)
\]
\[|v \cdot w|^2 \leq |v|^2 |w|^2\]
\[|v \cdot w| \leq |v||w|\]

Read the proof in De La Fuente. The Cauchy-Schwarz Inequality is essential in proving the triangle inequality in \(E^n\). Note that \(v \cdot w = |v||w| \cos \theta\) where \(\theta\) is the angle between \(v\) and \(w\):

\[\begin{array}{c}
v \setminus \theta \setminus \theta \setminus w \\
\downarrow \quad \theta \quad \uparrow \quad 0
\end{array}\]

Definition 5 Two norms \(\| \cdot \|\) and \(\| \cdot \|'\) on the same vector space \(V\) are said to be Lipschitz-equivalent if
\[
\exists m, M > 0 \quad \forall x \in V \quad m \|x\| \leq \|x\|' \leq M \|x\|
\]
Equivalently,
\[
\exists m, M > 0 \quad \forall x \in V, x \neq 0 \quad m \leq \frac{\|x\|'}{\|x\|} \leq M
\]

Theorem 6 (**10.8 on page 107 of de la Fuente) All norms on \(\mathbb{R}^n\) are Lipschitz-equivalent.

**The Theorem is correct, but the proof in de la Fuente has a problem.

However, infinite-dimensional spaces support norms which are not Lipschitz-equivalent. For example, on \(C([0, 1])\), let \(f_n\) be the
function 
\[ f_n(t) = \begin{cases} 
1 - nt & \text{if } t \in [0, \frac{1}{n}] \\
0 & \text{if } t \in (\frac{1}{n}, 1]
\end{cases} \]

Then 
\[ \frac{\|f_n\|_1}{\|f_n\|_\infty} = \frac{\frac{1}{2n}}{1} = \frac{1}{2n} \to 0 \]

**Definition 7** In a metric space \((X, d)\), define

\[ B_\varepsilon(x) = \text{open ball with center } x \text{ and radius } \varepsilon \]
\[ = \{y \in X : d(y, x) < \varepsilon\} \]

\[ B_\varepsilon[x] = \text{closed ball with center } x \text{ and radius } \varepsilon \]
\[ = \{y \in X : d(y, x) \leq \varepsilon\} \]

\[ S \subseteq X \text{ is bounded if} \]
\[ \exists x \in X, \beta \in \mathbb{R} \forall s \in S \ d(s, x) \leq \beta \]

\[ \text{diam} (S) = \sup\{d(s, s') : s, s' \in S\} \]

\[ d(A, x) = \inf_{a \in A} d(a, x) \]

\[ d(A, B) = \inf_{a \in A} d(B, a) \]
\[ = \inf\{d(a, b) : a \in A, b \in B\} \]

Note that \(d(A, x)\) cannot be a metric (since a metric is a function on \(X \times X\), the first and second arguments must be objects of the same type); in addition, \(d(A, B)\) does not define a metric on the space of subsets of \(X\). Another, more useful notion of the distance between sets is the Hausdorff distance, will probably see it in 201B

**Section 2.2: Convergence of sequences in metric spaces**

**Definition 8** Let \((X, d)\) be a metric space. A sequence \(\{x_n\}\)
converges to $x$ (written $x_n \to x$ or $\lim_{n \to \infty} x_n = x$) if

$$\forall \varepsilon > 0 \exists N(\varepsilon) \in \mathbb{N} \ n > N(\varepsilon) \Rightarrow d(x_n, x) < \varepsilon$$

This is exactly the same as the definition of convergence of a sequence of real numbers, except we replace $| \cdot |$ in $\mathbb{R}$ by the metric $d$.

**Theorem 9 (Uniqueness of Limits)** In a metric space $(X, d)$, if $x_n \to x$ and $x_n \to x'$, then $x = x'$.

**Proof:**

Suppose $\{x_n\}$ is a sequence in $X$, $x_n \to x$, $x_n \to x'$, $x \neq x'$. Since $x \neq x'$, $d(x, x') > 0$. Let

$$\varepsilon = \frac{d(x, x')}{2}$$

Then there exist $N(\varepsilon)$ and $N'(\varepsilon)$ such that

$$n > N(\varepsilon) \Rightarrow d(x_n, x) < \varepsilon$$

$$n > N'(\varepsilon) \Rightarrow d(x_n, x') < \varepsilon$$

Choose

$$n > \max\{N(\varepsilon), N'(\varepsilon)\}$$

Then

$$d(x, x') \leq d(x, x_n) + d(x_n, x')$$
\[
< \varepsilon + \varepsilon \\
= 2\varepsilon \\
= d(x, x') \\
\]

\[
d(x, x') < d(x, x')
\]
a contradiction. 

\[c\text{ is a cluster point of a sequence } \{x_n\}\text{ in a metric space } (X, d) \text{ if}
\]

\[\forall \varepsilon > 0 \ \{n : x_n \in B_\varepsilon (c)\} \text{ is an infinite set.}
\]

Equivalently,

\[\forall \varepsilon > 0, N \in \mathbb{N} \exists n > N \ x_n \in B_\varepsilon (c)
\]

**Example:**

\[x_n = \begin{cases} 
1 - \frac{1}{n} & \text{if } n \text{ even} \\
\frac{1}{n} & \text{if } n \text{ odd}
\end{cases}
\]

For \(n\) large and odd, \(x_n\) is close to zero; for \(n\) large and even, \(x_n\) is close to one. The sequence does not converge; the set of cluster points is \(\{0, 1\}\).

If \(\{x_n\}\) is a sequence and \(n_1 < n_2 < n_3 < \cdots\), then \(\{x_{n_k}\}\) is called a subsequence.

**Example:** \(x_n = \frac{1}{n}\), so \(\{x_n\} = (1, \frac{1}{2}, \frac{1}{3}, \ldots)\). If \(n_k = 2k\), then \(\{x_{n_k}\} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots)\).

**Theorem 10 (2.4 in De La Fuente, plus ...)** Let \((X, d)\) be a metric space, \(c \in X\), and \(\{x_n\}\) a sequence in \(X\). Then \(c\) is a cluster point of \(\{x_n\}\) if and only if there is a subsequence \(\{x_{n_k}\}\) such that \(\lim_{k \to \infty} x_{n_k} = c\).

**Proof:** Suppose \(c\) is a cluster point of \(\{x_n\}\). We inductively construct a subsequence that converges to \(c\). For \(k = 1\), \(\{n : x_n \in\)
\( B_1(c) \) is infinite, so nonempty; let
\[
    n_1 = \min \{ n : x_n \in B_1(c) \}
\]
Now, suppose we have chosen \( n_1 < n_2 < \cdots < n_k \) such that
\[
    x_{n_j} \in B_{\frac{1}{j}}(c) \text{ for } j = 1, \ldots, k
\]
\( \{ n : x_n \in B_{\frac{1}{k+1}}(c) \} \) is infinite, so it contains at least one element
bigger than \( n_k \), so let
\[
    n_{k+1} = \min \left\{ n : n > n_k, \ x_n \in B_{\frac{1}{k+1}}(c) \right\}
\]
Thus, we have chosen \( n_1 < n_2 < \cdots < n_k < n_{k+1} \) such that
\[
    x_{n_j} \in B_{\frac{1}{j}}(c) \text{ for } j = 1, \ldots, k, k+1
\]
Thus, by induction, we obtain a subsequence \( \{x_{n_k}\} \) such that
\[
    x_{n_k} \in B_{\frac{1}{k}}(c)
\]
Given any \( \varepsilon > 0 \), by the Archimedean property, there exists \( N(\varepsilon) > 1/\varepsilon \).
\[
    k > N(\varepsilon) \Rightarrow x_{n_k} \in B_{\frac{1}{k}}(c) \\
    \Rightarrow x_{n_k} \in B_{\varepsilon}(c)
\]
so
\[
    x_{n_k} \to c \text{ as } k \to \infty
\]
Conversely, suppose that there is a subsequence \( \{x_{n_k}\} \) converging to \( c \). Given any \( \varepsilon > 0 \), there exists \( K \in \mathbb{N} \) such that
\[
    k > K \Rightarrow d(x_{n_k}, c) < \varepsilon \Rightarrow x_{n_k} \in B_{\varepsilon}(c)
\]
Therefore,
\[
    \{ n : x_n \in B_{\varepsilon}(c) \} \supseteq \{ n_{K+1}, n_{K+2}, n_{K+3}, \ldots \}
\]
Since \( n_{K+1} < n_{K+2} < n_{K+3} < \cdots \), this set is infinite, so \( c \) is a cluster point of \( \{x_n\} \).

**Section 2.3: Sequences in \( \mathbb{R} \) and \( \mathbb{R}^m \)**

**Definition 11** A sequence of real number \( \{x_n\} \) is *increasing* (*decreasing*) if \( x_{n+1} \geq x_n \) \((x_{n+1} \leq x_n)\) for all \( n \).

**Definition 12** If \( \{x_n\} \) is a sequence of real numbers, \( \{x_n\} \) tends to infinity (written \( x_n \rightarrow \infty \) or \( \lim x_n = \infty \)) if

\[
\forall K \in \mathbb{R} \exists N(K) \ n > N(K) \Rightarrow x_n > K
\]

Similarly define \( \lim x_n = -\infty \).

We don’t say the sequence *converges* to infinity; the term “converge” is limited to the case of finite limits.

**Theorem 13 (Theorem 3.1’)** Let \( \{x_n\} \) be an increasing (decreasing) sequence of real numbers. Then \( \lim_{n \rightarrow \infty} x_n = \sup \{x_n : n \in \mathbb{N}\} \) \((\lim_{n \rightarrow \infty} x_n = \inf \{x_n : n \in \mathbb{N}\})\). In particular, the limit exists.

**Proof:** Read the proof in the book, and figure out how to handle the unbounded case.

**Lim Sups and Lim Infs Handout:**

Consider a sequence \( \{x_n\} \) of real numbers. Let

\[
\alpha_n = \sup \{x_k : k \geq n\} = \sup \{x_n, x_{n+1}, x_{n+2}, \ldots\}
\]

\[
\beta_n = \inf \{x_k : k \geq n\}
\]

Either \( \alpha_n = +\infty \) for all \( n \), or \( \alpha_n \in \mathbb{R} \) and \( \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots \).

Either \( \beta_n = -\infty \) for all \( n \), or \( \beta_n \in \mathbb{R} \) and \( \beta_1 \leq \beta_2 \leq \beta_3 \leq \cdots \).
Definition 14
\[
\limsup_{n \to \infty} x_n = \begin{cases} 
+\infty & \text{if } \alpha_n = +\infty \text{ for all } n \\
\lim \alpha_n & \text{otherwise.}
\end{cases}
\]
\[
\liminf_{n \to \infty} x_n = \begin{cases} 
-\infty & \text{if } \beta_n = -\infty \text{ for all } n \\
\lim \beta_n & \text{otherwise.}
\end{cases}
\]

Theorem 15 Let \( \{x_n\} \) be a sequence of real numbers. Then
\[
\lim_{n \to \infty} x_n = \gamma \in \mathbb{R} \cup \{-\infty, \infty\} \\
\iff \limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n = \gamma
\]

Return to Section 2.3:

Theorem 16 (Theorem 3.2, Rising Sun Lemma) Every sequence of real numbers contains an increasing subsequence or a decreasing subsequence or both.

Proof: Let \( S = \{s \in \mathbb{N} : \forall_{n>s} x_s > x_n\} \)
Either \( S \) is infinite, or \( S \) is finite.
If \( S \) is infinite, let
\[
n_1 = \min S \\
n_2 = \min (S \setminus \{n_1\}) \\
n_3 = \min (S \setminus \{n_1, n_2\}) \\
\vdots \\
n_{k+1} = \min (S \setminus \{n_1, n_2, \ldots, n_k\})
\]
Then \( n_1 < n_2 < n_3 < \cdots \).
\[
\begin{align*}
    x_{n_1} &> x_{n_2} \quad \text{since } n_1 \in S \text{ and } n_2 > n_1 \\
    x_{n_2} &> x_{n_3} \quad \text{since } n_2 \in S \text{ and } n_3 > n_2 \\
    &\vdots \\
    x_{n_k} &> x_{n_{k+1}} \quad \text{since } n_k \in S \text{ and } n_{k+1} > n_k \\
    &\vdots
\end{align*}
\]

so \( \{x_{n_k}\} \) is a strictly decreasing subsequence of \( \{x_n\} \).

If \( S \) is finite and nonempty, let \( n_1 = (\max S) + 1 \); if \( S = \emptyset \), let \( n_1 = 1 \). Then
\[
\begin{align*}
    n_1 &\not\in S \quad \text{so } \exists_{n_2>n_1} x_{n_2} \geq x_{n_1} \\
    n_2 &\not\in S \quad \text{so } \exists_{n_3>n_2} x_{n_3} \geq x_{n_2} \\
    &\vdots \\
    n_k &\not\in S \quad \text{so } \exists_{n_{k+1}>n_k} x_{n_{k+1}} \geq x_{n_k} \\
    &\vdots
\end{align*}
\]

so \( \{x_{n_k}\} \) is a (weakly) increasing subsequence of \( \{x_n\} \). ■

**Theorem 17 (Thm. 3.3, Bolzano-Weierstrass)** *Every bounded sequence of real numbers contains a convergent subsequence.*

**Proof:** Let \( \{x_n\} \) be a bounded sequence of real numbers. By the Rising Sun Lemma, find an increasing or decreasing subsequence \( \{x_{n_k}\} \). If \( \{x_{n_k}\} \) is increasing, then by Theorem 3.1’, \( \lim x_{n_k} = \sup\{x_{n_k} : k \in \mathbb{N}\} \leq \sup\{x_n : n \in \mathbb{N}\} < \infty \), since the sequence is bounded; since the limit is finite, the subsequence converges. Similarly, if the subsequence is decreasing, it converges. ■