Definition 1 Two sets $A, B$ in a metric space are separated if

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset$$

A set in a metric space is connected if it cannot be written as the union of two nonempty separated sets.

Example: $[0, 1)$ and $[1, 2)$ are disjoint but not separated:

$$[0, 1) \cap [1, 2) = [0, 1] \cap [1, 2) = \{1\} \neq \emptyset$$

$[0, 1)$ and $(1, 2]$ are separated:

$$[0, 1) \cap (1, 2] = [0, 1) \cap (1, 2] = \emptyset$$
$$[0, 1) \cap (1, 2] = [0, 1) \cap [1, 2] = \emptyset$$

Note that $d([0, 1), (1, 2]) = 0$ even though the sets are separated. Note that separation does not require that $\overline{A} \cap \overline{B} = \emptyset$.

$$[0, 1) \cup (1, 2]$$

is not connected.

Theorem 2 (9.2) A set $S$ of real numbers is connected if and only if it is an interval, i.e. given $x, y \in S$ and $z \in (x, y)$, then $z \in S$.

Proof: First, we show that $S$ connected implies that $S$ is an interval. We do this by proving the contrapositive: if $S$ is not an interval, it is not connected. If $S$ is not an interval, find

$$x, y \in S, \ x < z < y, \ z \notin S$$
Let 

\[ A = S \cap (-\infty, z), \quad B = S \cap (z, \infty) \]

Then

\[ \bar{A} \cap B \subseteq (-\infty, z) \cap (z, \infty) = (-\infty, z] \cap (z, \infty) = \emptyset \]

\[ A \cap \bar{B} \subseteq (-\infty, z) \cap (z, \infty) = (-\infty, z) \cap [z, \infty) = \emptyset \]

\[ A \cup B = (S \cap (-\infty, z)) \cup (S \cap (z, \infty)) \]

\[ = S \setminus \{z\} \]

\[ = S \]

\[ x \in A, \text{ so } A \neq \emptyset \]

\[ y \in B, \text{ so } B \neq \emptyset \]

so \( S \) is not connected. We have shown that if \( S \) is not an interval, then \( S \) is not connected; therefore, if \( S \) is connected, then \( S \) is an interval.

Now, we need to show that if \( S \) is an interval, it is connected. This is much like the proof of the Intermediate Value Theorem. See de la Fuente for the details.

**Theorem 3 (9.3)** Let \( X \) be a metric space, \( f : X \to Y \) continuous. If \( C \) is a connected subset of \( X \), then \( f(C) \) is connected.

**Proof:** This is problem 5(b) on Problem Set 3. The idea is in the diagram. Prove the contrapositive: if \( f(C) \) is not connected, then \( C \) is not connected.

**Corollary 4 (Intermediate Value Theorem)** If \( f : [a, b] \to \mathbb{R} \) is continuous, and \( f(a) < d < f(b) \), then there exists \( c \in (a, b) \) such that \( f(c) = d \).
\[ A = f^{-1}(P) \cap C \]
\[ B = f^{-1}(Q) \cap C \]
**Proof:** This is our third, and slickest, proof of the Intermediate Value Theorem. It is short because a substantial part of the proof was incorporated into the proof that $C \subseteq \mathbb{R}$ is connected if and only if $C$ is an interval, and the proof that if $C$ is connected, then $f(C)$ is connected. Here’s the proof: $[a, b]$ is an interval, so $[a, b]$ is connected, so $f([a, b])$ is connected, so $f([a, b])$ is an interval. $f(a) \in f([a, b])$, and $f(b) \in f([a, b])$, and $d \in [f(a), f(b)]$; since $f([a, b])$ is an interval, $d \in f([a, b])$, i.e. there exists $c \in [a, b]$ such that $f(c) = d$. Since $f(a) < d < f(b)$, $c \neq a$, $c \neq b$, so $c \in (a, b)$.

*Read on your own the material on arcwise-connectedness. Please note the discussion in the Corrections handout.*

**Section 2.10:** Read this on your own.

**Section 2.11: Continuity of Correspondences in $E^n$**

**Definition 5** A correspondence $\Psi : X \rightarrow Y$ is a function from $X$ to $2^Y$.

**Remark 6** See Item 1 on the Corrections handout. De la Fuente’s gives two inequivalent definitions of a correspondence on page 23. The first agrees with the definition we just gave, while the second requires that for all $x \in X$, $\Psi(x) \neq \emptyset$. In asserting the equivalence of the two definitions, he seems to believe, erroneously, that $\emptyset \not\in 2^Y$. In the literature, you will find the term correspondence defined in both ways, so you should check what any given author means by the term. In these lectures, we do not impose the requirement that $\Psi(x) \neq \emptyset$, since it will be convenient in Lecture 11 to consider a correspondence such that $\Psi(x) = \emptyset$ for some values of $x$. If $\Psi(x) \neq \emptyset$ for all $x$, we will say that $\Psi$ is “nonempty-valued.”

We want to talk about continuity of correspondences in a way analogous to continuity of functions. One way a function may be
discontinuous at a point \( x_0 \) is that it “jumps upward at the limit:”

\[
\exists_{x_n \to x_0} f(x_n) > \lim \sup f(x_n)
\]

It could also “jump downward at the limit:”

\[
\exists_{x_n \to x_0} f(x_n) < \lim \inf f(x_n)
\]

In either case, it doesn’t matter whether the sequence \( x_n \) approaches \( x_0 \) from the left or the right (or both). What should it mean for a set to “jump down” at the limit \( x_0 \)? It should mean the set suddenly gets smaller, i.e. it “implodes in the limit;” in other words there is a sequence \( x_n \to x_0 \) and points \( y_n \in \Psi(x_n) \) that are far from every point of \( \Psi(x_0) \). The set “jumps up” should mean that that the set suddenly gets bigger, i.e. it “explodes in the limit;” in other words, there is a point \( y \in \Psi(x_0) \) and a sequence \( x_n \to x \) such that \( y \) is far from every point of \( \Psi(x_n) \).

**Remark 7 Caution:** De la Fuente uses the term “explode” and “implode,” but not “at the limit.” For him, a set explodes if it suddenly gets bigger, which agrees with our use; however, instead of looking at whether the set explodes at the limit \( x_0 \), he looks instead at whether the set explodes as you move slightly away from the limit \( x_0 \), which is equivalent to imploding at the limit. Our approach follows the more conventional use in the literature, while de la Fuente’s use is the opposite.

**Remark 8** De la Fuente defines correspondences only with domain equalling a Euclidean space. In fact, we need correspondence defined on subsets of Euclidean space, so we need to modify his definition.
Function jumps up at $x_0$.

Function jumps down at $x_0$.

Correspondence jumps up at $x_0$.

Not UHC.

$U$ is not a neighborhood of $x_0$. 
Definition 9 \ Let \( X \subseteq \mathbb{E}^n, \ Y \subseteq \mathbb{E}^m \). Suppose \( \Psi : X \to Y \) is a correspondence.

- \( \Psi \) is upper hemicontinuous (uhc) at \( x_0 \in X \) if, for every open set \( V \supseteq \Psi(x_0) \), there is an open set \( U \) with \( x_0 \in U \) such that

\[
\Psi(x) \subseteq V \text{ for every } x \in U \cap X
\]

This says \( \Psi \) doesn’t “implode in the limit” at \( x_0 \);

- \( \Psi \) is lower hemicontinuous (lhc) at \( x_0 \in X \) if, for every open set \( V \) such that \( \Psi(x_0) \cap V \neq \emptyset \), there is an open set \( U \) with \( x_0 \in U \) such that

\[
\Psi(x) \cap V \neq \emptyset \text{ for every } x \in U \cap X
\]

This says \( \Psi \) doesn’t “explode in the limit” at \( x_0 \);

- \( \Psi \) is continuous at \( x_0 \in X \) if it is both uhc and lhc at \( x_0 \).

- \( \Psi \) is closed (has closed graph) if its graph

\[
\{(x, y) : y \in \Psi(x)\}
\]

is a closed subset of \( X \times \mathbb{E}^m \)

Note that the definition of lower hemicontinuity does not just replace \( \Psi(x_0) \subseteq V \) in the definition of upper hemicontinuity with \( V \subseteq \Psi(x_0) \); indeed, we will be very interested in correspondences in which \( \Psi(x) \) has empty interior, so there will often be no open sets \( V \) such that \( V \subseteq \Psi(x_0) \). Unfortunately, correspondences that arise in Economics are rarely continuous. The two most important concepts are upper hemicontinuity and closed graph; we will focus on these. See the drawings on the previous page.

**Example:** Consider the correspondence

\[
\Psi(x) = \begin{cases} 
\{1\} & \text{if } x \in (0, 1] \\
\{0\} & \text{if } x = 0
\end{cases}
\]
Ψ(0) = \{0\}. Let \( V = (-0.1, 0.1) \). Then \( \Psi(0) \subset V \), but no matter how close \( x \) is to 0,

\[
\Psi(x) = \left\{ \frac{1}{x} \right\} \not\subset V
\]

so \( \Psi \) is not uhc at 0. However, note that \( \Psi \) has closed graph.

**Example:** Consider the correspondence

\[
\Psi(x) = \begin{cases} \{1/x\} & \text{if } x \in (0,1] \\ \mathbb{R}_+ & \text{if } x = 0 \end{cases}
\]

\( \Psi(0) = [0, \infty) \), so any \( V \supseteq \Psi(0) \) contains \( \Psi(x) \) for all \( x \). Thus, \( \Psi \) is uhc, and has closed graph.

**Theorem 10** Let \( X \subseteq E^n \), \( Y \subseteq E^m \), \( f: X \to Y \) a function. Let \( \Psi(x) = \{f(x)\} \) for all \( x \in X \). Then \( \Psi(x) \) is uhc if and only if \( f \) is continuous.

**Proof:** Suppose \( \Psi \) is uhc. We consider the metric spaces \((X, d)\) and \((Y, d)\), where \( d \) is the Euclidean metric. Fix \( V \) open in \( Y \). Then

\[
f^{-1}(V) = \{x \in X : f(x) \in V\} = \{x \in X : \Psi(x) \subseteq V\}
\]

Thus, \( f \) is continuous if and only if \( f^{-1}(V) \) is open in \( X \) for each open \( V \) in \( Y \), if and only if \( \{x \in X : \Psi(x) \subseteq V\} \) is open in \( X \) for each open \( V \) in \( Y \), if and only if \( \Psi \) is uhc (as an exercise, think through why this last equivalence holds). \( \blacksquare \)

**Definition 11** Suppose \( X \subseteq E^m \), \( Y \subseteq E^n \). A correspondence \( \Psi: X \to Y \) is called *closed-valued* if \( \Psi(x) \) is a closed subset of \( E^n \) for all \( x \); \( \Psi \) is called *compact-valued* if \( \Psi(x) \) is compact for all \( x \).
The definition of upper hemicontinuity doesn’t handle very well correspondences which are not closed-valued; it is not hard to construct examples of pairs of correspondences which look equally well-behaved (or ill-behaved) in which one of the correspondences is uhc and the other is not. However, for closed-valued correspondences, things are much better.

**Theorem 12 (Not in de la Fuente)** Suppose $X \subseteq \mathbb{E}^n$ and $Y \subseteq \mathbb{E}^m$, and $\Psi : X \to Y$ is a correspondence.

- If $\Psi$ is closed-valued and uhc, then $\Psi$ has closed graph.

- If $Y$ is compact and $\Psi$ has closed graph, then $\Psi$ is uhc.

**Proof:** Suppose $\Psi$ is closed-valued and uhc. If $\Psi$ does not have closed graph, we can find a sequence $(x_n, y_n) \to (x_0, y_0)$, where $(x_n, y_n)$ lies in the graph of $\Psi$ (so $y_n \in \Psi(x_n)$) but $(x_0, y_0)$ does not lie in the graph of $\Psi$ (so $y_0 \notin \Psi(x_0)$). Since $\Psi$ is closed-valued, $\Psi(x_0)$ is closed; since $y_0 \notin \Psi(x_0)$, there is some $\varepsilon > 0$ such that $\Psi(x_0) \cap B_{2\varepsilon}(y_0) = \emptyset$, so $\Psi(x_0) \subseteq \mathbb{E}^n \setminus B_{2\varepsilon}[y_0]$. Let $V = \mathbb{E}^n \setminus B_{\varepsilon}[y_0]$; since $V$ is the complement of a closed set, $V$ is open, and it contains $\Psi(x_0)$. Since $\Psi$ is uhc, there is an open set $U$ with $x_0 \in U$ such that $x \in U \cap X \Rightarrow \psi(x) \subseteq V$. Since $(x_n, y_n) \to (x_0, y_0)$, $x_n \in U$ for $n$ sufficiently large, so $y_n \in \Psi(x_n) \subseteq V$, so $|y_n - y_0| \geq \varepsilon$, which shows that $y_n \not\to y_0$, so $(x_n, y_n) \not\to (x_0, y_0)$, a contradiction that shows that $\Psi$ is closed-graph.

Now, suppose $Y$ is compact and $\Psi$ has closed graph. Since $\Psi$ is closed-graph, it is closed-valued. Given $x_0 \in X$, let $V$ be any open set such that $V \supseteq \Psi(x_0)$. We need to show there exists an open set $U$ with $x_0 \in U$ such that $x \in U \cap X \Rightarrow \Psi(x) \subseteq V$. If not, we can find a sequence $x_n \to x_0$ and $y_n \in \Psi(x_n)$ such that $y_n \notin V$. Since $Y$ is compact, we can find a convergent subsequence
Then \((x_n, y_n) \to (x_0, y')\); since \(\Psi\) has closed graph, \(y' \in \Psi(x_0)\), so \(y' \in V\). Since \(V\) is open, \(y_n \in V\) for \(k\) sufficiently large, a contradiction. Thus, \(\Psi\) is uhc.

**Theorem 13 (11.2)** Suppose \(X \subseteq E^n\) and \(Y \subseteq E^m\). A compact-valued correspondence \(\Psi : X \to Y\) is uhc at \(x_0 \in X\) if and only if, for every sequence \(x_n \to x_0\), \(\{x_n\} \subseteq X\), and every sequence \(\{y_n\}\) such that \(y_n \in \Psi(x_n)\), there is a convergent subsequence \(\{y_{n_k}\}\) such that \(\lim_{k} y_{n_k} \in \Psi(x_0)\).

**Proof:** See de la Fuente. ■

**Remark 14** I don’t find the preceding sequential characterization of uhc to be very useful or intuitive, so I recommend that you bite the bullet and master the open set definition. However, the following sequential characterization of lhc is intuitive; it says that for any \(y_0 \in \Psi(x_0)\) and any \(x\) sufficiently close to \(x_0\), we may find \(y \in \Psi(x)\) such that \(y\) is close to \(y_0\).

**Theorem 15 (11.3)** A correspondence \(\Psi : X \to Y\) is lhc at \(x_0 \in X\) if and only if, for every sequence \(x_n \to x_0, \{x_n\} \subseteq X\), and every \(y_0 \in \Psi(x_0)\), there exists a companion sequence \(y_n\) with \(y_n \in \Psi(x_n)\) such that \(y_n \to y_0\).

**Proof:** See de la Fuente. ■