Exercise 1

Consider \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that \( f \in C^3(\mathbb{R}^2) \). Now let \( F(x, y, w, z) = f(x, y) - (w, z) \) and suppose that \( F(x, y, w, z) = 0 \) has solutions in \( \mathbb{R}^4 \). Let \( S \subset \mathbb{R}^4 \) be the set of solutions to this system. Show that there exists a set \( B \) such that \( B^c \) has measure zero and for \( (x, y, w, z) \in S \) where \( (w, z) \in B \), there is a local implicit function \( h : W \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) (W open) such that \( F(h(w, z), w, z) = 0 \) for all \( (w, z) \in W \) and \( h \in C^3(\mathbb{R}^2) \).

Exercise 2

Let \( f : [0, 1] \rightarrow [0, 1] \) be a correspondence defined as \( f(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{x+1}, & x \neq 0 \end{cases} \) for \( x \neq 0 \) and \( f(0) = \{1/2\} \). Does \( f \) have a fixed point? If yes, find the point(s). Does any of the fixed point theorems you have learned apply here? Explain. Answer the same questions for \( f(x) = \begin{cases} [\epsilon, 1/(x+1)], & x \in [0, 1] \end{cases} \) where \( 0 < \epsilon < 1/2 \).

Exercise 3

We say that a relation \( R \) on \( X \) is convex if whenever \( xRy \) and \( zRy \) then \( (\alpha x + (1 - \alpha)z) R y \) for all \( \alpha \in (0, 1) \). (if \( x \) and \( y \) are in \( \mathbb{R} \), \( \geq \) is an example of such relation). Let \( R_i \) be a convex relation on \( \mathbb{R}^n \) for \( i = 1, 2, ... m \), fix \( x \in \mathbb{R}^n \) and let \( B_i = \{ y - x : y R_i x, y \in \mathbb{R}^n \} \). Show that \( B_i \) is convex.

Let \( B = \bigcap_{i=1}^m B_i := \{ z_1 + z_2 + ... + z_m ; \text{such that } z_i \in B_i \text{ for all } i \} \). Show that \( B \) is convex. In the case where \( R \) is a "preference" relation (you will learn this later in Econ201B), \( 0 \notin B \) is equivalent to \( x \) being a Pareto optimal allocation. Show that in the case where \( 0 \notin B \), there exists \( p \neq 0 \) such that \( \inf(p \cdot B) \geq 0 \). This is how we construct prices in Econ201B.

Exercise 4

Show that if \( B \subset \mathbb{R}^n \) is open and convex, then \( B = \cap_{i \in I} S_i \), where \( \{S_i, i \in I\} \) is the set of all open half-spaces containing \( B \) (an open half-space in \( \mathbb{R}^n \) is a set \( S = \{ y \in \mathbb{R}^n : p \cdot y < c \} \) for some \( p \in \mathbb{R}^n, c \in \mathbb{R} \).
State whether the following functions are Lipschitz and prove your claim:

a) \( f(x) = \ln(x) \) for \( x > 0 \);

b) \( f(x) = \cos(x) \) for \( x \in \mathbb{R} \);

c) \( f : \mathbb{R} \rightarrow \mathbb{R}, f \) differentiable, such that \( \left| \frac{df}{dx} \right| \leq M \) for some \( M \in \mathbb{R} \);

If any of the functions above is not Lipschitz, what can you change to make them Lipschitz?

Consider the differential equation \( \frac{dy}{dt} = y(t) - \frac{1}{3} = 3 \) defined for all \( t \geq 0 \) and \( y(t_0) = 0 \). Does this differential equation have a solution? Is that solution unique? If yes, prove it. If not, explain why not and then modify the problem to make the solution unique.

Try to find a solution if it exists.

**Exercise 6**

Consider the following system of first order differential equations:

\[
\begin{align*}
x'(t) &= x^2 - y \\
y'(t) &= y(y - 1)
\end{align*}
\]

a) Plot the \( x'(t) = 0 \) and \( y'(t) = 0 \) curves on the \( x-y \) coordinate axes. Find the stationary point corresponding to \( x, y > 0 \).

b) Linearize the system using Taylor-series expansion around the \( x, y > 0 \) steady state. Write down the linearized equations.

c) Describe the behavior of the system and write down the general solution.