Econ 204 Section 4
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Key Words
Metric Space, Normed Vector Space, Euclidean Space, Lipschitz-Equivalent, Convergence, Cluster Point, Increasing(Decreasing) Sequence, Lim Sups(Lim Infs), Rising Sun Lemma, Bolzano-Weierstrass Theorem

Section 4.1 Metric Space

- Lecture 3 Definition 1 A metric space is a pair \((X, d)\), where \(X\) is a set and \(d : X \times X \to \mathbb{R}^+\), satisfying
  1. \(\forall x, y \in X \ d(x, y) \geq 0, d(x, y) = 0 \iff x = y\)
  2. \(\forall x, y \in X \ d(x, y) = d(y, x)\)
  3. (triangle inequality) \(\forall x, y, z \in X \ d(x, y) + d(y, z) \geq d(x, z)\)

Example 4.1.1 Let \(d(x, y) = \max\{|x - y|, 1\}\). Prove or disprove that \((\mathbb{R}, d)\) is a metric space.
Disproof:
Let \(x \in X\). Then \(d(x, x) = \max\{|x - x|, 1\} = \max\{0, 1\} = 1\). So \(d\) is not a metric.

Example 4.1.2 Let \(d(x, y) = \min\{|x - y|, 1\}\). Prove or disprove that \((\mathbb{R}, d)\) is a metric space.
Proof:: In fact this is called the standard bounded metric corresponding to \(d\).
Check the first two conditions for a metric. Do it by yourself.
Check the triangle inequality: \(d(x, z) \leq d(x, y) + d(y, z)\)
Now if either \(|x - y| \geq 1\) or \(|y - z| \geq 1\) then the right side of this inequality is at least 1;
since the left side is (by definition) at most 1, the inequality holds. It remains to consider the case in which \(|x - y| < 1\) and \(|y - z| < 1\). In this case, we have \(|x - z| \leq |x - y| + |y - z| = d(x, y) + d(y, z)|. Hence \(d(x, z) = \min\{|x - z|, 1\} \leq |x - z| \leq d(x, y) + d(y, z)\). The triangle inequality holds.

Example 4.1.3 Let \(X = [1, +\infty)\). Let \(d(x, y) = |\frac{1}{x} - \frac{1}{y}|\). Prove or disprove that \((X, d)\) is a metric space.
Proof:
Check the first two conditions for a metric
\(\forall x, y \in X, d(x, y) = |\frac{1}{x} - \frac{1}{y}| \geq 0\) and \(d(x, y) = |\frac{1}{x} - \frac{1}{y}| = 0 \iff x = y\)
\(\forall x, y \in X, d(x, y) = |\frac{1}{x} - \frac{1}{y}| = \frac{1}{x} - \frac{1}{y} = d(y, x)\)
Check the triangle inequality. We show that \(d(x, z) \leq d(x, y) + d(y, z)\) will depend upon the ordering of \(x, y,\) and \(z\).
Because \(d(x, z) = d(z, x)\), without loss of generality, we can assume \(x \leq z\).
Case 1. Suppose \(\frac{1}{x} \geq \frac{1}{y} \geq \frac{1}{z}\). Then
\(d(x, y) + d(y, z) = |\frac{1}{x} - \frac{1}{y}| + |\frac{1}{y} - \frac{1}{z}| = \frac{1}{x} - \frac{1}{y} + \frac{1}{y} - \frac{1}{z} = \frac{1}{x} - \frac{1}{z} = |\frac{1}{x} - \frac{1}{z}| = d(x, z)\)
Case 2. Suppose \(\frac{1}{x} \geq \frac{1}{y} \geq \frac{1}{z}\). Then
\(d(x, y) + d(y, z) = |\frac{1}{x} - \frac{1}{y}| + |\frac{1}{y} - \frac{1}{z}| = \frac{1}{x} - \frac{1}{y} + \frac{1}{y} - \frac{1}{z} = \frac{1}{x} - \frac{1}{z} \geq \frac{1}{z} + \frac{1}{x} - \frac{2}{y} = \frac{1}{x} - \frac{1}{z} = |\frac{1}{x} - \frac{1}{z}| = d(x, z)\)
Case 3. Suppose \(\frac{1}{y} \geq \frac{1}{z} \geq \frac{1}{x}\). Then
\(d(x, y) + d(y, z) = |\frac{1}{x} - \frac{1}{y}| + |\frac{1}{y} - \frac{1}{z}| = \frac{1}{y} - \frac{1}{x} + \frac{1}{y} - \frac{1}{z} = \frac{1}{x} - \frac{1}{z} \geq \frac{1}{x} + \frac{1}{z} - \frac{2}{y} = \frac{1}{x} - \frac{1}{z} = |\frac{1}{x} - \frac{1}{z}| = d(x, z)\)
So the triangle inequality holds.
Typically, showing the triangle inequality involves more effort. But do not forget to check the first two conditions.
Section 4.2 Normed Vector Space

- Lecture 2 Definition 2 Let $V$ be a vector space over $\mathbb{R}$. A norm on $V$ is a function $\| \|$ such that:
  1. $\forall x \in V \quad \| x \| \geq 0$
  2. $\forall x \in V \quad \| x \| = 0 \iff x = 0$
  3. (triangle inequality) $\forall x, y \in V \quad \| x + y \| \leq \| x \| + \| y \|$
  4. $\forall \alpha \in \mathbb{R}, x \in V \quad \| \alpha x \| = |\alpha| \| x \|$

**Example 4.2.1** $C([0,1])$ is the set of continuous functions from $[0,1]$ to $\mathbb{R}$. Show that $C([0,1])$ is a normed space with norm $\| f \| = \max_{x \in [0,1]} | f(x) |$

Solution:
Check the first two conditions by yourself
Check triangle inequality
$$\| f + g \| = \max_{x \in [0,1]} | f(x) + g(x) | \leq \max_{x \in [0,1]} | f(x) | + | g(x) | \leq \max_{x \in [0,1]} | f(x) | + \max_{x \in [0,1]} | g(x) | = \| f \| + \| g \|$$

**Example 4.2.8** $\mathbb{R}^n$ is a vector space. For all $x, y \in \mathbb{R}^n$, $\| x + y \| \leq \| x \| + \| y \|$. Let $f : \mathbb{R} \to \mathbb{R}^n$ be a function such that $f(x) = (x, x, \ldots, x)$. Show that $f$ is Lipschitz-equivalent on $[0,1]$. Solution:

**Section 4.3 Lipschitz-equivalent**

- Lecture 3 Definition 5 Two norms $\| \|$ and $\| \|'$ on the same vector space $V$ are said to be Lipschitz-equivalent if $\exists m, M > 0 \forall x \in V \quad m \| x \| \leq \| x \|' \leq M \| x \|$.
- Lecture 3 Theorem 6: All norms on $\mathbb{R}^n$ are Lipschitz-equivalent.

In exercise 6 of problem set 2, you are asked to reexamine the proof of De La Fuente.

**Section 4.4 Convergence and Cluster Point**

- Lecture 3 Definition 8: Let $(X, d)$ be a metric space. A sequence $x_n$ converges to $x$ if $\forall \varepsilon > 0 \exists N(\varepsilon) \in \mathbb{N}$ for all $N > N(\varepsilon) \Rightarrow d(x_n, x) < \varepsilon$. This is exactly the same as the definition of convergence of a sequence of real numbers, except we replace $| \cdot |$ in $\mathbb{R}$ by the metric $d$.

- Lecture 3 Definition Cluster Point: $c$ is a cluster point of a sequence $\{ x_n \}$ in a metric space $(X, d)$ if $\forall \varepsilon > 0 : \{ n : x_n \in B_\varepsilon(c) \}$ is an infinite set. Equivalently, $\forall \varepsilon > 0, \forall N \in \mathbb{N}, \exists n > N$ such that $x_n \in B_\varepsilon(c)$.

- Lecture 3 Theorem 10: Let $(X, d)$ be a metric space. $c \in X$ and $\{ x_n \}$ is a sequence in $X$. Then $c$ is a cluster point of $\{ x_n \}$ if and only if there is a subsequence $\{ x_{n_k} \}$ such that $\lim_{k \to \infty} x_{n_k} = c$.

**Example 4.4.1** Uniqueness of Cluster Point.
Prove that a convergent sequence in a metric space $(X, d)$ has exactly one cluster point.
Solution:
Clearly, the limit of a convergent sequence is a cluster point of the sequence, so a convergent sequence must have at least one cluster point.
Let $x_n$ be a convergent sequence in a metric space $(X, d)$, converging to $x$. Let $P$ be any point different from $x$, so $d(x, P) > 0$. We will show that $P$ is not a cluster point.
Let $\varepsilon = \frac{d(x, P)}{2}$, so $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) < \varepsilon$.
$$d(x_n, P) \geq d(x, P) - d(x_n, x) \geq 2\varepsilon - \varepsilon = \varepsilon$$
So $P$ is not a cluster point.

**Section 4.5 Sequences**
• Lecture 3 Definition 11: A sequence of real number $x_n$ is increasing (decreasing) if $x_{n+1} \geq x_n (x_{n+1} \leq x_n)$ for all $n$.

• Lecture 3 Theorem 13: Let $\{x_n\}$ be an increasing (decreasing) sequence of real numbers. The limit of $\{x_n\}$ exists.

• Lecture 3 Theorem 15 Lim Sups and Lim Infs Handout: Let $x_n$ be a sequence of real numbers. Then $\lim_{n \to \infty} x_n = \gamma \in \mathbb{R} \cup \{-\infty, \infty\} \iff \limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n = \gamma$.

• Lecture 3 Theorem 16 Rising Sun Lemma: Every sequence of real numbers contains an increasing subsequence or a decreasing subsequence or both.

• Lecture 3 Theorem 17 Bolzano-Weierstrass Theorem: Every bounded sequence of real numbers contains a convergent subsequence.

**Example 4.5.1** Lecture 3 Theorem 15 Lim Sups and Lim Infs Handout.

Prove this theorem for the case that $\gamma$ is finite.

Solution:

$(\Rightarrow)$ $x_n \to \gamma \in \mathbb{R}$ implies that $\forall \varepsilon > 0$ there exist $N(\varepsilon)$ such that $n \geq N(\varepsilon) \Rightarrow |x_n - \gamma| < \varepsilon$. This means that $\gamma + \varepsilon$ is an upper bound and $\gamma - \varepsilon$ is a lower bound for $\{x_k : k \geq N(\varepsilon)\}$.

Using $\alpha_n = \sup \{x_k : k \geq n\}$ and $\beta_n = \inf \{x_k : k \geq n\}$, we know that $\beta_n \leq \alpha_n$ (because a lower bound can’t be greater than an upper bound) and for $n > N(\varepsilon)$,

$$\gamma - \varepsilon \leq \beta_n \leq \alpha_n \leq \gamma + \varepsilon.$$ 

Since this is true for any $\varepsilon$, it must be true that $\alpha_n$ and $\beta_n$ both converge to $x$. This completes the proof that $\limsup x_n = \liminf x_n = \gamma$.

$(\Leftarrow)$ We will prove the contraposition. Suppose that $\lim_{n \to \infty} x_n \neq \gamma$. Then there exists an $\varepsilon > 0$ such that for all $N$, there is some $n \geq N$ such that $|x_n - \gamma| \geq \varepsilon$. This means that there are infinitely many $x_n$ outside of $B_\varepsilon(\gamma)$ and it must be the case that there are infinitely many of these above $\gamma + \varepsilon$, infinitely many below $\gamma - \varepsilon$ or both. If the former is true, then $\alpha_n \geq \gamma + \varepsilon$ for all $n$ which means that $\lim sup x_n$ must be greater than or equal to $\gamma + \varepsilon$. If the latter is true, then $\beta_n \leq \gamma - \varepsilon$ for all $n$, so $\liminf x_n$ must be less than or equal to $\gamma - \varepsilon$. In either case, it is not true that $\lim sup x_n = \liminf x_n = \gamma$, completing the proof.

**Example 4.5.2** Let $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2 + x_n}$. Prove that the sequence $\{x_n\}$ converges to 2.

Solution:

We show that the sequence is increasing and bounded, hence convergent. Then we calculate the limit.

Show that $\{x_n\}$ is strictly increasing by induction.

$x_2 = \sqrt{2 + \sqrt{2}} > \sqrt{2} = x_1$

Suppose $x_k - x_{k-1} > 0$ holds. $x_k^2 = 2 + x_{k-1}$

$x_{k+1} = \sqrt{2 + x_k} \Rightarrow x_{k+1}^2 = 2 + x_k$, So $(x_{k+1} + x_k)(x_{k+1} - x_k) = x_{k+1}^2 - x_k^2 = x_k - x_{k-1} > 0$. Since $x_n > 0$, $x_{k+1} - x_k > 0$

So $\{x_n\}$ is strictly increasing.

Show that $\{x_n\}$ is bounded between 0 and 3 by induction.

0 $\leq x_1 < 3$

Suppose 0 $< x_k < 3$ holds.

0 $< x_{k+1}^2 = 2 + x_k < 2 + 3 < 3^2 \Rightarrow x_{k+1} < 3$

So $\{x_n\}$ is bounded.

Hence $\{x_n\}$ converges to a finite real number $x$. $x_{n+1} = \sqrt{2 + x_n} \Rightarrow x = \sqrt{2 + x}$ $\Rightarrow x = 2$

**Example 4.5.3** Prove that every bounded sequence in $\mathbb{R}^2$ has a convergent subsequence.
Solution:
Let \((x_n, y_n)\) be a bounded sequence in \(\mathbb{R}^2\). Then, the coordinate sequences \(x_n\) and \(y_n\) must also be bounded sequences. By the Bolzano-Weierstrass theorem, there is a subsequence \(x_{n_k} \to \alpha\). Consider now the corresponding subsequence \(y_{n_k}\). By Bolzano-Weierstrass again, there is a further subsequence \(y_{n_{k_j}} \to \beta\). Since \(x_{n_{k_j}}\) is a subsequence of \(x_{n_j}\), it converges to \(\alpha\), too. It follows that the subsequence \((x_{n_{k_j}}, y_{n_{k_j}})\) converges to \((\alpha, \beta)\).

**Example 4.5.4** Prove \(-\sup a_n = \inf \{-a_n\}\)

Solution:
Note that \(a \leq b \Rightarrow -a \geq -b\). Let \(a = \inf\{-a_n\}\). Then, by definition, \(a \leq -a_m\) \forall \(m \geq n\) \Rightarrow \(-a \geq a_m, \forall m \geq n\). This implies that \(\sup a_n \leq -\inf\{-a_n\} = -a\). To show the reverse inequality, pick any \(\varepsilon > 0\). Then, by the definition of the infimum, \(\exists N > n\) such that \(-a_N < a + \varepsilon \Rightarrow a_N > a - \varepsilon \Rightarrow \sup a_n > -a - \varepsilon\). As \(\varepsilon > 0\) may be chosen to be arbitrarily small, we obtain \(\sup a_n \geq -\inf\{-a_n\}\) \Rightarrow \(\sup a_n = -\inf\{-a_n\}\). The proof that \(-\inf a_n = \sup\{-a_n\}\) follows along similar lines and is omitted.