Key Words
Open Set, Closed Set, Interior, Closure, Exterior, Boundary, Limits of Function, Continuity, Uniform Continuity, Lipschitz Functions, Homomorphism

Section 5.1 Open Set and Closed Set

- Lecture 4 Definition 1: Let \((X, d)\) be a metric space. A set \(A \subseteq X\) is open if \(\forall x \in A \exists \varepsilon > 0\) \(B_\varepsilon (x) \subseteq A\). A set \(C \subseteq X\) is closed if \(X \setminus C\) is open.
- Lecture 4 Definition 3:
  - \(\text{int} A\): the interior of \(A\), the largest open set contained in \(A\) (the union of all open sets contained in \(A\))
  - \(\overline{A}\): the closure of \(A\), the smallest closed set containing \(A\) (the intersection of all closed sets containing \(A\))
  - \(\text{ext} A\): the exterior of \(A\), the largest open set contained in \(X \setminus A\)
  - \(\partial A\): the boundary of \(A\), \((X \setminus A) \cap A\)
- Lecture 4 Theorem 2: Let \((X, d)\) be a metric space. The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open. The intersection of a finite collection of open sets is open.
- Lecture 4 Theorem 4: A set \(A\) in a metric space \((X, d)\) is closed if and only if \(\forall x \in A, \{x\} \rightarrow x \in X \Rightarrow x \in A\)

Example 5.1.1 Is it true that the intersection of an infinite collection of open sets is open. Prove the statement or give a counterexample.

Solution: False. Let \(A_n = (-1, \frac{1}{n})\). \(\cap_{n=1}^{\infty} A_n = (-1, 0]\) which is neither open nor closed. Notice that we can express a closed interval in \(\mathbb{R}\) as the intersection of open intervals. \([a, b] = \cap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})\)

Example 5.1.2 Prove that the intersection of an arbitrary collection of closed and that the union of a finite collection of closed sets is closed.

Solution: From de Morgan’s law we know that \(\neg (A \cap B) = \neg A \cup \neg B\). Then the result comes directly from Theorem 2 and the definition of closed set.

Example 5.1.3 State whether the following sets are open, closed, both, or neither:
1. \((x, 0) \mid x \in (0, 1)\) in \(\mathbb{R}^2\)
2. \((x, y, z) \mid 0 \leq x + y \leq 1, z = 0\) in \(\mathbb{R}^3\)
3. \(\{\frac{1}{n} \mid n \in \mathbb{N}\}\) in \(\mathbb{R}\)
4. \(\{\frac{1}{n} \mid n \in \mathbb{N}\}\) in \((0, \infty)\)
5. \(\mathbb{Q}\) in \(\mathbb{R}\)

Solution:

Example 5.1.4 What is the closure of \(\mathbb{Q}\) (the set of all rational numbers)

Solution:
\(\overline{\mathbb{Q}} = \mathbb{R}\).
Section 5.2

Continuity and Uniform Continuity

- Lecture 4 Definition 5: Let \((X, d)\) and \((Y, \rho)\) be metric spaces, \(f : X \to Y\). \(f\) is continuous **at a point** \(x_0 \in X\) if
  \[
  \forall \varepsilon > 0 \\exists \delta(x_0, \varepsilon) > 0 \quad d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon
  \]
  \(f\) is continuous if it is continuous at every element of its domain.

- Lecture 4 Definition 8: Suppose \(f : (X, d) \to (Y, \rho)\). \(f\) is uniformly continuous if
  \[
  \forall \varepsilon > 0 \\exists \delta(\varepsilon) > 0 \forall x_0 \in X \quad d(x, x_0) < \delta(\varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon
  \]

- Lecture 4 Theorem 6: Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces, \(f : X \to Y\). Then \(f\) is continuous if and only if \(\forall A \subseteq Y\ A\) open in \(Y\) \(\Rightarrow f^{-1}(A)\) is open in \(X\).

- Lecture 4 Theorem 7: Let \((X, d_X)\), \((Y, d_Y)\) and \((Z, d_Z)\) be metric spaces. If \(f : X \to Y\) and \(g : Y \to Z\) are continuous, then \(g \circ f : X \to Z\) is continuous.

Example 5.2.1 Uniformly Convergent

Let \(f_n\) be a sequence of real-valued functions from \(X\) to \(\mathbb{R}\). We say that \(f_n\) converges uniformly to \(f\) if for every \(\varepsilon > 0\), there exists an \(n \in \mathbb{N}\) such that for all \(x \in X\) and all \(n \geq N\), \(|f_n(x) - f(x)| < \varepsilon\). Define \(f_n : [0, 1] \to \mathbb{R}\) by the equation \(f_n(x) = x^n\). Show that the sequence \(f_n\) converges for each \(x \in [0, 1]\), but not uniformly.

Solution:
To show pointwise convergence, we will show that \(f_n \to 0\) for \(x \in [0, 1]\) and that \(f_n \to 1\) for \(x = 1\). First, let \(x \in [0, 1]\). Given \(\varepsilon > 0\), choose \(N(\varepsilon) > \log_x \varepsilon\). Then for \(n \geq N(\varepsilon)\), \(|f_n(x) - 0| = x^n \leq x^{\log_x \varepsilon} < x^{\log_x \varepsilon} = \varepsilon\). Second, let \(x = 1\). Then \(x^n = 1\) for all \(n\), so clearly \(x^n \to 1\).

The sequence \(f_n\) appears to be converging to the function

\[
  f(x) = \begin{cases} 
    0 & \text{if } x < 1 \\
    1 & \text{if } x = 1
  \end{cases}
\]
Now we will show that this convergence is not uniform. We will show that there is no single \( N(\varepsilon) \) that works for all \( x \in [0, 1] \). This is not surprising because the \( N(\varepsilon) \) we used above depended on \( x \). Let \( \varepsilon < 1/3 \). We will show that for any \( N \), there is some \( x \) such that \(|x^N - f(x)| > \varepsilon \). Because \( x^N \) is continuous for all \( n \) we know that \( x^N \) is continuous at \( x = 1 \). This means that there exists a \( \delta > 0 \) such that \(|1 - x| < \delta \Rightarrow |1 - x^N| < \varepsilon \).

Now choose \( x \in B_3(1) \cup [0, 1] \). Then \(|1 - x^N| < \varepsilon < 1/2 \) implies that \( x^N > 1/2 \). However, \(|x^N - f(x)| = |x^N - 0| = x^N > 1/2 > \varepsilon \), so it is not true that \( n \geq N \Rightarrow |x^N - f(x)| < \varepsilon \). We conclude that \( f_n \) does not converge uniformly to \( f \).

**Example 5.2.2** Let \( f_n : X \to Y \) be a sequence of continuous functions. Let \( x_n \) be a sequence of points of \( X \) converging to \( x \). Show that if the sequence \( f_n \) converges uniformly to \( f \), then \( f_n(x_n) \) converges to \( f(x) \).

**Solution:**

Our proof relies on the fact that \( f \) must itself be continuous (which we must prove). Once we establish this fact, the rest of the proof is as follows:

Given an \( \varepsilon > 0 \), continuity implies that we can choose \( \delta(\varepsilon) > 0 \) such that \( d(y, x) < \delta \Rightarrow |f(y) - f(x)| < \varepsilon \). Also, \( \{x_n\} \to x \) means that we can choose \( N_1(\delta) \) such that \( n > N_1(\delta) \) implies that \( d(x_n, x) < \delta \), so \( |f(x_n) - f(x)| < \varepsilon \). Finally, by the uniform convergence of \( \{f_n\} \), we can find \( N_2(\varepsilon) \) such that \( n > N_2(\varepsilon) \Rightarrow |f_n(x) - f(x)| < \varepsilon \) for all \( x \in X \). Let \( N = \max\{N_1(\delta(\varepsilon/2)), N_2(\varepsilon/2)\} \). Then \( n > N \) implies

\[
|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Now we must prove that \( f \) is continuous, using the fact that \( f_n \) is continuous for all \( n \) and \( \{f_n\} \) converges to \( f \) uniformly. Given \( \varepsilon > 0 \), find \( N(\varepsilon) \) such that \( n > N(\varepsilon) \Rightarrow \forall y \in X |f_n(y) - f(y)| < \varepsilon/3 \). Let \( n = N(\varepsilon) + 1 \). Since \( f_n \) is continuous, there exists \( \delta(\varepsilon) > 0 \) such that \( d(y, x) < \delta(\varepsilon) \Rightarrow |f_n(y) - f_n(x)| < \varepsilon/3 \). Then \( d(y, x) < \delta(\varepsilon) \) implies

\[
|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|,
\]

which is less than \( \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \).

**Example 5.2.3** Consider a subset \( A \) of an arbitrary metric space \((X, d)\). The distance between a point \( x \in X \) and the set \( A \) is defined as \( d(x, A) = \inf_{a \in A} d(x, a) \). Define the function \( f : X \to \mathbb{R} \) by \( f(x) = d(x, A) \), and let the metric on the range be Euclidean.

1. Show that \(|f(x) - f(y)| \leq d(x, y) \forall x, y \in X \).
2. Show that \( f \) is uniformly continuous.

**Solution:**

1. If \( f(x) = f(y) \), then the inequality holds trivially. Without loss of generality, let \( f(x) > f(y) \). If \( f(y) = 0 \), then \( y \) is in the closure of \( A \) so \( |f(x) - f(y)| = f(x) = d(x, A) = \inf_{a \in A} d(x, a) \leq d(x, y) \). Finally, if \( f(x) > f(y) > 0 \) we have to do a bit more to prove the inequality. First, note that for all \( \varepsilon > 0 \), \( \exists y_\varepsilon \in A \) such that \( d(y, A) \leq d(y, y_\varepsilon) < d(y, A) + \varepsilon \).

Furthermore, as \( \varepsilon \to 0 \), \( d(y, y_\varepsilon) \to d(y, A) \) and observe that \( d(x, y_\varepsilon) \geq d(x, A) \) for all \( \varepsilon > 0 \). Applying the triangle inequality, for all \( \varepsilon > 0 \) we have \( d(x, y_\varepsilon) \leq d(x, y) + d(y, y_\varepsilon) \). It follows then that \( d(x, A) - d(y, y_\varepsilon) \leq d(x, y) - d(y, y_\varepsilon) \leq d(x, y) \) As \( \varepsilon \to 0 \), \( d(x, A) - d(y, y_\varepsilon) \to d(x, A) - d(y, A) \). Hence \( d(x, A) - d(y, A) \leq d(x, y) \). So we have \( |f(x) - f(y)| = f(x) - f(y) = d(x, A) - d(y, A) \leq d(x, y) \).

2. Given \( \varepsilon > 0 \), take \( \delta = \varepsilon \). Now for any \( x, y \in X \), it follows that \( d(x, y) < \delta \Rightarrow |f(x) - f(y)| \leq d(x, y) < \delta = \varepsilon \). Note that does not depend on \( x \) or \( y \) so the function is uniformly continuous.

**Section 5.3** Lipschitz Functionoids and Homemorphism

- **Lecture 4 Definition 9** Let \( X, Y \) be normed vector space, \( E \subseteq X \). \( f : X \to Y \) is **lipschitz** on \( E \) if

\[
\exists K > 0 \forall x, z \in E \|f(x) - f(z)\|_Y \leq K\|x - z\|_X
\]
\( f \) is **locally Lipschitz** on \( E \) if \( \forall x_0 \in E \exists \varepsilon > 0 \) \( f \) is Lipschitz on \( B_\varepsilon(x_0) \cap E \)

Lipschitz \( \Rightarrow \) locally Lipschitz \( \Rightarrow \) continuous

Lipschitz \( \Rightarrow \) uniformly continuous

- Lecture 4 Definition 10: Let \((X, d)\) and \((Y, \rho)\) be metric spaces. A function \( f : X \to Y \) is called a homeomorphism if it is one-to-one and continuous, and its inverse function is continuous on \( f(X) \).

**Example 5.3.1**

Prove whether the following functions are Lipschitz and whether they are locally Lipschitz.

1. \( f(x) = x^2 \ x \in \mathbb{R} \)
2. \( f(x) = \sqrt{x} \ x \in [0, \infty) \)

Solution:

1. We prove that this function isn’t uniformly continuous, therefore it’s not Lipschitz.

WLOG, Choose \( x_0 > 0 \). Let \( \varepsilon = 1 > 0 \). Choose \( x = x_0 + \frac{1}{x_0} \) \( \Rightarrow |f(x_0) - f(x)| = \frac{1}{x_0^2} + 2 > \varepsilon \). So \( \delta(\varepsilon, x_0) \) must be chosen small enough so that \( \delta(\varepsilon, x_0) < \frac{1}{x_0} \) which converges to zero as \( x_0 \to \infty \). Hence \( f(x) \) is not uniformly continuous.

As for locally Lipschitz, let \( x \in B_\varepsilon(x) \). It turns out that it doesn’t matter which \( \varepsilon \) that we use; the slope of \( f \) only gets arbitrarily large if we have an infinite neighborhood (\( \varepsilon \to \infty \)), so let’s just fix \( \varepsilon \) as some positive number. Some reverse engineering shows that \( K = 2(|x| + \varepsilon) \) will work. Let’s check this. For \( y, z \in B_\varepsilon(x) \), \( ((f(y) - f(z))/(y-z)) = ((|y^2 - z^2|)/(|y-z|)) = (|(y-z)(y+z)|)/(|y-z|) = |y+z| < 2(|x| + \varepsilon) = K \).

Hence \( f(x) \) is locally Lipschitz.

2. Lipschitz implies locally Lipschitz, so not locally Lipschitz implies not Lipschitz. Suppose \( f \) is locally Lipschitz; then for the point \( x_0 = 0 \) we get an \( \varepsilon \) and a \( K \) from the definition. We will prove these can’t work. Reverse engineer the pair \( (y, 0) = (\min(\frac{x}{2}, \frac{1}{4K^2}), 0) \) and you get \( (y, 0) \in B_\varepsilon(0) \) but \( ((f(y)-f(0))/(y-0)) = \frac{\sqrt{y}}{y} = \frac{1}{\sqrt{y}} = \left(\min(\frac{x}{2}, \frac{1}{4K^2})\right)^{-\frac{1}{2}} = \max(\frac{1}{2}, 2K) \geq 2K > K \).

Hence \( f(x) \) is neither locally Lipschitz nor Lipschitz.