Handout #1: Derivation of the User Cost of Capital

Consider a firm wishing to maximize its value, defined as:

\[ V_t = \int_{s=t}^{\infty} e^{-r(s-t)} x_s ds \]

where \( x_s \) is the firm’s cash flow from real activities at date \( s \),

\[ x_s = (1 - \tau_s) p_s F(K_s) - q_s I_s (1 - k_s) + \tau_s \int_{u = s}^{\infty} D_u (s - u) q_u I_u du \]

In (2), \( K_s \) is the capital stock at date \( s \), \( I_s \) is the investment flow, \( p_s \) is the price of output, and \( q_s \) is the price of capital. The tax system has three components: \( \tau_s \) is the corporate tax rate at date \( s \), \( k_s \) is the initial subsidy to investment (e.g., an investment tax credit), and \( D_u (s - u) \) is the depreciation deduction at date \( s \) per dollar of investment at an earlier date \( u \). This deduction depends not only on the age of the asset, \( s - u \), but also on the tax rules that prevailed at date \( u \).

Inserting (2) into (1) yields:

\[ V_t = \int_{s=t}^{\infty} e^{-r(s-t)} \left( (1 - \tau_s) p_s F(K_s) - q_s I_s (1 - k_s) + \tau_s \int_{u = s}^{\infty} D_u (s - u) q_u I_u du \right) ds \]

\[ = \int_{s=t}^{\infty} e^{-r(s-t)} \left( (1 - \tau_s) p_s F(K_s) - q_s I_s (1 - k_s) + \tau_s \int_{u = s}^{t} D_u (s - u) q_u I_u du + \tau_s \int_{s = \infty}^{t} D_u (s - u) q_u I_u du \right) ds \]

\[ = \int_{s=t}^{\infty} e^{-r(s-t)} \left( (1 - \tau_s) p_s F(K_s) - q_s I_s (1 - k_s) + \tau_s \int_{u = t}^{\infty} D_u (s - u) q_u I_u du \right) ds + \bar{V} \]

where the second line breaks the flows of depreciation allowances into two pieces: those attributable to investment after date \( t \) and before date \( t \). The second piece affects the value of the firm at date \( t \), but not its decisions from date \( t \) onward, and so may be ignored. The remaining expression for firm value can be simplified by changing the order of integration for depreciation allowances (first over date of allowances, then over date of investment, rather than starting with date of investment), leading to:

\[ V_t = \int_{s=t}^{\infty} e^{-r(s-t)} \left( (1 - \tau_s) p_s F(K_s) - q_s I_s (1 - k_s) + q_s I_s \int_{u = t}^{\infty} e^{-r(u-s)} \tau_u D_u (u - s) du \right) ds + \bar{V}_t \]

\[ = \int_{s=t}^{\infty} e^{-r(s-t)} \left( (1 - \tau_s) p_s F(K_s) - q_s I_s (1 - \Gamma_s) \right) ds + \bar{V}_t \]

where \( \Gamma_s = k_s + \int_{u = t}^{\infty} e^{-r(u-t)} \tau_u D_u (u - s) du \) is the present value of tax benefits per dollar invested at date \( s \).
Expression (4) provides a general expression for the value of the firm at date $t$. Determining the optimal investment policy requires further specification of the firm’s technology. We assume that capital depreciates exponentially, so that:

\[(5) \quad \dot{K}_t = I_t - \delta K_t\]

in which $\delta$ is the rate of economic depreciation of capital, and that the full marginal cost of investment, $q$, is not affected by the level of investment. Then, inserting (5) into (4) and solving for an optimum based on the Euler equation,

\[\frac{\partial V_i}{\partial K} - \frac{d(\partial V_i/\partial \dot{K}_t)}{ds} = 0,\]

yields:

\[(6) \quad F'(K_s) = \frac{q^*_s}{p_s} \left( r + \delta - \frac{q^*_s}{q^*_s} \right); \quad \text{where } q^*_s = q_s (1 - \Gamma_s)\]

One may think of $q^*_s$ as the effective price of capital goods, taking into account the present value of tax benefits directly associated with investment.