Efficiency bounds for missing data models with semiparametric restrictions, supplemental material: details of calculations

This supplemental Appendix provides details of some of the more tedious calculations used to prove Theorem 3.1 of the paper “Efficiency bounds for missing data models with semiparametric restrictions,” by Bryan S. Graham.

G Details of calculations used in proof of Theorem 3.1

In order to calculate the bound the inverse of $\begin{pmatrix} M_{2\lambda}'V_{22}^{-1}M_{2\lambda} & M_{2\lambda}'V_{22}^{-1}M_{2\delta} \\ M_{2\delta}'V_{22}^{-1}M_{2\lambda} & M_{2\delta}'V_{22}^{-1}M_{2\delta} \end{pmatrix}$ is required (e.g., the penultimate equation in Appendix B). This inverse evaluates to

$$
\left(\begin{array}{c}
\left[\left(M_{2\lambda}'V_{22}^{-1}M_{2\lambda}\right)-\left(M_{2\lambda}'V_{22}^{-1}M_{2\delta}\right)\left(M_{2\delta}'V_{22}^{-1}M_{2\delta}\right)^{-1}\left(M_{2\delta}'V_{22}^{-1}M_{2\lambda}\right)\right]^{-1}\\
-\left[\left(M_{2\delta}'V_{22}^{-1}M_{2\delta}\right)-\left(M_{2\delta}'V_{22}^{-1}M_{2\lambda}\right)\left(M_{2\lambda}'V_{22}^{-1}M_{2\lambda}\right)^{-1}\left(M_{2\lambda}'V_{22}^{-1}M_{2\delta}\right)\right]^{-1}\\
\times\left(M_{2\delta}'V_{22}^{-1}M_{2\lambda}\right)\left(M_{2\lambda}'V_{22}^{-1}M_{2\lambda}\right)^{-1}
\end{array}\right.
$$

$$
\left.\begin{array}{c}
-\left(M_{2\lambda}'V_{22}^{-1}M_{2\lambda}\right)^{-1}\left(M_{2\lambda}'V_{22}^{-1}M_{2\delta}\right)\\
\times\left[\left(M_{2\delta}'V_{22}^{-1}M_{2\delta}\right)-\left(M_{2\delta}'V_{22}^{-1}M_{2\lambda}\right)\left(M_{2\lambda}'V_{22}^{-1}M_{2\lambda}\right)^{-1}\left(M_{2\lambda}'V_{22}^{-1}M_{2\delta}\right)\right]^{-1}\\
\left[\left(M_{2\delta}'V_{22}^{-1}M_{2\delta}\right)-\left(M_{2\delta}'V_{22}^{-1}M_{2\lambda}\right)\left(M_{2\lambda}'V_{22}^{-1}M_{2\lambda}\right)^{-1}\left(M_{2\lambda}'V_{22}^{-1}M_{2\delta}\right)\right]^{-1}
\end{array}\right)
$$

$$
=\left(\begin{array}{c}
\left[M_{2\lambda}'\left(V_{22}^{-1}-V_{22}^{-1}M_{2\delta}\left(M_{2\delta}'V_{22}^{-1}M_{2\delta}\right)^{-1}M_{2\delta}'V_{22}^{-1}\right)M_{2\lambda}\right]^{-1}\\
-\left[M_{2\delta}'\left(V_{22}^{-1}-V_{22}^{-1}M_{2\lambda}\left(M_{2\lambda}'V_{22}^{-1}M_{2\lambda}\right)^{-1}M_{2\lambda}'V_{22}^{-1}\right)M_{2\delta}\right]^{-1}\\
\times\left(M_{2\delta}'V_{22}^{-1}M_{2\lambda}\right)\left(M_{2\lambda}'V_{22}^{-1}M_{2\lambda}\right)^{-1}
\end{array}\right.
$$

$$
\left.\begin{array}{c}
-\left(M_{2\lambda}'V_{22}^{-1}M_{2\lambda}\right)^{-1}\left(M_{2\lambda}'V_{22}^{-1}M_{2\delta}\right)\\
\times\left[M_{2\delta}'\left(V_{22}^{-1}-V_{22}^{-1}M_{2\lambda}\left(M_{2\lambda}'V_{22}^{-1}M_{2\lambda}\right)^{-1}M_{2\lambda}'V_{22}^{-1}\right)M_{2\delta}\right]^{-1}\\
\left[M_{2\delta}'\left[V_{22}^{-1}-V_{22}^{-1}M_{2\lambda}\left(M_{2\lambda}'V_{22}^{-1}M_{2\lambda}\right)^{-1}M_{2\lambda}'V_{22}^{-1}\right]M_{2\delta}\right]^{-1}.
\end{array}\right).
$$

We then evaluate

$$
\begin{pmatrix} \left(\iota_L\otimes I_K\right)'M_{2\lambda} & \left(\iota_L\otimes I_K\right)'M_{2\delta} \end{pmatrix}\begin{pmatrix} M_{2\lambda}'V_{22}^{-1}M_{2\lambda} & M_{2\lambda}'V_{22}^{-1}M_{2\delta} \\ M_{2\delta}'V_{22}^{-1}M_{2\lambda} & M_{2\delta}'V_{22}^{-1}M_{2\delta} \end{pmatrix}^{-1}\begin{pmatrix} M_{2\lambda}'\left(\iota_L\otimes I_K\right) \\ M_{2\delta}'\left(\iota_L\otimes I_K\right) \end{pmatrix}
$$

as

$$
\begin{aligned}
&\left(\begin{array}{cc}
\begin{array}{c}
(\iota_L \otimes I_K)' M_{2\lambda} \left[M_{2\lambda}' \left(V_{22}^{-1} - V_{22}^{-1} M_{2\delta} \left(M_{2\delta}' V_{22}^{-1} M_{2\delta} \right)^{-1} M_{2\delta}' V_{22}^{-1} \right) M_{2\lambda} \right]^{-1} \\
- (\iota_L \otimes I_K)' M_{2\delta} \left[M_{2\delta}' \left(V_{22}^{-1} - V_{22}^{-1} M_{2\lambda} \left(M_{2\lambda}' V_{22}^{-1} M_{2\lambda} \right)^{-1} M_{2\lambda}' V_{22}^{-1} \right) M_{2\delta} \right]^{-1} \\
\times \left(M_{2\delta}' V_{22}^{-1} M_{2\lambda} \right) \left(M_{2\lambda}' V_{22}^{-1} M_{2\lambda} \right)^{-1}
\end{array}
&
\begin{array}{c}
- (\iota_L \otimes I_K)' M_{2\lambda} \left(M_{2\lambda}' V_{22}^{-1} M_{2\lambda} \right)^{-1} \left(M_{2\lambda}' V_{22}^{-1} M_{2\delta} \right) \\
\times \left[M_{2\delta}' \left(V_{22}^{-1} - V_{22}^{-1} M_{2\lambda} \left(M_{2\lambda}' V_{22}^{-1} M_{2\lambda} \right)^{-1} M_{2\lambda}' V_{22}^{-1} \right) M_{2\delta} \right]^{-1} \\
(\iota_L \otimes I_K)' M_{2\delta} \left[M_{2\delta}' \left[V_{22}^{-1} - V_{22}^{-1} M_{2\lambda} \left(M_{2\lambda}' V_{22}^{-1} M_{2\lambda} \right)^{-1} M_{2\lambda}' V_{22}^{-1} \right] M_{2\delta} \right]^{-1}
\end{array}
\end{array} \right) \\
&\times \left(\begin{array}{c} M_{2\lambda}' (\iota_L \otimes I_K) \\ M_{2\delta}' (\iota_L \otimes I_K) \end{array} \right) \\
= \;& (\iota_L \otimes I_K)' M_{2\lambda} \left[M_{2\lambda}' \left(V_{22}^{-1} - V_{22}^{-1} M_{2\delta} \left(M_{2\delta}' V_{22}^{-1} M_{2\delta} \right)^{-1} M_{2\delta}' V_{22}^{-1} \right) M_{2\lambda} \right]^{-1} M_{2\lambda}' (\iota_L \otimes I_K) \\
& - (\iota_L \otimes I_K)' M_{2\delta} \left[M_{2\delta}' \left(V_{22}^{-1} - V_{22}^{-1} M_{2\lambda} \left(M_{2\lambda}' V_{22}^{-1} M_{2\lambda} \right)^{-1} M_{2\lambda}' V_{22}^{-1} \right) M_{2\delta} \right]^{-1} \\
& \times \left(M_{2\delta}' V_{22}^{-1} M_{2\lambda} \right) \left(M_{2\lambda}' V_{22}^{-1} M_{2\lambda} \right)^{-1} M_{2\lambda}' (\iota_L \otimes I_K) \\
& - (\iota_L \otimes I_K)' M_{2\lambda} \left(M_{2\lambda}' V_{22}^{-1} M_{2\lambda} \right)^{-1} \left(M_{2\lambda}' V_{22}^{-1} M_{2\delta} \right) \\
& \times \left[M_{2\delta}' \left(V_{22}^{-1} - V_{22}^{-1} M_{2\lambda} \left(M_{2\lambda}' V_{22}^{-1} M_{2\lambda} \right)^{-1} M_{2\lambda}' V_{22}^{-1} \right) M_{2\delta} \right]^{-1} M_{2\delta}' (\iota_L \otimes I_K) \\
& + (\iota_L \otimes I_K)' M_{2\delta} \left[M_{2\delta}' \left[V_{22}^{-1} - V_{22}^{-1} M_{2\lambda} \left(M_{2\lambda}' V_{22}^{-1} M_{2\lambda} \right)^{-1} M_{2\lambda}' V_{22}^{-1} \right] M_{2\delta} \right]^{-1} M_{2\delta}' (\iota_L \otimes I_K) .
\end{aligned}
$$

This, and similar calculations, give the penultimate expression for $\mathcal{I}_{\mathrm{m}}^{\mathrm{f}}(\beta_0)$ given in the proof.

The task is now to use the expression for M and V given in the proof to evaluate $\mathcal{I}_{\mathrm{m}}^{\mathrm{f}}(\beta_0)$. We begin by evaluating

$\underset{M\times M}{M_{2\lambda}'V_{22}^{-1}M_{2\lambda}}$ as

$$
\begin{aligned}
& \begin{pmatrix} \tau_{11}\nabla_h q_{11}' & \cdots & 0 & \cdots & \tau_{I1}\nabla_h q_{I1}' & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_{1M}\nabla_h q_{1M}' & \cdots & 0 & \cdots & \tau_{IM}\nabla_h q_{IM}' \end{pmatrix} \\
& \times \begin{pmatrix} \frac{1}{\tau_{11}}\left\{\frac{\Sigma_{11}}{\rho_{11}}\right\}^{-1} & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & & & & \vdots \\ 0 & & \frac{1}{\tau_{1M}}\left\{\frac{\Sigma_{1M}}{\rho_{1M}}\right\}^{-1} & & & & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & & & & \frac{1}{\tau_{I1}}\left\{\frac{\Sigma_{I1}}{\rho_{I1}}\right\}^{-1} & & 0 \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & \frac{1}{\tau_{IM}}\left\{\frac{\Sigma_{IM}}{\rho_{IM}}\right\}^{-1} \end{pmatrix} \\
& \times \begin{pmatrix} \tau_{11}\nabla_h q_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_{1M}\nabla_h q_{1M} \\ \vdots & & \vdots \\ \tau_{I1}\nabla_h q_{I1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_{IM}\nabla_h q_{IM} \end{pmatrix} \\
= & \begin{pmatrix} \nabla_h q_{11}'\left\{\frac{\Sigma_{11}}{\rho_{11}}\right\}^{-1} & \cdots & 0 & \cdots & \nabla_h q_{I1}'\left\{\frac{\Sigma_{I1}}{\rho_{I1}}\right\}^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & \nabla_h q_{1M}'\left\{\frac{\Sigma_{1M}}{\rho_{1M}}\right\}^{-1} & \cdots & 0 & \cdots & \nabla_h q_{IM}'\left\{\frac{\Sigma_{IM}}{\rho_{IM}}\right\}^{-1} \end{pmatrix} \\
= & \times \begin{pmatrix} \tau_{11}\nabla_h q_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_{1M}\nabla_h q_{1M} \\ \vdots & & \vdots \\ \tau_{I1}\nabla_h q_{I1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_{IM}\nabla_h q_{IM} \end{pmatrix} \\
= & \begin{pmatrix} \sum_{i=1}^{I}\tau_{i1}\nabla_h q_{i1}'\left\{\frac{\Sigma_{i1}}{\rho_{i1}}\right\}^{-1}\nabla_h q_{i1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{i=1}^{I}\tau_{iM}\nabla_h q_{iM}'\left\{\frac{\Sigma_{iM}}{\rho_{iM}}\right\}^{-1}\nabla_h q_{iM} \end{pmatrix} \\
= & \begin{pmatrix} \varsigma_1\mathbb{E}\left[\left(\frac{\partial q}{\partial h'}\right)'\left[\frac{\Sigma}{p}\right]^{-1}\left(\frac{\partial q}{\partial h'}\right)\middle| X_2 = x_{2,1}\right] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varsigma_M\mathbb{E}\left[\left(\frac{\partial q}{\partial h'}\right)'\left[\frac{\Sigma}{p}\right]^{-1}\left(\frac{\partial q}{\partial h'}\right)\middle| X_2 = x_{2,M}\right] \end{pmatrix}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\underset{K\times M}{\left(\iota_L \otimes I_K\right)'} M_{2\lambda} &= -\begin{pmatrix} I_K & \cdots & I_K \end{pmatrix}' \begin{pmatrix} H_1 \\ \vdots \\ H_I \end{pmatrix} \\
&= -\sum_{i=1}^{I} \begin{pmatrix} \tau_{i1} \nabla_h q_{i1} & \cdots & \tau_{iM} \nabla_h q_{iM} \end{pmatrix} \\
&= -\left(\varsigma_1 \mathbb{E}\left[\left(\tfrac{\partial q}{\partial h'}\right)' \middle| X_2 = x_{2,1} \right] \quad \cdots \quad \varsigma_M \mathbb{E}\left[\left(\tfrac{\partial q}{\partial h'}\right)' \middle| X_2 = x_{2,M} \right] \right),
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\underset{K\times J}{\left(\iota_L \otimes I_K\right)'} M_{2\delta} &= -\begin{pmatrix} I_K & \cdots & I_K \end{pmatrix}' \begin{pmatrix} \tau_1 \nabla_\delta q_1 \\ \vdots \\ \tau_L \nabla_\delta q_L \end{pmatrix} = -\sum_{i=1}^{I} \sum_{m=1}^{M} \tau_{im} \nabla_\delta q_{im} \\
&= -\mathbb{E}\left[\mathbb{E}\left[\left(\frac{\partial q}{\partial \delta'} \right) \middle| X_2 \right] \right].
\end{aligned}
$$

Putting these three expressions together gives $\left(\iota_L \otimes I_K\right)' M_{2\lambda} \underset{K\times K}{\left(M_{2\lambda}' V_{22}^{-1} M_{2\lambda} \right)} M_{2\lambda}' \left(\iota_L \otimes I_K\right)$ equal to

$$
\begin{aligned}
& \sum_{m=1}^{M} \varsigma_m \mathbb{E}\left[\frac{\partial q}{\partial h'} \middle| X_2 = x_{2,m} \right] \mathbb{E}\left[\left(\frac{\partial q}{\partial h'} \right)' \left[\frac{\Sigma}{p} \right]^{-1} \left(\frac{\partial q}{\partial h'} \right) \middle| X_2 = x_{2,m} \right]^{-1} \mathbb{E}\left[\frac{\partial q}{\partial h'} \middle| X_2 = x_{2,m} \right]' \\
= \; & \mathbb{E}\left[\mathbb{E}\left[\frac{\partial q}{\partial h'} \middle| X_2 \right] \mathbb{E}\left[\left(\frac{\partial q}{\partial h'} \right)' \left[\frac{\Sigma}{p} \right]^{-1} \left(\frac{\partial q}{\partial h'} \right) \middle| X_2 \right]^{-1} \mathbb{E}\left[\frac{\partial q}{\partial h'} \middle| X_2 \right]' \right] \\
= \; & \mathbb{E}\left[H_0\left(X_2\right) \Upsilon_0^h \left(X_2\right)^{-1} H_0\left(X_2\right)' \right].
\end{aligned}
$$

Next evaluate $\underset{M\times J}{M_{2\lambda}' V_{22}^{-1} M_{2\delta}}$ as

$$
\begin{aligned}
& \begin{pmatrix} \tau_{11}\nabla_h q_{11}' & \cdots & 0 & \cdots & \tau_{I1}\nabla_h q_{I1}' & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_{1M}\nabla_h q_{1M}' & \cdots & 0 & \cdots & \tau_{IM}\nabla_h q_{IM}' \end{pmatrix} \\
& \times \begin{pmatrix} \frac{1}{\tau_{11}}\left\{\frac{\Sigma_{11}}{\rho_{11}}\right\}^{-1} & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & & & & \vdots \\ 0 & & \frac{1}{\tau_{1M}}\left\{\frac{\Sigma_{1M}}{\rho_{1M}}\right\}^{-1} & & & & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & & & & \frac{1}{\tau_{I1}}\left\{\frac{\Sigma_{I1}}{\rho_{I1}}\right\}^{-1} & & 0 \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & \frac{1}{\tau_{IM}}\left\{\frac{\Sigma_{IM}}{\rho_{IM}}\right\}^{-1} \end{pmatrix} \\
& \times \begin{pmatrix} \tau_1 \nabla_\delta q_1 \\ \vdots \\ \tau_L \nabla_\delta q_L \end{pmatrix} \\
= & \begin{pmatrix} \sum_{i=1}^{I} \tau_{i1}\nabla_h q_{i1}' \left\{\frac{\Sigma_{i1}}{\rho_{i1}}\right\}^{-1} \nabla_\delta q_{i1} \\ \vdots \\ \sum_{i=1}^{I} \tau_{iM}\nabla_h q_{iM}' \left\{\frac{\Sigma_{iM}}{\rho_{iM}}\right\}^{-1} \nabla_\delta q_{iM} \end{pmatrix} \\
= & \begin{pmatrix} \varsigma_1 \mathbb{E}\left[\left(\frac{\partial q}{\partial h'}\right)' \left\{\frac{\Sigma}{p}\right\}^{-1} \left(\frac{\partial q}{\partial \delta'}\right) \middle| X_2 = x_{2,1} \right] \\ \vdots \\ \varsigma_M \mathbb{E}\left[\left(\frac{\partial q}{\partial h'}\right)' \left\{\frac{\Sigma}{p}\right\}^{-1} \left(\frac{\partial q}{\partial \delta'}\right) \middle| X_2 = x_{2,M} \right] \end{pmatrix}
\end{aligned}
$$

and hence we evaluate $(\iota_L \otimes I_K)' \underbrace{\left(M_{2\delta} - M_{2\lambda}\left(M_{2\lambda}' V_{22}^{-1} M_{2\lambda}\right)^{-1} M_{2\lambda}' V_{22}^{-1} M_{2\delta}\right)}_{K\times J}$ equal to

$$
\begin{aligned}
& -\left[\mathbb{E}\left[\mathbb{E}\left[\left.\frac{\partial q}{\partial \delta'}\right| X_2\right]\right]\right. \\
& -\left(\begin{array}{ccc} \varsigma_1 \mathbb{E}\left[\left.\frac{\partial q}{\partial h'}\right| X_2 = x_{2,1}\right] & \cdots & \varsigma_M \mathbb{E}\left[\left.\frac{\partial q}{\partial h'}\right| X_2 = x_{2,M}\right] \end{array}\right) \\
& \times diag\left\{\begin{array}{ccc} \varsigma_1 \mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left[\frac{\Sigma}{p}\right]^{-1}\left(\frac{\partial q}{\partial h'}\right)\right| X_2 = x_{2,1}\right] & \cdots & \varsigma_M \mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left[\frac{\Sigma}{p}\right]^{-1}\left(\frac{\partial q}{\partial h'}\right)\right| X_2 = x_{2,M}\right] \end{array}\right\}^{-1} \\
& \times \left(\begin{array}{c} \varsigma_1 \mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left\{\frac{\Sigma}{p}\right\}^{-1}\left(\frac{\partial q}{\partial \delta'}\right)\right| X_2 = x_{2,1}\right] \\ \vdots \\ \varsigma_M \mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left\{\frac{\Sigma}{p}\right\}^{-1}\left(\frac{\partial q}{\partial \delta'}\right)\right| X_2 = x_{2,M}\right] \end{array}\right) \\
= & -\mathbb{E}\left[\mathbb{E}\left[\left.\frac{\partial q}{\partial \delta'}\right| X_2\right]\right. \\
& \left. -\mathbb{E}\left[\left.\frac{\partial q}{\partial h'}\right| X_2\right] \mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left[\frac{\Sigma}{p}\right]^{-1}\left(\frac{\partial q}{\partial h'}\right)\right| X_2\right]^{-1} \mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left\{\frac{\Sigma}{p}\right\}^{-1}\left(\frac{\partial q}{\partial \delta'}\right)\right| X_2\right]\right] \\
= & -\mathbb{E}\left[\frac{\partial q_0(X)}{\partial \delta'} - \left(\frac{\partial q_0(X)}{\partial h'}\right) \Upsilon_0^h (X_2)^{-1} \Upsilon_0^{h\delta}(X_2)\right] \\
= & -\mathbb{E}\left[G_0(X)\right].
\end{aligned}
$$

Similarly we have

$$
\underbrace{M_{2\delta}' V_{22}^{-1} M_{2\delta}^{-1}}_{J\times J} = \mathbb{E}\left[\mathbb{E}\left[\left.\left(\frac{\partial q}{\partial \delta'}\right)' \left\{\frac{\Sigma}{p}\right\}^{-1}\left(\frac{\partial q}{\partial \delta'}\right)\right| X_2\right]\right] = \mathbb{E}\left[\Upsilon_0^{\delta}(X_2)\right]
$$

and hence $\underbrace{M_{2\delta}' V_{22}^{-1} M_{2\lambda}\left(M_{2\lambda}' V_{22}^{-1} M_{2\lambda}\right)^{-1} M_{2\lambda}' V_{22}^{-1} M_{2\delta}^{-1}}_{J\times J}$ equal to

$$
\begin{aligned}
& \left(\begin{array}{ccc} \varsigma_1 \mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left\{\frac{\Sigma}{p}\right\}^{-1}\left(\frac{\partial q}{\partial \delta'}\right)\right| X_2 = x_{2,1}\right]' & \cdots & \varsigma_M \mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left\{\frac{\Sigma}{p}\right\}^{-1}\left(\frac{\partial q}{\partial \delta'}\right)\right| X_2 = x_{2,M}\right]' \end{array}\right) \\
& \times \left(\begin{array}{ccc} \varsigma_1 \mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left[\frac{\Sigma}{p}\right]^{-1}\left(\frac{\partial q}{\partial h'}\right)\right| X_2 = x_{2,1}\right] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varsigma_M \mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left[\frac{\Sigma}{p}\right]^{-1}\left(\frac{\partial q}{\partial h'}\right)\right| X_2 = x_{2,M}\right] \end{array}\right)^{-1} \\
& \times \left(\begin{array}{c} \varsigma_1 \mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left\{\frac{\Sigma}{p}\right\}^{-1}\left(\frac{\partial q}{\partial \delta'}\right)\right| X_2 = x_{2,1}\right] \\ \vdots \\ \varsigma_M \mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left\{\frac{\Sigma}{p}\right\}^{-1}\left(\frac{\partial q}{\partial \delta'}\right)\right| X_2 = x_{2,M}\right] \end{array}\right) \\
= & \mathbb{E}\left[\mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left\{\frac{\Sigma}{p}\right\}^{-1}\left(\frac{\partial q}{\partial \delta'}\right)\right| X_2\right]' \mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left[\frac{\Sigma}{p}\right]^{-1}\left(\frac{\partial q}{\partial h'}\right)\right| X_2\right]^{-1} \mathbb{E}\left[\left.\left(\frac{\partial q}{\partial h'}\right)' \left\{\frac{\Sigma}{p}\right\}^{-1}\left(\frac{\partial q}{\partial \delta'}\right)\right| X_2\right]\right] \\
= & \mathbb{E}\left[\Upsilon_0^{h\delta}(X_2)' \Upsilon_0^h (X_2)^{-1} \Upsilon_0^{h\delta}(X_2)\right],
\end{aligned}
$$

which then gives $M_{2\delta}'V_{22}^{-1}M_{2\delta}^{-1} - M_{2\delta}'V_{22}^{-1}M_{2\lambda}\left(M_{2\lambda}'V_{22}^{-1}M_{2\lambda}\right)^{-1}M_{2\lambda}'V_{22}^{-1}M_{2\delta}^{-1}$ equal to

$$
\begin{aligned}
& \mathbb{E}\Bigg[\mathbb{E}\left[\left(\frac{\partial q}{\partial \delta'}\right)'\left\{\frac{\Sigma}{p}\right\}^{-1}\left(\frac{\partial q}{\partial \delta'}\right)\Bigg| X_2\right] \\
& - \mathbb{E}\left[\left(\frac{\partial q}{\partial h'}\right)'\left\{\frac{\Sigma}{p}\right\}^{-1}\left(\frac{\partial q}{\partial \delta'}\right)\Bigg| X_2\right]'\mathbb{E}\left[\left(\frac{\partial q}{\partial h'}\right)'\left[\frac{\Sigma}{p}\right]^{-1}\left(\frac{\partial q}{\partial h'}\right)\Bigg| X_2\right]^{-1}\mathbb{E}\left[\left(\frac{\partial q}{\partial h'}\right)'\left\{\frac{\Sigma}{p}\right\}^{-1}\left(\frac{\partial q}{\partial \delta'}\right)\Bigg| X_2\right]\Bigg] \\
= \; & \mathbb{E}\left[\Upsilon_0^{\delta}(X_2) - \Upsilon_0^{h\delta}(X_2)'\,\Upsilon_0^{h}(X_2)^{-1}\,\Upsilon_0^{h\delta}(X_2)\right] \\
= \; & \mathcal{I}_{\mathrm{m}}^{\mathrm{f}}(\delta_0)\,.
\end{aligned}
$$

The result then follows directly.