Lecture 8: Using Network Structure to Identify Peer Spillovers

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The notion that an individual's choices are affected by the behavior and/or attributes of her peers is a natural one. Individual's may have a taste for conformity, such that they experience disutility when their actions deviate too far from established social norms (e.g., Akerlof, 1997). Since we often learn about new technologies, products and ideas from our friends, peer groups may also play an important role in the spread of new technologies (e.g., Conley and Udry, 2010). Finally, for certain behaviors there may be complementaries in actions across peers. For example, the returns to effort may be increasing in the effort of one's classmates, teammates or colleagues.

This note describes methods of inferring the presence, and identifying the size of, peer group effects using network data. The focus is on the so-called "linear-in-means" model of social interactions. This model has served as the organizing framework of much of the empirical literature on peer group effects (especially in the economics of education field, e.g., Angrist and Lang (2004)).

Formal econometric analysis of the linear-in-means model begins with Charles Manski's (1993) well-known "reflection-problem" paper. Subsequent researchers have extended Manski's basic analysis in various ways (e.g., Brock and Durlauf, 2001; Bramoulle, Djebbari and Fortin, 2009). This last paper, in particular, shows how the availability of detailed network data on who is connected to whom, substantially increases the set of conditions under which positive identification results are possible. In a recent working paper, Blume, Brock, Durlauf and Jayaraman (2012) survey and extend prior work. My exposition is synoptic, drawing on all the papers cited above, and introducing a few (very) small new ideas along the way.

Game theoretic analyses of models of social interactions that yield linear best reply functions are extensively surveyed in Jackson and Zenou (forthcoming). Such models provide rigorous micro-foundations for empirical analyses based on the linear-in-means model.

Linear-in-Means Model

Consider a network consisting of N agents. The econometrician observes *all* ties within this network. These ties are encoded in the $N \times N$ adjacency matrix $\mathbf{D}$. For simplicity I assume that ties are binary-valued and undirected. For each agent the econometrician also observes the K-vector of attributes X_i, as well as her chosen action Y_i. Let $\mathbf{X}$ and $\mathbf{Y}$ be the $N \times K$ and $N \times 1$ matrix and vector of agent attributes and equilibrium actions respectively.

Let $\mathbf{G} = \mathrm{diag}\left(\mathbf{D}\iota_{\mathrm{N}}\right)^{-1}\mathbf{D}$ be the **row-normalized network adjacency matrix** (i.e., the network adjacency matrix where each element of the i^{th} row is divided by D_{i+}, the i^{th} agent's degree). Note that all rows of this matrix sum to 1 by construction. The matrix is row-stochastic.

Let

$$
\begin{aligned}
\mathbf{G}_i\mathbf{y} &= \sum_{j \neq i} G_{ij} y_j \stackrel{def}{\equiv} \bar{y}_{n(i)} \\
\mathbf{G}_i\mathbf{X} &= \sum_{j \neq i} G_{ij} X_j \stackrel{def}{\equiv} \bar{X}_{n(i)}
\end{aligned}
$$

respectively equal the average action of player i's peers under the (perhaps hypothetical) action profile $\mathbf{y}$ and the average of her peers' attribute vectors. Here $\mathbf{G}_i$ denotes the i^{th} row of $\mathbf{G}$.

Assume that the utility agent i receives from action profile $\mathbf{y}$ given network structure $(\mathbf{D})$ and agent attributes $(\mathbf{X})$ is

$$
\begin{aligned}
u_i\left(\mathbf{y};\mathbf{D},\mathbf{X}\right) &= v_i\left(\mathbf{D},\mathbf{X}\right) y_i - \frac{1}{2} y_i^2 + \beta \bar{y}_{n(i)} y_i \\
&= v_i\left(\mathbf{D},\mathbf{X}\right) y_i - \frac{1}{2} y_i^2 + \beta \mathbf{G}_i \mathbf{y} y_i \qquad (1)
\end{aligned}
$$

with $|\beta| < 1$ and $v_i\left(\mathbf{D},\mathbf{X}\right)$ equal to

$$
\begin{aligned}
v_i\left(\mathbf{D},\mathbf{X}\right) &= X_i'\gamma + \bar{X}_{n(i)}'\delta + A + U_i \\
&= X_i'\gamma + \left(\mathbf{G}_i\mathbf{X}\right)'\delta + A + U_i
\end{aligned}
$$

Assume that the observed action $\mathbf{Y}$ corresponds to a Nash equilibrium where no agent can increase her utility by changing her action given the actions of all other agents in the network. The econometrician observes the triple $(\mathbf{Y},\mathbf{X},\mathbf{D})$; she does not observe A, nor does she observe $\mathbf{U}$, the $N \times 1$ vector of individual-level heterogeneity terms. Unlike the econometrician, we assume that agents *do* observe $(A,\mathbf{U})$.

The utility function (1) posits the existence of two types of social interactions or peer group effects. First, the marginal utility associated with an increase in y_i is increasing in the average action of one's peers, $\bar{y}_{n(i)}$. Specifically,

$$\frac{\partial^2 u_i\left(\mathbf{y}, \mathbf{D}, \mathbf{X}\right)}{\partial y_i \partial \bar{y}_{n(i)}} = \beta.$$

That is, own- and peer-effort are complements. In the terminology of Manski (1993), the magnitude of β indexes the strength of any **endogenous social interactions.**

Second the marginal utility associated with an increase in y_i may vary with peer attributes:

$$\frac{\partial^2 u_i\left(\mathbf{y}, \mathbf{D}, \mathbf{X}\right)}{\partial y_i \partial \bar{X}'_{n(i)}} = \delta.$$

Manski (1993) terms this type of interaction an exogenous or **contextual effect**.

A third, and key, feature of (1) is what Manski (1993) calls **correlated effects**:

$$\frac{\partial^2 u_i\left(\mathbf{y}, \mathbf{D}, \mathbf{X}\right)}{\partial y_i \partial A} = 1.$$

Agents located in networks with high values of A will choose higher actions.

Endogenous, contextual and correlated effects all tend to cause outcomes across members of a common network to covary. Attributing this covariance to true spillovers, whether endogenous or contextual, versus heterogeneity is of policy relevance. Spillovers raise the possibility that rewirings of the network – the addition or subtraction of links – could improve the distribution of outcomes.

The identification problem is to recover $\theta = (\beta, \gamma', \delta')'$ from $F\left(\mathbf{Y}, \mathbf{D}, \mathbf{X}\right)$. Concretely, we will assume that the econometrician has available a dataset formed from a random sample of networks. Identification and inference when data from only a single network are available raises complicated, and largely unsolved, issues.

Equilibrium behavior

The first order condition for optimal behavior associated with (1) generates the following best response function:

$$Y_i = A + \beta \bar{Y}_{n(i)} + X'_i \gamma + \bar{X}'_{n(i)} \delta + U_i \tag{2}$$

for $i = 1, \dots, N$. Equation (2) is called the **linear-in-means** model of social interactions (e.g., Brock and Durlauf, 2001). An agent's best reply varies with the average action of those to whom she is directly connected $(\bar{Y}_{n(i)})$, her own observed attributes (X_i), the average attributes of her direct peers $(\bar{X}_{n(i)})$, the unobserved network effect, A, and unobserved own attributes (U_i).

Observe that (2) defines an $N \times 1$ system of simultaneous equations. A least squares fit of Y_i onto a constant, $\bar{Y}_{n(i)}$, X and $\bar{X}_{n(i)}$ will not provide consistent estimates of $\theta_0 = (A_0, \beta_0, \gamma_0', \delta_0')'$. Define the index set

$$\mathcal{N}(i) = \{j \,:\, D_{ij} = 1\}$$

with cardinality N_i. By construction U_i will be correlated with the N_i player actions Y_j with $j \in \mathcal{N}(i)$ (since Y_i is a component of each of these players' best response functions). Hence U_i will covary with $\bar{Y}_{n(i)}$ and the least squares estimator will be inconsistent. Manski (1993) calls this feature of the linear-in-means model the **reflection problem**.

It is convenient, for what follows, to write the system defined by (2) in matrix form:

$$\mathbf{Y} \;=\; A\iota_N + \mathbf{X}\gamma + \mathbf{G}\mathbf{X}\delta + \beta\mathbf{G}\mathbf{Y} + \mathbf{U}. \tag{3}$$

Derivation of the reduced form action vector

Observe that for $|\beta| < 1$ the matrix $I_N - \beta\mathbf{G}$ is strictly (row) diagonally dominant. By the Levy-Desplanques Theorem (cf., Horn and Johnson, 2013) it is therefore non-singular. Non-singularity of $(I_N - \beta\mathbf{G})$ allows us to solve for the equilibrium action vector as a function of $\mathbf{D}$, $\mathbf{X}$, A and $\mathbf{U}$ alone.

Solving (3) for $\mathbf{Y}$ yields the reduced form

$$\mathbf{Y} = A\left(I_N - \beta\mathbf{G}\right)^{-1}\iota_N + \left(I_N - \beta\mathbf{G}\right)^{-1}\left(\mathbf{X}\gamma + \mathbf{G}\mathbf{X}\delta\right) + \left(I_N - \beta\mathbf{G}\right)^{-1}\mathbf{U}. \tag{4}$$

It is helpful to simplify (4) in a number of ways. First, using the series expansion

$$\left(I_N - \beta\mathbf{G}\right)^{-1} = \sum_{k=0}^{\infty}\beta^k\mathbf{G}^k,$$

as well as the fact that $\mathbf{G}\iota_N = \iota_N$ (and hence that $\mathbf{G}^k\iota_N = \iota_N$ for $k \geq 1$) we get the simplification

$$\begin{aligned}
A\left(I_N-\beta\mathbf{G}\right)^{-1}\iota_N &= A\left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^k\right]\iota_N \\
&= A\left(1+\beta+\beta^2+\beta^3+\cdots\right)\iota_N \\
&= \frac{A}{1-\beta}\iota_N.
\end{aligned}$$

Using this result and re-arranging (4) yields

$$\begin{aligned}
\mathbf{Y} &= \frac{A}{1-\beta}\iota_N+\left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^k\right](\mathbf{X}\gamma+\mathbf{GX}\delta)+\left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^k\right]\mathbf{U} \\
&= \frac{A}{1-\beta}\iota_N+\left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^k\right]\mathbf{X}\gamma+\left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^{k+1}\right]\mathbf{X}\delta+\left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^k\right]\mathbf{U} \\
&= \frac{A}{1-\beta}\iota_N+\mathbf{X}\gamma+\left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^{k+1}\right]\mathbf{X}\gamma\beta+\left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^{k+1}\right]\mathbf{X}\delta+\left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^k\right]\mathbf{U} \\
&= \frac{A}{1-\beta}\iota_N+\mathbf{X}\gamma+\left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^{k+1}\mathbf{X}\right](\gamma\beta+\delta)+\left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^k\right]\mathbf{U}. \qquad (5)
\end{aligned}$$

The social multiplier

Equation (5) provides some insight into what various researchers have called the social multiplier. For simplicity assume that $\delta=0$, so that the only type of peer influence is the endogenous effect. Further assume that X_i is a scalar. Now consider a policy which increases the i^{th} agent's value of X_i by Δ. We can conceptualize the full effect of this increase on the network's distribution of outcomes as occurring in "waves". In the initial wave only agent i's outcome increases. The change in the entire action vector is therefore

$$\triangle\gamma\mathbf{c}_i,$$

where $\mathbf{c}_i$ is an N-vector with a one in its i^{th} element and zeros elsewhere.

In the second wave all of agent i's friends experience outcome increases. This is because their best reply actions change in response to the increase in agent $i's$ action in the initial wave. The action vector in wave two therefore changes by

$$\triangle\gamma\beta\mathbf{G}\mathbf{c}_i.$$

In the third wave the outcomes of agent i's friends' friends change (this includes a direct feedback effect to agent i). In wave three we get a further change in the action vector of

$$\Delta\gamma\beta^2\mathbf{G}^2\mathbf{c}_i.$$

In the k^{th} wave we have a change in the action vector of

$$\triangle\gamma\beta^{k-1}\mathbf{G}^{k-1}\mathbf{c}_i.$$

Observing the pattern of geometric decay we see that the "long-run" or full effect of the change in X_i on the entire distribution of outcomes is given by

$$\triangle\gamma\left(I_N - \beta\mathbf{G}\right)^{-1}\mathbf{c}_i. \tag{6}$$

Observe that, if the cost of perturbing X_i does not vary with i, the planner can use the form of $\mathbf{G}$ to efficiently target interventions. This is the policy content of endogenous social effects in the linear-in-means model.

Identification of θ

Let $\bar{\mathbf{X}}$ be the $N \times K$ matrix with i^{th} row $\bar{X}'_{n(i)}$. Observe that $\mathbf{GX} = \overline{\mathbf{X}}$; this is a matrix consisting of the average of friends' characteristics for each of the $i = 1, \ldots, N$ agents. Now observe that $\mathbf{G}^2\mathbf{X} = \mathbf{G}\bar{\mathbf{X}}$ is a matrix consisting of an average of your friends' friends' average attributes. Likewise $\mathbf{G}^3\bar{\mathbf{X}}$ is an average of your friends' friends' average of their friends' average attributes (and so on in increasingly unmanageable mouthfuls).

Re-arranging we get

$$\sum_{k=0}^{\infty}\beta^k\mathbf{G}^{k+1}\mathbf{X} = \overline{\mathbf{X}} + \sum_{k=1}^{\infty}\beta^k\mathbf{G}^k\bar{\mathbf{X}}$$

which gives our final reduced form expression

$$\mathbf{Y} = \frac{A}{1-\beta}\iota_N + \mathbf{X}\gamma + \bar{\mathbf{X}}\left(\gamma\beta + \delta\right) + \left[\sum_{k=1}^{\infty}\beta^k\mathbf{G}^k\bar{\mathbf{X}}\right]\left(\gamma\beta + \delta\right) + \left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^k\right]\mathbf{U}. \tag{7}$$

Equation (7) indicates that, in equilibrium, an agent's action will vary with own attributes, those of her peers, as well as those of her peers' peers and so on. Bramoulle, Djebbari and Fortin (2009) show that the influence of indirect links on equilibrium actions can, under certain network configurations, allow for consistent estimation of θ.

Equation (7) has a form reminiscent of the reduced form of a linear dynamic panel data model with a K-vector of strictly exogenous regressors. It is well-known that dynamics induce a distributed lag structure on the coefficients on lags of X_t in the reduced form of this model (e.g., Arellano, 2003).

To understand Bramoulle, Djebbari and Fortin's (2009) result define $\bar{\mathbf{Y}} = \mathbf{GY}$ to be the $N \times 1$ vector of peer average actions. Multiplying (7) by $\mathbf{G}$ yields an N-vector of "first stage" equations equal to

$$\bar{\mathbf{Y}} = \frac{A}{1-\beta}\iota_M + \bar{\mathbf{X}}\gamma + \left[\sum_{k=0}^{\infty} \beta^k \mathbf{G}^{k+1}\bar{\mathbf{X}}\right](\gamma\beta + \delta) + \left[\sum_{k=0}^{\infty} \beta^k \mathbf{G}^{k}\right]\bar{\mathbf{U}} \tag{8}$$

Recall, using some of the definitions given above, our structural equation (i.e., the $N \times 1$ vector of agent-specific best response functions) of

$$\mathbf{Y} = A\iota_N + \beta\bar{\mathbf{Y}} + \mathbf{X}\gamma + \bar{\mathbf{X}}\delta + \mathbf{U}. \tag{9}$$

Let $\bar{X}_{n(i)}^{\mathrm{ff}} = \mathbf{G}_i\mathbf{G}\mathbf{X}$ be the average of agent i's friends' friends' average attributes. We will initially work with the following assumption.

Assumption 1. *The econometrician (i) observes a random sequence of networks indexed by c with the size of network c equal to N_c and with action profile $\mathbf{Y}_c$, adjacency matrix $\mathbf{D}_c$ and attribute matrix $\mathbf{X}_c$ and (ii) $\mathbb{E}\left[\mathbf{U}_c \middle| \mathbf{D}_c, \mathbf{X}_c, N_c\right] = 0$.*

Part (ii) of Assumption 1 is strong. It, effectively, imposes strong restrictions on the network formation process. Under Assumption 1 the moment

$$\mathbb{E}\left[U_{ci}\bar{X}_{cn(i)}\right] = \mathbb{E}\left[U_{ci}\mathbf{G}_{ci}\mathbf{X}_c\right]$$

is mean zero. There are many reasons to be doubtful of this restriction. Under (1) agents with high values of U_i have a high marginal utility of action. Such agents may also be attractive friends, if so they will have high degree, D_{i+}. In such situations $\mathbf{G}_i\mathbf{X}$ may help to predict U_i and part (ii) of Assumption 1 will fail.

Unfortunately I am aware of no simple mechanisms which can guarantee that Assumption 1 holds. Say the researcher observes $\mathbf{D}$ and then randomly assigns X_i to each agent. This assignment mechanism yields the independence relationship

$$(\mathbf{U}, \mathbf{D}) \perp \mathbf{X},$$

but this does not imply the additional restriction

$$\mathbf{U} \perp \mathbf{D} | \mathbf{X},$$

which would together be sufficient for part (ii) of Assumption 1. More exotic experimental designs, that both manipulate $\mathbf{D}$ and $\mathbf{X}$ might work, but real work in thinking through this type of intuition formally is needed.

Under Assumption 1 the following $1+3K$ moment restrictions holds at the population vector θ_0 (note $\dim(\theta) = 2+2K$)

$$\mathbb{E}\left[\begin{pmatrix} \iota_{N_c} & \mathbf{G}_c\bar{\mathbf{X}}_c & \mathbf{X}_c & \bar{\mathbf{X}}_c \end{pmatrix}' \left(\mathbf{Y} - A_0\iota_{N_c} - \beta_0\bar{\mathbf{Y}} - \mathbf{X}\gamma_0 - \bar{\mathbf{X}}\delta_0\right)\right] = 0 \tag{10}$$

As long as I_{N_c}, $\mathbf{G}_c$ and $\mathbf{G}_c^2$ are linearly independent and $\gamma\beta + \delta \neq 0$, then a GMM estimator based on the sample analog of (10) will be consistent for θ_0. See Bramoulle, Djebbari and Fortin (2009, Proposition 1) . Operationally a linear instrumental variables fit of Y_{ci} onto a constant, $\bar{Y}_{cn(i)}$, X_{ci} and $\bar{X}_{cn(i)}$ with $\bar{X}^{\text{ff}}_{cn(i)}$ serving as an excluded instrument for $\bar{Y}_{cn(i)}$ and standard errors "clustered" at the network level will yield consistent estimates of θ_0 and asymptotically valid standard error estimates (cf., De Giorgi, Pellizzari and Redaelli, 2010). The intuition is that the characteristics of my friends' friends influence their actions and hence me, but do not directly influence me. More on this below!

Bramoulle, Djebbari and Fortin (2009) provide some sufficient conditions for linear independence of I_{N_c}, $\mathbf{G}_c$ and $\mathbf{G}_c^2$ as well as several counter-examples. I begin with (a variant of) Manski's (1993) famous non-identification result.

Relationship to Manski's (1993) non-identification result

Early analyses of the linear-in-means model focused on the case where $\mathbf{G}_c$ is of the form

$$\mathbf{G}_c = \left(\iota_{N_c}\iota_{N_c}' - I_{N_c}\right)\frac{1}{N_c - 1}.$$

This corresponds to a network where all individuals are linked to one another. A canonical example is a classroom of students, where all students are presumed to influence one another. This form of $\mathbf{G}_c$ is associated with what is sometimes called the "leave-own-out linear-in-means" model.

Under this network structure we have

$$\begin{aligned}
\mathbf{G}_c^2 &= \left(\iota_{N_c}\iota_{N_c}' - I_{N_c}\right)\left(\iota_{N_c}\iota_{N_c}' - I_{N_c}\right)\left(\frac{1}{N_c-1}\right)^2 \\
&= \left(N_c\iota_{N_c}\iota_{N_c}' - 2\iota_{N_c}\iota_{N_c}' + I_{N_c}\right)\left(\frac{1}{N_c-1}\right)^2 \\
&= \left[\left(N_c-2\right)\left(\iota_{N_c}\iota_{N_c} - I_{N_c}\right) + \left(N_c-1\right)I_{N_c}\right]\left(\frac{1}{N_c-1}\right)^2 \\
&= \frac{1}{N_c-1}I_{N_c} + \frac{N_c-2}{N_c-1}\mathbf{G}_c.
\end{aligned}$$

If groups/networks vary in size, then I_{N_c}, $\mathbf{G}_c$ and $\mathbf{G}_c^2$ will be linearly independent (cf., Lee, 2007). However if groups are equal in size identification will fail. Manski (1993) essentially considers the case where $N_c \to \infty$, which gives $\mathbf{G}_c^2 = \mathbf{G}_c$. When all groups are fairly large, this equality will be approximately true and identification will be weak.

The identifying power of intransitive triads

Bramoulle, Djebbari and Fortin (2009) note that if the pair (i,j) are not connected then $D_{ij} = 0$. However if they share some friends in common, then $(i,j)^{th}$ element of $\mathbf{D}^2$, which equals $\sum_k D_{ik}D_{kj}$, will be non-zero. Thus the presence of intransitive triads, in at least some networks, guarantees linear independence of I_{N_c}, $\mathbf{G}_c$ and $\mathbf{G}_c^2$. See De Giorgi, Pellizzari and Redaelli (2010) for the same observations and an empirical example. Intransitivity is sufficient for identification even when all networks are equally sized and/or large. This, in my view, is really the key take-away of their paper for empirical researchers.

Network effects

Until now we have assumed that A is constant across networks. We can relax this assumption and replace condition (ii) of Assumption 1 with $\mathbb{E}\left[\mathbf{U}_c \middle| \mathbf{D}_c, \mathbf{X}_c, N_c, \mathbf{A}_c\right] = 0$. This yields the modified reduced form

$$\mathbf{Y}_c = \frac{A_c}{1-\beta}\iota_{N_c} + \mathbf{X}_c\gamma + \bar{\mathbf{X}}_c\left(\gamma\beta+\delta\right) + \left[\sum_{k=1}^{\infty}\beta^k\mathbf{G}_c^k\bar{\mathbf{X}}_c\right]\left(\gamma\beta+\delta\right) + \left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}_c^k\right]\mathbf{U}_c. \tag{11}$$

Subtracting (8) from this equation yields

$$\begin{aligned}
\mathbf{Y}_c - \bar{\mathbf{Y}}_c &= \left(\mathbf{X}_c - \bar{\mathbf{X}}_c\right)\gamma + \left[\sum_{k=0}^{\infty} \beta^k \mathbf{G}^k \left(I_{N_c} - \mathbf{G}_c\right)\bar{\mathbf{X}}\right](\gamma\beta + \delta) + \left[\sum_{k=0}^{\infty} \beta^k \mathbf{G_c}^k\right]\left(\mathbf{U}_c - \bar{\mathbf{U}}_c\right) \\
&= \left(\mathbf{X}_c - \bar{\mathbf{X}}_c\right)\gamma + \left(I_{N_c} - \mathbf{G}_c\right)\bar{\mathbf{X}}(\gamma\beta + \delta) + \left[\sum_{k=1}^{\infty} \beta^k \mathbf{G}^k \left(I_{N_c} - \mathbf{G}_c\right)\bar{\mathbf{X}}\right](\gamma\beta + \delta) \\
&\quad + \left[\sum_{k=0}^{\infty} \beta^k \mathbf{G_c}^k\right]\left(\mathbf{U}_c - \bar{\mathbf{U}}_c\right).
\end{aligned}$$

The presence of $\mathbf{G}^k \left(I_{N_c} - \mathbf{G}_c\right)\bar{\mathbf{X}}$ in this transformation of the reduced form for $k = 1, \ldots, N$ facilitates identification of θ_0. A sufficient condition for identification in this model is linear independence of $I_{N_c} - \mathbf{G}_c$, $\left(I_{N_c} - \mathbf{G}_c\right)\mathbf{G}_c$ and $\left(I_{N_c} - \mathbf{G}_c\right)\mathbf{G}_c^2$. This will hold if I_{N_c}, $\mathbf{G}_c$, $\mathbf{G}_c^2$ and $\mathbf{G}_c^3$ are linearly independent. Bramoulle, Djebbari and Fortin (2009, Corollary 1) show that a sufficient condition for this latter condition is that the diameter of the network is at least three. This condition is satisfied by many real world networks.

Let $\bar{Y}^{\text{ff}}_{cn(i)}$ equal the i^{th} element of of $\mathbf{G}_c^2\mathbf{Y}_c$. This gives the average of my friends' averages of their friends behavior. Let $\bar{X}^{\text{fff}}_{cn(i)}$ be the i^{th} element of $\mathbf{G}_c^3\mathbf{X}$. The i^{th} row of this matrix coincides with a (weighted) average of agent characteristics up to three degrees away from i.

If the linear independence condition holds, then a linear instrumental variables fit of $Y_{ci} - \bar{Y}_{cn(i)}$ onto $\bar{Y}_{cn(i)} - \bar{Y}^{\text{ff}}_{cn(i)}$, $X_{ci} - \bar{X}_{cn(i)}$ and $\bar{X}_{cn(i)} - \bar{X}^{\text{ff}}_{cn(i)}$ with $\bar{X}^{\text{ff}}_{cn(i)} - \bar{X}^{\text{fff}}_{cn(i)}$ serving as an excluded instrument for $\bar{Y}_{cn(i)} - \bar{Y}^{\text{ff}}_{cn(i)}$ and standard errors "clustered" at the network level will yield consistent estimates of θ_0 and asymptotically valid standard error estimates.

Some thoughts on empirical work

Under the maintained assumption of linear best-reply behavior, identification of θ_0 requires maintaining fairly strong assumptions about the network formation process. These will be credible in some settings, but not in others. Condition $\mathbb{E}\left[\mathbf{U}_c \middle| \mathbf{D}_c, \mathbf{X}_c, N_c\right] = 0$ provides a useful way for assessing the plausibility of empirical work: can I predict the idiosyncratic component of behavior using network structure, agent characteristics and/or network size?

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