C Supplemental Web Appendix: Details of Calculations (Not for Publication)

Calculation details for proof of Lemma 4

To derive (24), as given in the proof of Lemma 4, observe that tedious calculation gives

$$
\nabla\varphi_{\mathbf{A}}(\mathbf{A}) = \begin{pmatrix}
\frac{\sum_{j\neq 1}\frac{\exp\left(W_{1j}'\beta\right)\exp\left(W_{1j}'\beta+A_1(\beta)\right)}{\left[\exp\left(-A_j(\beta)\right)+\exp\left(W_{1j}'\beta+A_1(\beta)\right)\right]^2}}{\sum_{j\neq 1}\frac{\exp\left(W_{1j}'\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(W_{1j}'\beta+A_1(\beta)\right)}} & -\frac{\frac{\exp(W_{12}'\beta)\exp(-A_2(\beta))}{\left[\exp(-A_2(\beta))+\exp\left(W_{12}'\beta+A_1(\beta)\right)\right]^2}}{\sum_{j\neq 1}\frac{\exp\left(W_{1j}'\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(W_{1j}'\beta+A_1(\beta)\right)}} & \cdots & -\frac{\frac{\exp(W_{1N}'\beta)\exp(-A_N(\beta))}{\left[\exp(-A_N(\beta))+\exp\left(W_{1N}'\beta+A_1(\beta)\right)\right]^2}}{\sum_{j\neq 1}\frac{\exp\left(W_{1j}'\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(Z_{1j}'\beta+A_1(\beta)\right)}} \\
-\frac{\frac{\exp(W_{12}'\beta)\exp(-A_1(\beta))}{\left[\exp(-A_1(\beta))+\exp\left(W_{12}'\beta+A_2(\beta)\right)\right]^2}}{\sum_{j\neq 2}\frac{\exp\left(W_{2j}'\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(W_{2j}'\beta+A_2(\beta)\right)}} & \frac{\sum_{j\neq 2}\frac{\exp\left(W_{2j}'\beta\right)\exp\left(W_{2j}'\beta+A_2(\beta)\right)}{\left[\exp\left(A_j(\beta)\right)+\exp\left(W_{2j}'\beta+A_2(\beta)\right)\right]^2}}{\sum_{j\neq 2}\frac{\exp\left(W_{2j}'\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(W_{2j}'\beta+A_2(\beta)\right)}} & \cdots & -\frac{\frac{\exp(W_{2N}'\beta)\exp(-A_N(\beta))}{\left[\exp(-A_N(\beta))+\exp\left(W_{2N}'\beta+A_2(\beta)\right)\right]^2}}{\sum_{j\neq 2}\frac{\exp\left(W_{2j}'\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(W_{2j}'\beta+A_2(\beta)\right)}} \\
\vdots & & \ddots & \vdots \\
-\frac{\frac{\exp(W_{1N}'\beta)\exp(-A_1(\beta))}{\left[\exp(-A_1(\beta))+\exp\left(W_{1N}'\beta+A_N(\beta)\right)\right]^2}}{\sum_{j\neq N}\frac{\exp\left(W_{Nj}'\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(W_{Nj}'\beta+A_N(\beta)\right)}} & \cdots & \cdots & \frac{\sum_{j\neq N}\frac{\exp\left(W_{Nj}'\beta\right)\exp\left(W_{Nj}'\beta+A_N(\beta)\right)}{\left[\exp\left(A_j(\beta)\right)+\exp\left(W_{Nj}'\beta+A_N(\beta)\right)\right]^2}}{\sum_{j\neq N}\frac{\exp\left(Z_{Nj}'\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(Z_{Nj}'\beta+A_N(\beta)\right)}}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\frac{\sum_{j\neq 1} r_{1j}p_{1j}}{\sum_{j\neq 1} r_{1j}} & -\frac{r_{12}(1-p_{12})}{\sum_{j\neq 1} r_{1j}} & \cdots & -\frac{r_{1N}(1-p_{1N})}{\sum_{j\neq 1} r_{1j}} \\
-\frac{r_{21}(1-p_{12})}{\sum_{j\neq 2} r_{2j}} & \frac{\sum_{j\neq 2} r_{2j}p_{2j}}{\sum_{j\neq 2} r_{2j}} & \cdots & -\frac{r_{2N}(1-p_{2N})}{\sum_{j\neq 2} r_{2j}} \\
\vdots & & \ddots & \vdots \\
-\frac{r_{N1}(1-p_{1N})}{\sum_{j\neq N} r_{Nj}} & -\frac{r_{2N}(1-p_{2N})}{\sum_{j\neq N} r_{Nj}} & \cdots & \frac{\sum_{j\neq N} r_{Nj}p_{Nj}}{\sum_{j\neq N} r_{Nj}}
\end{pmatrix},
$$

where the second equality follows from the definition

$$
r_{ij}\left(\beta,\mathbf{A},W_{ij}\right) = \frac{\exp\left(W_{ij}'\beta\right)}{\exp\left(-A_j\right)+\exp\left(W_{ij}'\beta+A_i\right)} = \exp\left(A_i\right)p_{ij},
$$

and the relationships

$$\frac{\frac{\exp\left(W_{ij}'\beta\right)\exp(-A_j(\beta))}{\left[\exp(-A_j(\beta))+\exp\left(W_{ij}'\beta+A_i(\beta)\right)\right]^2}}{\sum_{j\neq i}\frac{\exp\left(W_{ij}'\beta\right)}{\exp(-A_j(\beta))+\exp\left(W_{ij}'\beta+A_i(\beta)\right)}} = \frac{r_{ij}\left(1-p_{ij}\right)}{\sum_{j\neq i} r_{ij}} = \frac{p_{ij}\left(1-p_{ij}\right)}{\sum_{j\neq i} p_{ij}},$$

and

$$\frac{\sum_{j\neq i}\frac{\exp\left(W_{ij}'\beta\right)\exp\left(W_{ij}'\beta+A_i(\beta)\right)}{\left[\exp(-A_j(\beta))+\exp\left(W_{ij}'\beta+A_i(\beta)\right)\right]^2}}{\sum_{j\neq i}\frac{\exp\left(W_{ij}'\beta\right)}{\exp(-A_j(\beta))+\exp\left(W_{ij}'\beta+A_i(\beta)\right)}} = \frac{\sum_{j\neq i} r_{ij}p_{ij}}{\sum_{j\neq i} r_{ij}} = \frac{\sum_{j\neq i} p_{ij}^2}{\sum_{j\neq i} p_{ij}}.$$

Calculation details for proof of Lemma 6

To derive the bound for R_p appearing in the proof of Lemma 6 observe that

$$\frac{\partial}{\partial \mathbf{A}\partial \mathbf{A}'} s_{\mathbf{A}ij}^{(p)}\left(\beta_0, \mathbf{A}\left(\beta_0\right)\right) = -p_{ij}\left(1-p_{ij}\right)\left(1-2p_{ij}\right)T_{ij}T_{ij}'T_{p,ij}$$

and hence that $\sum_{i=1}^{N}\sum_{j<i}\frac{\partial}{\partial \mathbf{A}\partial \mathbf{A}'} s_{\mathbf{A}ij}^{(p)}\left(\beta_0, \mathbf{A}\left(\beta_0\right)\right)$ equals

$$-\begin{pmatrix} \sum_{j\neq 1} p_{1j}\left(1-p_{1j}\right)\left(1-2p_{1j}\right)T_{p,1j} & \cdots & p_{1N}\left(1-p_{1N}\right)\left(1-2p_{1N}\right)T_{p,1N} \\ \vdots & \ddots & \vdots \\ p_{1N}\left(1-p_{1N}\right)\left(1-2p_{1N}\right)T_{p,1N} & \cdots & \sum_{j\neq N} p_{Nj}\left(1-p_{Nj}\right)\left(1-2p_{Nj}\right)T_{p,Nj} \end{pmatrix}.$$

So that

$$\iota_N'\left[\sum_{i=1}^{N}\sum_{j<i}\frac{\partial}{\partial \mathbf{A}\partial \mathbf{A}'} s_{\mathbf{A}ij}^{(p)}\left(\beta_0, \mathbf{A}\left(\beta_0\right)\right)\right]\iota_N = 2\sum_{i=1}^{N}\sum_{j\neq i} p_{ij}\left(1-p_{ij}\right)\left(1-2p_{ij}\right)T_{p,ij}.$$

Finally observe that $\sum_{i=1}^{N}\sum_{j\neq i} T_{p,ij} = 2\left(N-1\right).$

Calculation details for proof of Theorem 1

To derive (32) use iterated expectations to show that

$$\begin{aligned} L(\beta) &= \mathbb{E}\left[|S_{ij,kl}|\left\{S_{ij,kl}W'_{ij,kl}\beta - \ln\left[1+\exp\left(S_{ij,kl}W'_{ij,kl}\beta\right)\right]\right\}\right] \\ &= \Pr\left(S_{ij,kl}\in\{-1,1\}\right)\mathbb{E}\left[\left. S_{ij,kl}W'_{ij,kl}\beta - \ln\left[1+\exp\left(S_{ij,kl}W'_{ij,kl}\beta\right)\right]\right| S_{ij,kl}\in\{-1,1\}\right] \\ &= \Pr\left(S_{ij,kl}\in\{-1,1\}\right) \\ &\quad \times\mathbb{E}\left[\mathbb{E}\left[\left. S_{ij,kl}W'_{ij,kl}\beta - \ln\left[1+\exp\left(S_{ij,kl}W'_{ij,kl}\beta\right)\right]\right| X_i, X_j, X_k, X_l, S_{ij,kl}\in\{-1,1\}\right]\right. \\ &\quad \left.|S_{ij,kl}\in\{-1,1\}\right]. \end{aligned}$$

Evaluating the innermost expectation then yields

$$\begin{aligned} &\mathbb{E}\left[\left. S_{ij,kl}W'_{ij,kl}\beta - \ln\left[1+\exp\left(S_{ij,kl}W'_{ij,kl}\beta\right)\right]\right| X_i, X_j, X_k, X_l, S_{ij,kl}\in\{-1,1\}\right] \\ =\ &\left\{W'_{ij,kl}\beta - \ln\left[1+\exp\left(W'_{ij,kl}\beta\right)\right]\right\}q_{ij,kl} \\ &+\left\{-W'_{ij,kl}\beta - \ln\left[1+\exp\left(-W'_{ij,kl}\beta\right)\right]\right\}\left[1-q_{ij,kl}\right] \\ =\ &\ln\left\{q_{ij,kl}(\beta)\right\}q_{ij,kl} + \ln\left\{1-q_{ij,kl}(\beta)\right\}\left[1-q_{ij,kl}\right] \\ =\ &-\left\{q_{ij,kl}\ln\left(\frac{q_{ij,kl}}{q_{ij,kl}(\beta)}\right)+\left[1-q_{ij,kl}\right]\ln\left(\frac{1-q_{ij,kl}}{1-q_{ij,kl}(\beta)}\right)\right\} \\ &+q_{ij,kl}\ln\left(q_{ij,kl}\right)+\left[1-q_{ij,kl}\right]\ln\left(1-q_{ij,kl}\right) \\ =\ &-D_{KL}\left(q_{ij,kl}\|\, q_{ij,kl}(\beta)\right)-\mathbf{S}\left(q_{ij,kl}\right). \end{aligned}$$

Fixing i and j and averaging with respect to independent random draws k and l, from the population of agents, yields

$$\begin{aligned} \mathbb{E}\left[S_{ij,kl}\right| i,j,\mathbf{X},\mathbf{A}] &= D_{ij}\Pr\left(D_{kl}=1, D_{ik}=0, D_{jl}=0\right| i,j,\mathbf{X},\mathbf{A}) \\ &\quad -\left(1-D_{ij}\right)\Pr\left(D_{kl}=0, D_{ik}=1, D_{jl}=1\right| i,j,\mathbf{X},\mathbf{A}) \\ &= D_{ij}\mathbb{E}\left[p_{kl}\left(1-p_{ik}\right)\left(1-p_{jl}\right)\right| i,j,\mathbf{X},\mathbf{A}] \\ &\quad -\left(1-D_{ij}\right)\mathbb{E}\left[\left(1-p_{kl}\right)p_{ik}p_{jl}\right| i,j,\mathbf{X},\mathbf{A}]. \end{aligned} \tag{55}$$

An implication of (55) is that $C\left(\bar{s}_{2,ij},\bar{s}_{2,kl}\right|\mathbf{X},\mathbf{A}) = 0$ unless ij and kl correspond to the same dyad. This is an implication of independent edge formation *conditional* on $\mathbf{X}$ and $\mathbf{A}$.

To derive (35) observe that

$$\begin{aligned}
\sum_{i<j}^{N} \binom{N}{4}^{-1} \sum_{k<l<m<n} \phi_{klmn,ij} &= \sum_{i<j}^{N} \binom{N}{4}^{-1} \binom{N-2}{2} \bar{s}_{2,ij} \\
&= \sum_{i<j}^{N} \left\{ \frac{4!\,(N-4)!}{N!} \right\} \left\{ \frac{(N-2)!}{2!\,(N-4)!} \right\} \bar{s}_{2,ij} \\
&= \sum_{i<j}^{N} \left\{ \frac{12}{N\,(N-1)} \right\} \bar{s}_{2,ij}.
\end{aligned}$$

To derive the form of $\mathbb{C}\left(U_N^*, U_N\right)$ given in the proof note that

$$\begin{aligned}
6\binom{N}{4}^{-1}\binom{N-2}{2}\Delta_{2,N} &= 6\frac{4!\,(N-4)!}{N!}\frac{(N-2)!}{2!\,(N-4)!}\Delta_{2,N} \\
&= \frac{72}{N\,(N-1)}\Delta_{2,N}.
\end{aligned}$$

Calculation details for proof of Theorem 4

Probability limit of concentrated Hessian: The expression for $H_{N,\beta\beta}+H_{N,\beta\mathbf{A}}V_N^{-1}H_{N,\beta\mathbf{A}}$, the approximate Hessian of the concentrated log-likelihood given in (48), may be calculated as follows

$$
\begin{aligned}
& H_{N,\beta\beta}+H_{N,\beta\mathbf{A}}V_N^{-1}H_{N,\beta\mathbf{A}} \\
= & -\sum_{i=1}^{N}\sum_{j<i}p_{ij}\left(1-p_{ij}\right)W_{ij}W_{ij}' \\
& +\begin{pmatrix} -\sum_{j\neq 1}p_{1j}\left(1-p_{1j}\right)W_{1j} & \cdots & -\sum_{j\neq N}p_{Nj}\left(1-p_{Nj}\right)W_{Nj}\end{pmatrix} \\
& \times\mathrm{diag}\left\{\frac{1}{\sum_{j\neq 1}p_{1j}\left(1-p_{1j}\right)},\ldots,\frac{1}{\sum_{j\neq N}p_{Nj}\left(1-p_{Nj}\right)}\right\}' \\
& \times\begin{pmatrix} -\sum_{j\neq 1}p_{1j}\left(1-p_{1j}\right)W_{1j}' \\ \vdots \\ -\sum_{j\neq N}p_{Nj}\left(1-p_{Nj}\right)W_{Nj}'\end{pmatrix} \\
= & -\sum_{i=1}^{N}\sum_{j<i}p_{ij}\left(1-p_{ij}\right)W_{ij}W_{ij}' \\
& \begin{pmatrix} \frac{-\sum_{j\neq 1}p_{1j}(1-p_{1j})W_{1j}}{\sum_{j\neq 1}p_{1j}(1-p_{1j})} & \cdots & \frac{-\sum_{j\neq N}p_{Nj}\left(1-p_{Nj}\right)W_{Nj}}{\sum_{j\neq N}p_{Nj}\left(1-p_{Nj}\right)}\end{pmatrix}\begin{pmatrix} -\sum_{j\neq 1}p_{1j}\left(1-p_{1j}\right)W_{1j}' \\ \vdots \\ -\sum_{j\neq N}p_{Nj}\left(1-p_{Nj}\right)W_{Nj}'\end{pmatrix} \\
= & -\left\{\sum_{i=1}^{N}\sum_{j<i}p_{ij}\left(1-p_{ij}\right)W_{ij}W_{ij}'-\sum_{i=1}^{N}\frac{\left(\sum_{j\neq i}p_{ij}\left(1-p_{ij}\right)W_{ij}\right)\left(\sum_{j\neq i}p_{ij}\left(1-p_{ij}\right)W_{ij}\right)'}{\sum_{j\neq i}p_{ij}\left(1-p_{ij}\right)}\right\}.
\end{aligned}
$$

Analysis of remainder term in (50): Let $f\left(v\right)=\frac{\exp(v)}{1+\exp(v)}$ be the logit function. To bound the third term in (50) I begin by calculating the derivative of $f\left(v\right)\left(1-f\left(v\right)\right)\left(1-2f\left(v\right)\right)=f\left(v\right)-3f\left(v\right)^2+2f\left(v\right)^3$ with respect to v. Using the fact that $f'\left(v\right)=f\left(v\right)\left(1-f\left(v\right)\right)$ I get

$$
\begin{aligned}
\frac{\partial}{\partial v}\left\{f\left(v\right)\left(1-f\left(v\right)\right)\left(1-2f\left(v\right)\right)\right\} &= f\left(v\right)\left(1-f\left(v\right)\right)-6f\left(v\right)^2\left(1-f\left(v\right)\right)+6f\left(v\right)^3\left(1-f\left(v\right)\right) \\
&= f\left(v\right)\left(1-f\left(v\right)\right)\left(1-6f\left(v\right)+6f\left(v\right)^2\right) \\
&= f\left(v\right)\left(1-f\left(v\right)\right)\left(1-6f\left(v\right)\left(1-f\left(v\right)\right)\right).
\end{aligned}
$$

Using condition (16) then gives

$$
\sup_{1\leq i,j\leq N}\left|p_{ij}\left(1-p_{ij}\right)\left(1-6p_{ij}\left(1-p_{ij}\right)\right)W_{ij}\right|\leq\frac{1}{4}\left(1-6\kappa\left(1-\kappa\right)\right)\times\sup_{w\in\mathbb{W}}\left|w\right|.
$$

Expanding the fourth term in (50) I get

$$\begin{aligned}
&= \sum_{k=1}^{N}\sum_{l=1}^{N}\left(\hat{A}_k(\beta_0)-A_k(\beta_0)\right)\left(\hat{A}_l(\beta_0)-A_l(\beta_0)\right) \\
&\quad \times\left[\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^3}{\partial A_k \partial A_l \partial \mathbf{A}'}s_{\beta ij}\left(\beta_0,\bar{\mathbf{A}}(\beta_0)\right)\right] \\
&= -\sum_{k=1}^{N}\sum_{l\neq k}\begin{pmatrix} 0 \\ \vdots \\ \left(\hat{A}_k-A_k\right)\left(\hat{A}_l-A_l\right)p_{kl}(1-p_{kl})(1-6p_{kl}(1-p_{kl}))W_{kl}' \\ \vdots \\ \left(\hat{A}_k-A_k\right)\left(\hat{A}_l-A_l\right)p_{kl}(1-p_{kl})(1-6p_{kl}(1-p_{kl}))W_{kl}' \\ \vdots \\ 0 \end{pmatrix}' \\
&= -2\begin{pmatrix} \left(\hat{A}_1-A_1\right)\sum_{j\neq 1}\left(\hat{A}_j-A_j\right)p_{1j}(1-p_{1j})(1-6p_{1j}(1-p_{1j}))W_{1j}' \\ \vdots \\ \left(\hat{A}_N-A_N\right)\sum_{j\neq N}\left(\hat{A}_j-A_j\right)p_{Nj}(1-p_{Nj})(1-6p_{Nj}(1-p_{Nj}))W_{1j}' \end{pmatrix}'.
\end{aligned}$$

Multiplying this by the $N \times 1$ vector $\hat{\mathbf{A}} - \mathbf{A}$ yields the $K \times 1$ vector

-

$$2\sum_{i=1}^{N}\sum_{j\neq i}\left(\hat{A}_i-A_i\right)^2\left(\hat{A}_j-A_j\right)(1-p_{ij})(1-6p_{ij}(1-p_{ij}))W_{ij}$$

which gives (51) of the main text.

Derivation of asymptotic bias: To derive (52) it is convenient to proceed regressor by regressor. Observe that the k^{th} element of the third term appearing in (50) is, for $k = 1, \ldots, K$,

$$\frac{1}{2}\frac{1}{\sqrt{n}}\left[\sum_{l=1}^{N}\left(\hat{A}_l(\beta_0)-A_l(\beta_0)\right)\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l \partial \mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}(\beta_0)\right)\right]\left(\hat{\mathbf{A}}(\beta_0)-\mathbf{A}(\beta_0)\right) \quad (56)$$

The probability limit of (56) equals (52). To simplify (56) and, derive this limit, start by observing that, for $l = 1, \ldots, N$,

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l \partial \mathbf{A}'}s_{\beta ij}\left(\beta_0, \mathbf{A}\left(\beta_0\right)\right) &= -\sum_{i=1}^{N}\sum_{j<i}p_{ij}\left(1-p_{ij}\right)\left(1-2p_{ij}\right)W_{ij}T_{ij}'T_{l,ij} \\
&= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}p_{ij}\left(1-p_{ij}\right)\left(1-2p_{ij}\right)W_{ij}T_{ij}'T_{l,ij} \\
&\quad -\Big(\; p_{1l}\left(1-p_{1l}\right)\left(1-2p_{1l}\right)W_{1l} \\
&\quad \cdots \quad p_{l-1l}\left(1-p_{l-1l}\right)\left(1-2p_{l-1l}\right)W_{l-1l} \\
&\quad \sum_{j\neq l}p_{lj}\left(1-p_{lj}\right)\left(1-2p_{lj}\right)W_{lj} \\
&\quad p_{l+1l}\left(1-p_{l+1l}\right)\left(1-2p_{l+1l}\right)W_{l+1l} \\
&\quad \cdots \quad p_{Nl}\left(1-p_{Nl}\right)\left(1-2p_{Nl}\right)W_{Nl} \; \Big).
\end{aligned}$$

Next, using (31) from the proof of Lemma 6 and recalling that e_l is a conformable selection vector with a 1 in its l^{th} element and zeros elsewhere, gives

$$\hat{A}_l\left(\beta_0\right) - A_l\left(\beta_0\right) = -e_l' H_{N,\mathbf{AA}}^{-1}\begin{pmatrix} D_{1+} - p_{1+} \\ \vdots \\ D_{N+} - p_{N+}\end{pmatrix} + o_p\left(1\right).$$

which allows the k^{th} element of the third term in (50) to be replaced with its asymptotic equivalent

$$\begin{aligned}
\frac{1}{2}\frac{1}{\sqrt{n}}\Bigg[\sum_{l=1}^{N}\begin{pmatrix} D_{1+} - p_{1+} \\ \vdots \\ D_{N+} - p_{N+}\end{pmatrix}' H_{N,\mathbf{AA}}^{-1}e_l\left\{\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l \partial \mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0, \mathbf{A}\left(\beta_0\right)\right)\right\} \\
H_{N,\mathbf{AA}}^{-1}\begin{pmatrix} D_{1+} - p_{1+} \\ \vdots \\ D_{N+} - p_{N+}\end{pmatrix}\Bigg]. \qquad (57)
\end{aligned}$$

Applying the trace operator to (57) and cycling elements yields

$$\frac{1}{2}\frac{1}{\sqrt{n}}\sum_{l=1}^{N}\mathrm{Tr}\left(\left\{\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial\mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right\}H_{N,\mathbf{AA}}^{-1}\begin{pmatrix}D_{1+}-p_{1+}\\ \vdots\\ D_{N+}-p_{N+}\end{pmatrix}\begin{pmatrix}D_{1+}-p_{1+}\\ \vdots\\ D_{N+}-p_{N+}\end{pmatrix}'H_{N,\mathbf{AA}}^{-1}e_l\right),$$

which, after taking expectations conditional on $\mathbf{X}$ and $\mathbf{A}_0$, gives

$$-\frac{1}{2}\frac{1}{\sqrt{n}}\sum_{l=1}^{N}\mathrm{Tr}\left(\left\{\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial\mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right\}H_{N,\mathbf{AA}}^{-1}e_l\right) \tag{58}$$

The difference between (57) and its expectation (58) is $o_p\left(1\right)$. To see this observe that the diagonal elements of the $N\times N$ matrix

$$\left[\sum_{i=1}^{N}\sum_{j<i}s_{\mathbf{A}ij}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]\left[\sum_{i=1}^{N}\sum_{j<i}s_{\mathbf{A}ij}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]'=\begin{pmatrix}D_{1+}-p_{1+}\\ \vdots\\ D_{N+}-p_{N+}\end{pmatrix}\begin{pmatrix}D_{1+}-p_{1+}\\ \vdots\\ D_{N+}-p_{N+}\end{pmatrix}'. \tag{59}$$

consist of the terms $\left(D_{i+}-p_{i+}\right)^2$ for $i=1,\ldots,N$. Fix i, order the balance of units arbitrarily, and define $l_{j|i}=\left(D_{ij}-p_{ij}\right)\left(D_{i+}-p_{i+}\right)-p_{ij}\left(1-p_{ij}\right)$; note that $\left\{l_{j|i}\right\}_{j=1}^{\infty}$ is a martingale difference sequence (with $\mathbb{E}\left[\left.l_{j|i}\right|l_{1|i},\ldots,l_{j-1|i}\right]=0$ and bounded moments). A law of large numbers for martingale difference sequences therefore gives (recalling that a $+$ denotes summation over the omitted subscript)

$$\frac{1}{N-1}\left(D_{i+}-p_{i+}\right)^2\ \xrightarrow{p}\ \lim_{N\rightarrow\infty}\left\{\frac{\sum_{j\neq i}p_{ij}\left(1-p_{ij}\right)}{N-1}\right\}.$$

A similar argument can be used to characterize the probability limits of the off-diagonal elements of (59)

$$\frac{1}{N-1}\left(D_{i+}-p_{i+}\right)\left(D_{k+}-p_{k+}\right)\ \xrightarrow{p}\ \lim_{N\rightarrow\infty}\left\{\frac{p_{ik}\left(1-p_{ik}\right)}{N-1}\right\}.$$

Together these results imply that $H_{N,\mathbf{AA}}^{-1}\left[\sum_{i=1}^{N}\sum_{j<i}s_{\mathbf{A}ij}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]\left[\sum_{i=1}^{N}\sum_{j<i}s_{\mathbf{A}ij}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]'=-I_N+o_p\left(1\right)$ and hence (58).

To evaluate (58) it is convenient to be able to replace $H_{N,\mathbf{AA}}^{-1}$ with $-V_N$:

$$
\begin{aligned}
&-\frac{1}{2\sqrt{n}}\sum_{l=1}^{N}\mathrm{Tr}\left(\left\{\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial \mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right\}H_{N,\mathbf{AA}}^{-1}e_l\right)\\
= &\ \frac{1}{2\sqrt{n}}\sum_{l=1}^{N}\mathrm{Tr}\left(\left[\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial \mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]V_N^{-1}e_l\right)\\
&+\frac{1}{2\sqrt{n}}\sum_{l=1}^{N}\mathrm{Tr}\left(\left[\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial \mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]\left(-H_{N,\mathbf{AA}}^{-1}-Q_N\right)e_l\right)\\
&+\frac{1}{2\sqrt{n}}\sum_{l=1}^{N}\mathrm{Tr}\left(\left[\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial \mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]\left(Q_{N,\mathbf{AA}}-V_N^{-1}\right)e_l\right). \qquad (60)
\end{aligned}
$$

The first term in (60) coincides with the k^{th} element of the bias expression given in the statement of the theorem. Evaluating this term yields

$$
\begin{aligned}
\frac{1}{2}\frac{1}{\sqrt{n}}\sum_{l=1}^{N}\mathrm{Tr}\left(\left[\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial \mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]V_N^{-1}e_l\right) &=\\
\frac{1}{2}\frac{1}{\sqrt{n}}\sum_{l=1}^{N}\left[\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial \mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]&\\
\times\begin{pmatrix}0\\ \vdots\\ 0\\ \frac{1}{\sum_{j\neq l}p_{lj}\left(1-p_{lj}\right)}\\ 0\\ \vdots\\ 0\end{pmatrix} &=\\
&=\\
-\frac{1}{2}\frac{1}{\sqrt{n}}\sum_{l=1}^{N}\frac{\sum_{j\neq l}p_{lj}\left(1-p_{lj}\right)\left(1-2p_{lj}\right)W_{k,lj}}{\sum_{j\neq l}p_{lj}\left(1-p_{lj}\right)}.&
\end{aligned}
$$

The second and third terms are asymptotically negligible. Equation (52) follows directly.