Portfolio diversification and internalization of production externalities through majority voting

Hervé Crès
HEC School of Management

Mich Tvede
HEC School of Management
University of Copenhagen

September 2005

Abstract

A general equilibrium model with uncertainty and production externalities is studied. In absence of markets for externalities, we look for governances and conditions under which majority voting among shareholders is likely to give rise to efficient internalization. Two observations lead the analysis: On the one hand, the shareholders with the right incentives for efficient internalization are those who hold the (perfectly diversified) market portfolio. On the other hand, the financial market clearing conditions puts the market portfolio at the center of gravity of the equilibrium portfolio distribution: thanks to the market mechanism, the market portfolio becomes the mean equilibrium portfolio. Combining these two observations, we propose two mean voter theorems to show that in a stakeholder democracy (one stakeholder, one vote) efficient internalization is likely to come out of a voting process among shareholders: Randomizing over exogeneous technology and endowment shocks, we show that perfect internalization is a 50%-majority stable equilibrium in large economies.

Keywords: Production externalities, majority voting, portfolio diversification, general equilibrium, stakeholder democracy, shareholder governance, mean voter.
1 Introduction

We address the problem of managing externalities between firms (or production externalities for short) when there is no markets to trade them. Due to this incompleteness, markets in general fail in internalizing externalities. Nevertheless, depending on their portfolios, shareholders may be interested in some degree of internalization: Suppose that firm $j$ can increase its profit at the expense of the profit of firm $k$, then a shareholder with a portfolio that includes shares in both firms is interested in some degree of internalization, because the dividend of the portfolio comes in part from firm $j$ and in part from firm $k$. Therefore if the ‘preferences’ of firms represent some aggregation of the shareholders’ interests rather than some objective function such as profit maximization, then thanks to portfolio diversification markets may promote internalization of external effects.

Typically portfolios of shareholders are not identical, some shareholders have more shares in firm $j$ than in firm $k$ while it is the opposite for some other shareholders. Therefore shareholders usually disagree on how firms should be managed. The present paper shows that the disagreement between shareholders in markets with production externalities is similar to the disagreement between shareholders in incomplete financial markets to the extent that the conflict of interests may be seen as a conflict on the prices for which profit should maximized.

Two extreme positions of a shareholder clarify the stakes. If a shareholder in firm $j$ is perfectly diversified (holding the same proportion of shares in each firm), then clearly the shareholder is interested in the overall performance of the production sector rather than the performance of each firm, so the shareholder favors that firm $j$ internalizes perfectly. And if a shareholder in firm $j$ is completely undiversified (holding shares in only that firm), then he/she is interested only in the performance of firm $j$ rather than the overall performance of the production sector, so the shareholder favors profit maximization with respect to market prices. More generally, if the shareholder has less (more) shares in firm $k$ than in firm $j$, then the shareholder favors less (more) than perfect internalization for firm $j$. But the following intuition drives our main argument: at the financial market equilibrium, if there exists a shareholder with less shares in firm $k$ than in firm $j$ — therefore biased toward less than perfect internalization for firm $j$ —, then there must exist a shareholder with more shares in firm $k$ than in firm $j$ —therefore biased toward more than perfect internalization for firm $j$—; and under some well designed governance these biases neutralize each other. Furthermore, thanks to the financial market clearing condition, on average shareholders hold the (perfectly diversified) market portfolio, which happen to be the one for which there is no bias: it gives the right incentives for efficient internalization. The question therefore becomes: which governance gives the managing power to the shareholders who hold the market portfolio?
The theory of social choice offers several ‘mean voter’ arguments that help answer this question. We show that the stakeholder democracy (one stakeholder\(^1\), one vote) is the voting-based governance most likely to promote efficient internalization. It is benchmarked through the design of an even better performing governance (although much more complex, see Section 5). It is also compared to the classical shareholder governance (one share, one vote). There is a very significant difference between the latter and the stakeholder democracy. Consider the group of shareholders in firm \(j\): while they own 100 percent of shares in firm \(j\), they typically own less (or more, in case of no short-sale constraints) than 100 percent of the shares in firm \(k\). Therefore the mean shareholder in firm \(j\) is typically not interested in perfect internalization, but in either less or more than perfect internalization. Conversely, at the market equilibrium, the mean stakeholder owns the same fraction of firm \(j\) and firm \(k\), so he/she is interested in perfect internalization.

So, deviating from the standard approach, in the present paper firms do not have a specified objective function. Rather the behavior of firms is modeled as protecting some aggregate of the shareholders’ (or stakeholders) interests in the following sense: the firm chooses a production plan such that no alternative production plan makes a (super) majority of them better off. Based on this characterization we generalize the observation of Hansen and Lott (1996) that if all shareholders are perfectly diversified, then perfect internalization is the only outcome. Hardly surprising, the conditions ensuring that all shareholders are perfectly diversified are quite similar to the conditions under which muti-fund separation theorems are valid (see Proposition 1 and Corollary 2). Moreover we give conditions on the distribution of externalities between firms, the distribution of portfolios between shareholders and the (super) majority voting rule such that even though shareholders are not perfectly diversified (so they have different interests) perfect internalization is protected (see Theorem 3 and Theorem 4).

The problem of providing a public good through majority voting shows some similarity to ours. Since provision of public goods based on the market mechanism typically fails, mechanisms based on majority voting appear as natural and legitimate alternatives to the market mechanism. Whether such collective-decision mechanisms implement efficient allocations is a question that has been studied since Bowen (1943). Indeed Bowen shows that if the marginal rates of substitution of the voters are symmetrically distributed and all voters pay the same fraction of cost associated with the provision of the public good, then Pareto optimal allocations happen to be the optimal choice of the median voter. Hence Pareto optimal allocations are stable under the 50\%-majority voting rule. The argument of Bowen has been extended by Bergstrom (1979) (see also Barlow (1970)).

\(^1\)A stakeholder of firm \(j\) is an investor who has shares in firm \(j\) and/or in a firm impacted by production decisions in firm \(j\).
The analysis of Bowen might be useful for understanding the contributions of the present paper. Consider an economy with a private good (used as numéraire) which can be transformed into a public good by a decreasing returns to scale technology. At a given allocation let \( p_i \) be the marginal rate of substitution (gradient) between the public and the private goods for consumer/voter \( i \), and let \( p \) be the marginal rate of technical substitution for the firm. The so-called Bowen-Lindahl-Samuelson (necessary) condition for the allocation to be optimal is \( \sum_i p_i = p \). Now suppose that every voter pays the same fraction of costs, so if the number of voters is \( n \), then the marginal tax rate is \( (1/n)p \). Therefore at an optimal allocation the mean gradient \( (1/n) \sum_i p_i \) is equal to the marginal tax rate. Clearly the distribution of gradients is one-dimensional, so a median voter exists. Hence if gradients are symmetrically distributed, then at an optimal allocation the mean gradient is the median gradient, so the unique political equilibrium is the optimal allocation.

Bowen’s approach has two caveats: (1) in general, the median is not the mean; (2) as soon as there are more than one public good, the problem lies in a multi-dimensional setup and the concept of median voter is tricky to generalize\(^2\); and since Plott (1967) it is known that an equilibrium usually does not exist under the 50%-majority voting rule\(^3\). However there is a comforting observation: caveat (2) makes caveat (1) less embarrassing after all. There is more: among the successful attempts in the theory of social choice to go beyond the one-dimensional setting, Caplin and Nalebuff (1988, 1991) provide a rather strong argument fingerling the mean voter as a likely winner of the voting process. They show that under some conditions on individual preferences and on their distribution, the ideal candidate of the mean voter is a stable outcome, hence a political equilibrium, for super majority rules with a not too conservative rate of super majority (inferior to 64%); moreover, under some additional conditions, the ideal candidate of the mean voter is the min-max, so it is the political outcome that is stable under the lowest possible rate of super majority. Another strand of the social choice litterature promoting the idea that the candidate of the mean voter is the best placed for winning an election is the one on probabilistic voting (see, e.g., Hinich (1977), Coughlin (1992)). We propose here two new arguments (Theorem 3 and Theorem 4) supporting the same idea: Randomizing over exogeneous technology and endowment shocks, we show that perfect internalization is a 50%-majority stable equilibrium in large economies.

We also argue that a stakeholder democracy appears to be better than the traditional shareholder governance to protect perfect internalization. Production externalities are

\(^2\)Greenberg (1979) may be seen as a way to generalize the median voter argument to multi-dimensional settings.

\(^3\)Grandmont (1978) gives an extension of the conditions on preferences, and on the distribution of preferences under which a 50%-majority political equilibrium exists in a multi-dimensional setup.
just one instance where markets fail in leading the shareholders to unanimously support an efficient production plan. Another instance is incomplete financial markets. Collective decision mechanisms have been studied for that problem too. A mechanism based on Lindahl pricing (side payments) was proposed by Drèze (1974), and then by Grossman and Hart (1979), in order to recover (constrained) efficiency. Along a different avenue, following Gevers (1974), mechanisms based on majority voting have naturally been proposed and studied (see, e.g., Benninga and Muller (1979), Drèze (1985), Sadanand and Williamson (1991), De Marzo (1993), Kelsey and Milne (1996), Tvede and Crès (2005)). Crès and Tvede (2004) reconcile these two approaches along the line proposed in the present paper: the Drèze (1974) criterion to recover (the first order conditions of constrained) efficiency indicates that production should be optimized with respect to the gradients of the mean shareholder; so, unlike in the present paper, the shareholder governance seems to be the right one to promote (constrained) efficiency.

The paper is constructed as follows: Section 2 introduces the model, defines the market and political equilibrium concepts, sets the assumptions and provides the conditions for efficient internalization. In Section 3 the fundamental structure of the political problem is exposed; it first focuses on perfect individual portfolio diversification: conditions are given under which shareholders are unanimous, a unanimity which always results in efficient internalization; it then turns toward the more general case where individual portfolio are not perfectly diversified: we provide a measure for the degree of disagreement between shareholders, and based on that, a characterization of what are ‘good’ distributions of portfolios for the political process in the firm. Then Section 4 provides our main results, obtained in stochastic environments, which underline the efficiency property of the stakeholder democracy. The latter is benchmarked in Section 5 through the design of a better performing governance.

2 The model

Consider an economy with 2 dates, \( t \in \{0, 1\} \), 1 state at the first date \( s = 0 \), and \( S \) states at the second date \( s \in \{1, \ldots, S\} \). There are: 1 commodity at every state, a finite number of consumers with \( i \in I \) where \( I = \{1, \ldots, I\} \) and \( J \) firms where \( J = S \) with \( j \in J \) where \( J = \{1, \ldots, J\} \). Consumers are characterized by their identical consumption sets \( \mathcal{X} = \mathbb{R}^{S+1} \), initial endowments \( \omega_i \in \mathbb{R}^{S+1} \), utility functions \( u_i : \mathcal{X} \to \mathbb{R} \), and initial portfolio of shares in firms \( \delta_i = (\delta_{i1}, \ldots, \delta_{iJ}) \), where \( \delta_{ij} \in \mathbb{R} \) and \( \sum_{i \in \mathcal{I}} \delta_{ij} = 1 \) for all \( j \).

The classical approach of firms in the financial equilibrium literature is that they choose their output (or return) vector, interpreted as a financial asset. But here, since there are external effects, firms cannot choose that asset directly. They choose a vector of actions,
which can be interpreted as inputs, or investment/returns in some states only, or some organizational parameter that they control, etc. Hence firm $j$ is characterized by its sets of action $A_j$ and production function $F_j : A \to \mathbb{R}^{S+1}$ where $A = \prod_{j \in J} A_j$, so if the action of firm $k$ is $a_k$ then $y_j = F_j(a_1, \ldots, a_J)$ is the production plan of firm $j$. The set of possible action is described by the map $G_j : \mathbb{R}^{n_j} \to \mathbb{R}^S$ such that $A_j = \{a_j \in \mathbb{R}^{n_j} | G_j(a_j) \leq 0\}$, where $n_j$ is the number of actions under control by firm $j$.

**Individual programs and equilibrium concepts**

Let $q = (q_1, \ldots, q_J)$ where $q_j \in \mathbb{R}$ is the price of shares in firm $j$, be the price system. Consumers choose consumption plans $x_i \in \mathcal{X}$ and portfolios $\theta_i \in \mathbb{R}^J$. Firms choose action $a_j \in A_j$.

The program of consumer $i$ given a price system for shares and a collection of individual actions $(q, a)$ where $a = (a_1, \ldots, a_J)$ is

$$
\max_{x_i, \theta_i} u_i(x_i)
$$

s.t. \begin{align}
    x_i^0 - \omega_i^0 &= \sum_j q_j \delta_{ij} - \sum_j (q_j - y_j^0) \theta_{ij} \\
    x_i^s - \omega_i^s &= \sum_j y_j^s \theta_{ij} \text{ for all } s \geq 1.
\end{align}

(1)

We stick here to the case where there is no redundant asset. Hence, under the assumption of ‘sincere trading’ (compatible with the traditional competitive analysis), no one tries to become pivotal in the voting process within, say, firm $j$ by financing an infinitely long position in firm $j$ through some infinitely short positions in other firms: there is no strategic considerations in the portfolio choice. Shareholders self-select only according to their insurance needs. This assumption is reinforced by our focusing on the so-called ‘stakeholder democracy’, where investor cannot gain power in the political process by strategically rebalancing their portfolio. Before introducing our full-fledged political and economic equilibrium concept (see Definition 2), let us define stock market equilibria.

**Definition 1** For a collection of individual actions $a$, a **stock market equilibrium with fixed actions** denoted $\text{SME}(a)$, is a price system for shares and a collection of individual consumption plans and portfolios $(\bar{q}, \bar{x}, \bar{\theta})$ where $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_I)$ and $\bar{\theta} = (\bar{\theta}_1, \ldots, \bar{\theta}_I)$, such that:

- consumers maximize their utilities: $(\bar{x}_i, \bar{\theta}_i)$ is a solution to the program of consumer $i$ given $(\bar{q}, a)$;
- markets clear: $\sum_i \bar{x}_i = \sum_i \omega_i + \sum_j F_j(a)$ and $\sum_i \bar{\theta}_i = \sum_i \delta_i$. 

6
The study will be restricted to the case of complete financial markets. Therefore at a SME(a) the gradients of the consumers/shareholders are collinear. Let \( \bar{\mathbf{p}} \in \mathbb{R}^{S+1} \) denote the common normalized identical gradient vector (by normalizing its date zero component to one).

Although financial markets are complete, there is market incompleteness in the present model because there are no market for externalities. Hence shareholders disagree on how firms should be managed. In particular, there is no unanimity of shareholders for profit maximization with respect to \( \bar{\mathbf{p}} \). Indeed the only case where a shareholder wants a firm to maximize profits with respect to \( \bar{\mathbf{p}} \) is the case where he is not affected by the externalities generated by the firm, e.g. when his portfolio is totally undiversified in the sense that he only has shares in the considered firm. In order to formalize this intuition, consider the following construction.

Assuming differentiability, for all \( j \) and \( k \) let \( \bar{p}_{jk} \in \mathbb{R}^{n_j} \) denote the ‘price’ vector of marginal externalities of firm \( j \) on firm \( k \), so
\[
\bar{p}_{jk} = \bar{p}D_{aj}F_k(a).
\]
(2)

It vector prices the action of firm \( j \) according to their marginal impact on firm \( k \). Profit maximization in production plans with respect to \( \bar{p} \) is equivalent to profit maximization in actions with respect to \( \bar{p}_{jj} \). Next, let \( \bar{p}_{ij} \) denote the ‘price’ vector of marginal changes in action on consumer \( i \), so \( \bar{p}_{ij} \) is the vector that firm \( j \) should use to price actions according to consumer \( i \):
\[
\bar{p}_{ij} = \sum_k \bar{\theta}_{ik}\bar{p}_{jk}.
\]
(3)

From this construction it becomes clear that externalities make shareholders disagree on the optimal production plans for the firms: Each consumer wants the action of firm \( j \) to be chosen optimally with respect to his idiosyncratic pricing vector \( \bar{p}_{ij} \).

The fact that markets fail to push shareholders to agree on the way to price actions in firms raises a political problem that naturally leads to a the study of whether collective decision mechanisms can help reduce this market failure. In the present paper we focus on majority voting mechanisms. But the first obstacle that arises is the generic non-existence of 50%-majority voting equilibria in multidimensional setups. A way out is to consider super majority voting rules. The behavior of the firms is modelled through the following stability concept: the firm provides a production plan that aggregates its shareholders’ preferences in the sense that no alternative production plan makes a (super) majority of

\[\text{This is the case also with incomplete financial markets: A shareholder wants profit to be maximized with respect to his idiosyncratic equilibrium gradient; and at equilibrium shareholders’ gradients are typically not collinear because utility maximization only make gradients orthogonal to the subspace of possible income transfers, which has codimesion two at least in case of incomplete financial markets.}\]
them better off. This stability concept allows us to define preferences for the firms as follows.

Let \( A_{ij}(x_i, \theta_i, a) \subset A_j \) denote the set of actions for firm \( j \) that at the consumption bundle, portfolio and collection of individual actions \((x_i, \theta_i, a)\) make consumer \( i \) better off, so

\[
A_{ij}(x_i, \theta_i, a) = \{ a'_j \in A_j | u_i(x_i + \sum_k \theta_{ik}[F_k(a'_j, a_{-j}) - F_k(a)]) > u_i(x_i) \}.
\]

Next, at \((x, \theta, a)\) let \( I_j(x, \theta, a, a'_j - a_j) \) denote the set of consumers who are better off with action \( a'_j \) than with action \( a_j \) for firm \( j \), so

\[
I_j(x, \theta, a, a'_j - a_j) = \{ i \in I | a'_j \in A_{ij}(x_i, \theta_i, a) \}.
\]

Finally let \( \mu_j = (\mu_{ij}, \cdots, \mu_{IJ}) \) where \( \mu_{ij} \geq 0 \), be a collection of individual voting weights for decision making in firm \( j \). Then for a rate of majority \( \rho \in [0, 1] \) preferences of firms are described by correspondences \( P^\rho_j : X^I \times \mathbb{R}^{IJ} \times A \times \mathbb{R}^I \rightarrow A_j \) defined by

\[
P^\rho_j(x, \theta, a, \mu_j) = \begin{cases} 
\emptyset & \text{for } \sum_i \mu_{ij} = 0 \\
\{ a'_j \in A_j | \frac{\sum_{i \in I_j(x, \theta, a, a'_j - a_j)} \mu_{ij}}{\sum_i \mu_{ij}} > \rho \} & \text{for } \sum_i \mu_{ij} > 0.
\end{cases}
\]

Thus \( a_j \) is a solution to the program of firm \( j \) if \( P^\rho_j(x, \theta, a, \mu_j) = \emptyset \). So we can now introduce our full-fledged political and economic equilibrium concept, where decisions in firms are endogenized through a voting mechanism.

**Definition 2** \((\bar{q}, \bar{x}, \bar{\theta}, \bar{a})\) is a \( \rho \)-majority stable equilibrium for governance \( \mu \), denoted \( \rho\text{-MSE}(\mu) \), if

- \((q, x, \theta)\) is a SME(\( \bar{a} \)), and;

- \(\bar{a}_j\) is a solution to the program of firm \( j \), so \( P^\rho_j(\bar{x}, \bar{\theta}, \bar{a}, \mu) = \emptyset \).

A collection of individual consumption bundles and actions \((x, a)\) is denoted a state and a collection of individual consumption bundles, portfolios and actions \((x, \theta, a)\) is denoted an extended state.
**Assumptions**

Let us now turn toward our body of assumptions. Most of them are classical in the general equilibrium literature, or mere adaptations of classical assumptions to the current setup (with actions instead of production plans). A remark though: to found and strengthen the differentiable approach leading to the definition of the firms’ and investors’ pricing vectors (see Equations (2) and (3)), we need assumptions ensuring the *principle of minimal differentiation* (see Lemma 1 below). The latter principle states that in the political process for the choice of an action, when a challenger wants to maximize its support against the status quo, infinitesimal changes of action perform better than large changes. This is secured mainly by appropriate convexity assumptions.

Consumer $i$ is supposed to satisfy the following assumptions:

(A.1) $u_i \in C^1(\mathcal{X}, \mathbb{R})$ with $Du_i(x) \in \mathbb{R}^{S+1}$ for all $x \in \mathcal{X}$.

(A.2) $u_i$ is quasi-concave and $u_i^{-1}(r)$ is bounded from below for all $r \in \mathbb{R}$.

Both assumptions are standard (see Balasko (1988)).

For firm $j$ the set of actions is supposed to satisfy the following assumptions:

(A.3) $\mathcal{A}_j$ is compact and convex.

(A.4) $G_j \in C^1(\mathbb{R}^{n_j}, \mathbb{R})$ with $DG_j(b) \neq 0$ for all $b \in \mathbb{R}^{n_j}$.

Assumption (A.4) implies that the set of actions is a $n_j$-dimensional manifold. For firm $j$ the production function is supposed to satisfy the following assumptions

(A.5) $F_j : \mathcal{A} \to \mathbb{R}^{S+1}$ is concave in each variable.

(A.6) $F_j \in C^1(\mathcal{A}, \mathbb{R}^{S+1})$.

Assumptions (A.3) and (A.5) ensure that, for fixed actions $a_{-k}$ of all firms but firm $k$, the production set of firm $j$

$$\mathcal{Y}_j(a_{-k}) = \{Y_j \in \mathbb{R}^{S+1}| Y_j = F_j(a_k, a_{-k}) \text{ for some } a_k \in \mathcal{A}_k\}$$

is concave.

The production sector is supposed to satisfy the following assumptions:

(A.7) For all $a = (a_1, \ldots, a_J)$ the matrix

$$Y(a) = \begin{pmatrix} F_1^1(a) & \cdots & F_1^J(a) \\ \vdots & \ddots & \vdots \\ F_S^1(a) & \cdots & F_S^J(a) \end{pmatrix}$$

has full rank.
For all actions \( a = (a_j, a_{-j}) \) and firms \( j \) if \( a_j \in \text{int} A_j \) then there exists \( a'_j \in A_j \) such that \( \sum_k F_k(a'_j, a_{-j}) \geq \sum_k F_k(a_j, a_{-j}) \) and \( \sum_k F_k(a'_j, a_{-j}) \neq \sum_k F_k(a_j, a_{-j}) \).

For all \( a \), for all \( j \), the matrix \( D_{aj}F(a) \) has rank \( n_j \).

Assumption (A.7) excludes that firms are able to replicate production plans of each other. (A.8) ensures that only collections of individual actions in the boundaries of action sets produce efficient production plans. Finally (A.9) excludes superfluous actions: a change in the action of a firm must produce a change in the production plan of some firm.

All assumptions are supposed to be satisfied in the sequel.

Let us close this subsection by showing that our assumptions ensure the principle of minimal differentiation: infinitesimal changes of action perform better than large changes against the statu quo. This principle is secured by three assumptions: (1) convexity of the sets of feasible actions, so for any technologically feasible change of action \( \Delta a_j \) for firm \( j \), all infinitesimal changes \( da_j = \epsilon \Delta a_j \) where \( \epsilon \to 0 \), are also feasible; (2) convexity of the production functions, so for any technologically feasible change of production plan \( \Delta y = F(a'_j, a_{-j}) - F(a_j, a_{-j}) \) for firm \( j \), all infinitesimal changes \( dy_j = \epsilon (F(a'_j, a_{-j}) - F(a_j, a_{-j})) \) where \( \epsilon \to 0 \) are also feasible, and; (3) quasi-concavity of utility functions, so if a change of a production plan \( \Delta y \) makes consumer \( i \), then all infinitesimal changes \( dy = \epsilon \Delta y \) where \( \epsilon \to 0 \) also makes the consumer better off.

Let \( T_j(a_j) \subset \mathbb{R}^{n_j} \) be defined by

\[
T_j(a_j) = \begin{cases} 
\mathbb{R}^{n_j} & \text{for } a_j \in \text{int} A_j \\
\{ b_j \in \mathbb{R}^{n_j} | DG_j(a_j) \cdot b_j \leq 0 \} & \text{for } a_j \in \text{bd} A_j.
\end{cases}
\]

The principle takes the following form:

**Lemma 1** For all \( a_j, a'_j \in A_j \) and \( a_{-j} \in \prod_{k \neq j} A_k \),

\[ F(a'_j, a_{-j}) - F(a_j, a_{-j}) \in D_{aj} F(a_j, a_{-j}) T_j(a_j). \]

**Proof:** Clearly \( A_j \subset \{a_j\} + T_j(a_j) \) according to (A.3). So

\[ \mathcal{Y}(a_{-j}) \subset \{ F(a_j, a_{-j}) \} + D_{aj} F(a_j, a_{-j}) T_j(a_j) \]

according to (A.5). Therefore \( F(a'_j, a_{-j}) - F(a_j, a_{-j}) \in D_{aj} F(a_j, a_{-j}) T_j(a_j) \).

\[ Q.E.D \]

**Efficiency conditions**

Let us first define Pareto optimal states.
Definition 3 A state \((x, a) \in X^I \times A\) is Pareto optimal if there does not exist another state \((x', a') \in X^I \times A\) such that:

- \(\sum_i x'_i \leq \sum_i \omega_i + \sum_j F_j(a')\).
- \(u_i(x'_i) \geq u_i(x_i)\) for all \(i\) with “\(>\)” for at least one consumer.

The following lemma provides the usual necessary conditions for Pareto optimality.

Lemma 2 A state \((x, a) \in X^I \times A\) is Pareto optimal only if:

- There exist a normalized vector of ‘state prices’ \(p \in \{1\} \times \mathbb{R}^S_{++}\), and a collection of individual multipliers \((\nu_i)_i\), where \(\nu_i > 0\), such that \(Du_i(x_i) = \nu_i p\).
- Let \(p_j \in \mathbb{R}^n_j\) be defined by \(p_j = DG_j(a_j)\) and let \((p_j)_k\) be defined by Equation (2), then
  \[
  \sum_{k \in J} p_{jk} \in \langle p_j \rangle \tag{4}
  \]
  where \(\langle p_j \rangle\) is the span of \(p_j\).

Moreover if the aggregate production function \(a \rightarrow \sum_j F_j(a)\) is concave, then the conditions are sufficient.

Proof: If the state \((\bar{x}, \bar{a})\) is Pareto optimal, then it is a solution of the following optimization program:

\[
\max_{x, a} \quad u_1(x_1) \\
\text{s.t.} \quad \begin{align*}
  u_i(x_i) & \leq u_i(\bar{x}_i) \quad \text{for } i \geq 2 \\
  G_j(a_j) & \leq 0 \quad \text{for all } j \\
  \sum_i(x_i - \omega_i) & \leq \sum_j F_j(a)
\end{align*}
\]

Thanks to assumptions (A.1) and (A.8), one has at the solution: \(G_j(a_j) = 0\) for all \(j\), and \(\sum_i(x_i - \omega_i) = \sum_j F_j(a)\). First order derivatives of the Lagrangean with respect to \(x\) and \(a\) give the conditions of the lemma.

Q.E.D

Corollary 1 For a SME\((a, (\bar{q}, \bar{x}, \bar{\theta}), (\bar{q}, \bar{x}, \bar{\theta}), \text{the state } (\bar{x}, \bar{a})\text{ is Pareto optimal only if Equation (4) holds.}
3  Fundamentals of firms’ politics

In this section we focus on the structural aspects of the political game inside the firms stemming from the fact that, at stock market equilibria, there typically is some disagreements between shareholders on the aim the firm should pursue. Firstly we introduce two extreme cases where the degree of conflict is maximal, resp. minimal. Secondly we characterize coalitions which are never unanimous when proposed some alternative to the status quo; based on this characterization, we provide a measure of the degree of disagreement between shareholders. Thirdly this construction is applied to provide a characterization of what are ‘good’ distributions of portfolios for the political process in the firm.

Illustrations

At a SME(a), consider a SME(a) at which the pricing vectors \((\bar{p}_{ij})_{i \in I}\) are linearly independent. Then shareholders completely disagree on the price with respect to which the firm should optimize its action’s choice. Indeed there exist alternatives to \(a_j\) such that all shareholders but one are better off; moreover no other action is stable for a lower rate of majority. Hence, in a stakeholder democracy, only super majority rules with rate \(\rho > (I - 1)/I\) support perfect internalization. This result is in line with the result in Greenberg (1979): \((I - 1)/I\) is the lowest super majority rule that ensures existence of equilibrium in voting models where the dimension of conflict is \(I - 1\), as in the present case.

Alternatively, at the other extreme, consider the case where shareholders have collinear pricing vectors \(\bar{p}_{ij}\) pointing in the same direction. This happens, e.g., when they are perfectly diversified: for each shareholder there exists a \(\tau_i > 0\) such that \(\bar{\theta}_i = \tau_i \mathbf{1}_J\) where \(\mathbf{1}_J \in \mathbb{R}^J\) is the market portfolio\(^5\). Then all shareholders agree that every firm should maximize its action with respect to the efficient pricing vector \(\bar{p}_j = \sum_k \bar{p}_{jk}\). Indeed, market clearing yields: \(\sum_i \bar{p}_{ij} = \sum_k \bar{p}_{jk} = \bar{p}_j\), and the \(\bar{p}_{ij}\)'s being collinear, for all \(i\) \(\bar{p}_{ij}\) is collinear to \(\bar{p}_j\) pointing in the same direction. Therefore we make the following remarkable observation that underlines the special role played by the market portfolio in the political game:

\(^5\)The condition that all shareholders are perfectly diversified can be weakened. Suppose that firms are partitioned into \(L\) clusters, \(J_1, J_2, \ldots, J_L\), with \(J_\ell \cap J_{\ell'} = \emptyset\) if \(\ell \neq \ell'\) and \(\bigcup_\ell J_\ell = J\), so that if two firms pertain to two different clusters, they do not inflict external effects upon each other: hence \(\bar{p}_{jk} = 0\) whenever \(j \in J_\ell\) and \(k \in J_{\ell'}\) with \(\ell \neq \ell'\). It is easy to define the finest clustering of the economy. Then a weaker sufficient condition for shareholders’ unanimity for efficient internalization is that shareholder are all perfectly diversified within clusters: there exists a \(L\)-vector \(\tau_i = (\tau_{iL})\) such that \(\bar{\theta}_i = (\tau_{i1} \mathbf{1}_{J_1}, \ldots, \tau_{iL} \mathbf{1}_{J_L})\). Such a generalization of the paper is trivial and will be omitted, for the sake of lightness of notation, by sticking to only one cluster.
Observation 1 If all shareholders agree, then they unanimously support perfect internalization.

Conditions on consumers under which they unanimously support efficient internalization are now studied below.

Collinear pricing vectors:

We consider here economies where one firm, say firm 1, does not inflict not receive any production external effects, and whose only possible action leads to the provision of the riskless security \((0, 1)\) \(\in \mathbb{R}^{S+1}\). So firm 1 forms a cluster on its own and all other firms form a second cluster. Such an economy is known in the finance literature as a bond-equity economy. For shareholders’ unanimity for efficient internalization to obtain, it is sufficient that shareholders be perfectly diversified within the second cluster. This is exactly what happens when consumers have von Neumann-Morgenstern additively separable utility functions with linear risk tolerance and the same marginal risk tolerance, as proved in Cass and Stiglitz (1970) (see also Magill and Quinzii (1997), section 16, for a more recent, integrated treatment of that case).

\[(A.10)\] Utility functions are

\[u_i(x_i) = u_{i0}(x_i^0) + \sum_{s=1}^{S} \pi_s u_{i1}(x_i^s)\]

where \(u_{i0}\) and \(u_{i1}\) are strictly increasing and strictly concave. The risk tolerance

\[T_i(x) = -\frac{u_{i1}'(x)}{u_{i1}(x)}, \quad x \in \mathbb{R}\]

is linear: there exists \((\alpha_i, \beta_i)\) \(\in \mathbb{R}_+ \times \mathbb{R}\) such that \(T_i(x) = \alpha_i + \beta_i x\) on the relevant domain, i.e., whenever \(\alpha_i + \beta_i x > 0\). And agents all have the same marginal risk tolerance: \(\beta_i = \beta\) for all \(i\).

This assumption yields the hyperbolic constant absolute risk aversion (HARA) class of utility functions:

\[u_{i1}(x) = \begin{cases} 
(\alpha_i + \beta x)^{1-1/\beta} & \text{if } \beta \neq 0, \beta \neq 1 \\
1/\beta(1 - 1/\beta) & \text{if } \beta = 0 \\
-\alpha_i e^{-x/\alpha_i} & \text{if } \beta = 1 \\
\log(\alpha_i + x) & \text{if } \beta = 1 
\end{cases}\]

The case \(\beta \geq 0\) includes power functions with power less than 1, the log and the negative exponential. When \(\beta > 0\) and \(\alpha_i = 0\) the utility function exhibits constant relative risk aversion. The quadratic case corresponds to \(\beta = -1\).
Proposition 1 Under the additional assumption (A.10), at a SME(a), the equilibrium allocations satisfy a linear sharing rule and the two-fund separation property holds: for all $i$, the portfolio $\bar{\theta}_i$ is of the form $(\bar{\theta}_i, t_i 1_{J-1})$, with $\sum_i \bar{\theta}_i = \sum_i t_i = 1$; i.e., agent $i$ invests $(\bar{\theta}_i - \delta_i)$ in the riskless bond and $t_i$ in the market portfolio.

Proof: Classical result (see, e.g., proposition 16.15 in Magill and Quinzii (1997)). Q.E.D

Corollary 2 Assume (A.10), a $\rho$-MSE($\mu$), $(\bar{q}, \bar{x}, \bar{\theta}, \bar{a})$, satisfies the first-order conditions for Pareto optimality if:

- $\rho < 1$ for governances where only shareholders with positive amounts of shares can participate in the voting process$^6$;
- $\rho < 0.5$ in a stakeholder democracy where all shareholders with $t_i \neq 0$ have some voting right.

Proof: Consider a $\rho$-MSE($\mu$): $(\bar{q}, \bar{x}, \bar{\theta}, \bar{a})$. Then $(\bar{q}, \bar{x}, \bar{\theta})$ is a SME($\bar{a}$) and therefore, thanks to Corollary 1, one only has to check that equations (4) hold; thanks to Proposition 1, for all $j \in J \setminus \{1\}$, for all $i \in I$, $\bar{p}_{ij} = t_i \sum_{k \geq 2} \bar{p}_{j k}$: all shareholders such that $t_i > 0$ are unanimous on the way action of firm $j$ should be priced. So are shareholders such that $t_i < 0$.

Suppose that in firm $j$, equations (4) do not hold. Then there exists an infinitesimal change of action $da_j \in T_{\bar{a}} A_j$, the tangent space at $\bar{a}$ to $A_j$ (which is orthogonal to $\bar{p}_j$), such that for all $i$ with $t_i > 0$, $\bar{p}_{ij} \cdot da_j > 0$. Hence $I_j(\bar{x}, \bar{\theta}, \bar{a}, da_j) = \{ i \in I \mid t_i > 0 \}$ and $I_j(\bar{x}, \bar{\theta}, \bar{a}, -da_j) = \{ i \in I \mid t_i < 0 \}$.

If only shareholders with positive amounts of shares can participate in the voting process, $\sum_{i \in I} \mu_{ij} = \sum_{i \in I_j(\bar{x}, \bar{\theta}, \bar{a}, da_j)} \mu_{ij}$, therefore $\bar{a}_j + da_j \in P^\rho_j(\bar{x}, \bar{\theta}, \bar{a}, \mu_j)$ as soon as $\rho < 1$, a contradiction to the assumption that $(\bar{q}, \bar{x}, \bar{\theta}, \bar{a})$ is a $\rho$-MSE($\mu$).

Consider a stakeholder democracy. Then, since $\rho < 0.5$, either $\bar{a}_j + da_j \in P^\rho_j(\bar{x}, \bar{\theta}, \bar{a}, \mu_j)$ or $\bar{a}_j - da_j \in P^\rho_j(\bar{x}, \bar{\theta}, \bar{a}, \mu_j)$ (or both) depending on which of the two groups $I_j(\bar{x}, \bar{\theta}, \bar{a}, da_j)$ and $I_j(\bar{x}, \bar{\theta}, \bar{a}, -da_j)$ has the highest aggregate voting weight. (If the two groups have the same aggregate voting weight, then we have both.) Hence a contradiction.

Q.E.D

$^6$Note that $\rho < 1$ includes infra majority voting rules, and that such governances include the traditional ‘one share-one vote’ and ‘one shareholder-one vote’.
Political stability

Consider a SME, \( \mathcal{E} = (\tilde{q}, \tilde{x}, \tilde{\theta}) \) and let \( p_j \) be the supporting price of action \( a_j \). The following construction shows that if the current action \( a_j \) is optimized with respect to a supporting price \( p_j \) which somewhat *averages* the idiosyncratic prices \( \bar{p}_{ij} \) of the members of the coalition (in the sense that \( p_j \) is in the positive convex cone of the \( \bar{p}_{ij} \)'s), then there does not exist a change \( \Delta a_j \) of action that is *unanimously* supported by the coalition members.

Let us begin with some drawings. Figures 1.a and 1.b show two possible political configurations within firm \( j \) (where \( n_j = 3 \)): \( p_j \) is the supporting price of the action \( a_j \) and there are five shareholders having individual pricing vectors \( (\bar{p}_{ij})_{1 \leq i \leq 5} \). (To avoid three-dimensional pictures, without loss of generality all pricing vectors are supposed to be ‘normalized’ so that they lie in a two-dimensional hyperplane.)

Figure 1.a represents an optimistic scenario. Indeed, the maximal coalitions \( \mathcal{C} \) such that \( p_j \) does not lie inside the convex hull of the \( (p_{ij})_{i \in \mathcal{C}} \) are of size 3: \( \{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 1\}, \{5, 1, 2\} \). Hence, in the political game, any change of action will divide the electoral population in three voters against two, as shown by the five dotted lines, one for each possible division. In such a configuration, in a stakeholder democracy, we will say that the ‘score’ of \( p_j \) with respect to the family \( (\bar{p}_{ij})_{1 \leq i \leq 5} \) is \( 3/5 \), i.e. 60% (see Definition 4 below). Theorem 1 below formalizes this approach.

Figure 1.b on the other hand is a more pessimistic scenario: \( p_j \) does not lie inside...
the convex hull of the \((p_{ij})_{i \in C}\) where \(C = \{1, 2, 3, 4\}\). Hence there exists a change of action (indicated by the vertical dotted line) which rallies 4 votes against the status quo \(a\) supported by \(p_j\). Here the ‘score’ of \(p_j\) with respect to the family \((\tilde{p}_{ij})_{1 \leq i \leq 5}\) is 4/5, i.e. 80%. This configuration is more pessimistic to the extent that if one wants, for the sake of productive efficiency, to guarantee the political stability of \(p_j\) (which in these figures satisfies Equations (4)), then a super majority rate of more than 80% has to be adopted (see Corollary 3 below); hence a very conservative voting rule.

Let us now formalize that. Some pieces of notation are needed: For a given finite collection of \(H\) vectors \(V = (v_h)_{h \in H}\) where \(v_h \in \mathbb{R}^n\), let \(K(V)\) denote the convex cone generated by \(V\)

\[
K(V) = \{v \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R}^H : v = \sum_h \lambda_h v_h \text{ and } \lambda \geq 0\},
\]

and let \(K_+(V)\) denote the strictly positive convex cone

\[
K_+(V) = \{v \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R}^H : v = \sum_h \lambda_h v_h, \lambda \geq 0 \text{ and } \lambda \neq 0\}.
\]

**Theorem 1** At a SME(\(a\)), for a coalition of consumers \(C \subset I\), there does not exist a possible change of action \(\Delta a_j\) for firm \(j\) (so \(p_j \cdot \Delta a_j \leq 0\)) that is unanimously supported by all members of \(C\) if and only if \(p_j \in K((\tilde{p}_{ij})_{i \in C})\) or \(0 \in K_+(((\tilde{p}_{ij})_{i \in C})\).

**Proof:** Let the \(n_j \times J\) matrix \(\bar{P}_j\) be defined by \(\bar{P}_j = (\bar{P}_{jk})_k\). For firm \(j\), a change of action \(\Delta a_j\) where \(\Delta a_j \in \mathbb{R}^{n_j}\) is feasible if and only if

\[
p_j^t \Delta a_j \leq 0
\]

where \(p_j^t \Delta a_j = 0\) corresponds to an efficient change and \(p_j^t \Delta a_j < 0\) corresponds to an inefficient change.

For a coalition \(C \subset I\) of consumers let the \(J \times |C|\) matrix \(\bar{\theta}_C\) be defined by \(\bar{\theta}_C = (\bar{\theta}_i)_{i \in C}\). Then coalition \(C\) supports a change \(\Delta a_j\) if and only if

\[
\bar{\theta}_C^t \bar{P}_j^t \Delta a_j > 0.
\]

From Theorem 22.2 in Rockafellar (1970) it follows that either there exists a solution to (which is \(\Delta a_j \in \mathbb{R}^{n_j}\) such that)

\[
(a) \begin{cases}
p_j^t \Delta a_j \leq 0 \\
-\bar{\theta}_C^t \bar{P}_j^t \Delta a_j < 0
\end{cases}
\]

or there exists a solution to (which is \(\mu \in \mathbb{R}\) and \(\lambda \in \mathbb{R}^{|C|}\) such that)
\[
(b) \quad \begin{align*}
p_j \mu - \tilde{P}_j \tilde{\theta}_C \lambda &= 0 \\
I_{|C|} \lambda &\geq 0 \\
\mu &\geq 0 \\
(1, \ldots, 1) \lambda &> 0
\end{align*}
\]

where \( I_{|C|} \) is the \(|C| \times |C|\) identity matrix.

Clearly there exists a solution to \((b)\) if and only if there exists a solution to

\[
\begin{align*}
\tilde{P}_j \tilde{\theta}_C \lambda &= 0 \\
I_{|C|} \lambda &\geq 0 \\
(1, \ldots, 1) \lambda &> 0
\end{align*}
\]

or

\[
\begin{align*}
\tilde{P}_j \tilde{\theta}_C \lambda &= p_j \\
I_{|C|} \lambda &\geq 0
\end{align*}
\]

However \((b.1)\) is equivalent to the zero price vector being in the strictly positive convex cone generated by the individual pricing vector of the members of \(C\). And \((b.2)\) is equivalent to \(p_j\) being the convex cone generated by individual pricing vector of the members of \(C\).

\(Q.E.D\)

Obviously as a consequence of Theorem 1 we have \(p_j \notin K((\tilde{p}_{ij})_{i \in C})\) and \(0 \notin K_+((\tilde{p}_{ij})_{i \in C})\) if and only if there exists a change that coalition \(C\) unanimously supports. Therefore Theorem 1 enables us to define the rate of super majority that is necessary and sufficient (within firm \(j\)) for the current action to be majority stable, as well as the .

**Definition 4** At a SME\((a)\) the score of a price \(p_j\) with respect to the collection \((\tilde{p}_{ij})_{i \in I}\) (also called the score of action \(a_j\)) is defined as the maximum size of a coalition unanimously supporting some change within firm \(j\):

\[
\rho_\mu(p_j; (\tilde{p}_{ij})_{i \in I}) = \max \left\{ \frac{\sum_{i \in C} \mu_{ij}}{\sum_{i \in I} \mu_{ij}} \mid C \subset I \text{ and } p_j \notin K((\tilde{p}_{ij})_{i \in C}) \text{ and } 0 \notin K_+((\tilde{p}_{ij})_{i \in C}) \right\}.
\]

**Corollary 3** For a SME\((a)\) action \(a_j\) is \(\rho\)-majority stable within firm \(j\) if and only if \(\rho \geq \rho_\mu(p_j; (\tilde{p}_{ij})_{i \in I})\). A SME\((a)\) is a \(\rho\)-MSE\((\mu)\) if and only if \(\rho \geq \rho_\mu(a) = \max_j \rho_\mu(p_j; (\tilde{p}_{ij})_{i \in I})\).
Proof: Immediate.

Q.E.D

In case all consumers are unanimous in a firm (because, e.g., they all have the market portfolio), the score is 0 if \( p_j \) is collinear to \( \bar{p}_j \), 1 otherwise. In that case, efficient internalization is the only \( \rho \)-MSE, and it is so even for infra majority rules, whatever the governance. The score of a supporting price with respect to individual pricing vectors measures the degree of disagreement between shareholders, at a SME(\( a \)). The score as defined here is in the line of the traditional Simpson-Kramer approach: it is the maximum size of a coalition that unanimously supports some alternative action to the status quo. And the ‘best’ initial position for the status quo is the one with lowest score: the so-called ‘min-max’.

**Definition 5** The min-max score over all actions \( a \) is \( \rho^*_\mu = \min_{a \in A} \rho_\mu(a) \). The min-max set is \( A^* = \{ a \in A | \rho_\mu(a) = \rho^*_\mu \} \).

A first classical class of social choice results (see, e.g., Grandmont (1978)) allows us to underline once again the special role played by the market portfolio (and thus perfect internalization) in the political process, even in case consumers are not unanimous. This deals with the case where the collection \( (\bar{p}_{ij})_{i \in I} \) is axially balanced: within each firm \( j \) there exists a vector \( p^*_j \) such that every agent \( i \in I \) can be pairwise matched with another one\(^7\), \( \bar{i} \in I \), such that \( \bar{p}_{ij} + \bar{p}_{ij} = \lambda_i p^*_j \) and \( \bar{i} \) is matched with \( i \). Then the following result can easily be shown.

**Proposition 2** Suppose that at a SME(\( a \)), \( E \), for all \( j \) the collection of pricing vectors \( (\bar{p}_{ij})_{i \in I} \) is axially balanced; then \( E \) is a 0.5-MSE(\( \mu^d \)) if and only if it satisfies the first-order condition (4) of Pareto optimality.

**From pricing vectors to portfolios: the role of diversification**

The special structure of the present set-up allows us to study the political stability of SME(\( \bar{a} \))’s, for efficient actions \( \bar{a} \), in all firms at once by working with portfolios rather than with pricing vectors.

**Theorem 2** Let \( \bar{a} \) be a vector of efficient choices. At a SME(\( \bar{a} \)), for a coalition of consumers \( C \subset I \), there does not exist a possible change of action \( \Delta a \) in any firm that is unanimously supported by all members of \( C \) if \( 1_J \in K((\bar{\theta}_i)_{i \in C}) \) or \( 0 \in K_+((\bar{\theta}_i)_{i \in C}) \).

\(^7\) An agent such that \( \bar{p}_{ij} \) is collinear to \( p^*_j \) can be matched with himself.
Proof: If there exists a solution to
\[
\begin{align*}
\bar{\theta}_c \lambda &= 0 \\
I_{|C|} \lambda &\geq 0 \\
(1, \ldots, 1) \lambda &> 0
\end{align*}
\]
resp.
\[
\begin{align*}
\bar{\theta}_c \lambda &= 1_J \\
I_{|C|} \lambda &\geq 0
\end{align*}
\]
then there exists a solution to (b.1) resp. (b.2) in the proof of Theorem 1. The fact that (b.4) implies (b.2) follows from \(\sum_k \bar{p}_{jk} = \bar{p}_j\) (\(\bar{p}_j\) is a supporting price of \(\bar{a}_j\)). Hence if there exists a solution to (b.3) or (b.4), then there exists a solution to (b) and there exists no solution to (a) both in the proof of Theorem 1.

Q.E.D

Note that Theorem 2 only depends on portfolios, and not on consumers’ pricing vectors (which in turn depend on firms’ pricing vectors and portfolios) as in Theorem 1. However Theorem 2 provides a sufficient condition for a coalition not to support any change in any firm, while Theorem 1 provides a necessary and sufficient condition for a coalition not to support any change in some firm.

As in Definition 4 we define the score of a portfolio \(\theta\) with respect to the collection \((\bar{\theta}_i)_{i\in \mathcal{I}}\), as:
\[
\rho_\mu(\theta; (\bar{\theta}_i)_{i\in \mathcal{I}}) = \max \left\{ \frac{\sum_{i\in \mathcal{C}} \mu_{ij}}{\sum_i \mu_{ij}} \mid j \in J, C \subset \mathcal{I} \text{ and } \theta \notin K((\bar{\theta}_i)_{i\in \mathcal{C}}) \text{ and } 0 \notin K_+(((\bar{\theta}_i)_{i\in \mathcal{C}}) \right\}.
\]

**Corollary 4** Suppose that \(\mathcal{E}\) is efficient. Then \(\mathcal{E}\) is a \(\rho\)-MSE(\(\mu\)) if \(\rho \geq \rho_\mu(1_J; (\bar{\theta}_i)_{i\in \mathcal{I}})\).

**Proof:** The proof of Theorem 2 yields that \(\rho_\mu(\bar{p}_j; (\bar{p}_{ij})_{i\in \mathcal{I}}) \leq \rho_\mu(1_J; (\bar{\theta}_i)_{i\in \mathcal{I}})\). Then apply Corollary 3.

Q.E.D

Ideally from an economic point of view, one would like the market portfolio to have a very small score (i.e., close to 0.5). This would entail that efficient actions are protected by small rates of super majority. The score is small (say 55%) if the market portfolio lies in the convex cone of (the collection of) portfolios of all coalitions of 55% or more of the stakeholders. Intuitively, the market portfolio being the mean of the individual portfolios, there is hope that such a property hold. It is enough that the equilibrium distribution
of individual portfolios be not too ‘vicious’. In the following section, randomizing over endowments, we give conditions under which, in large economies, efficient internalization is protected by the 50%-majority rule because the distribution of individual portfolios shows up enough symmetry. Next, we show how this result can be preserved even for the most ‘vicious’ distribution of individual portfolios by randomizing over technology shocks.

4 The political stability of efficient internalization

Ideally, from an economic viewpoint, we would like efficient internalization to come out of the voting mechanism. To study the relative political stability of the efficient internalization amounts to study the stability property, in the space $\mathbb{R}^{n}$, of the efficient pricing vector $\sum_k \bar{p}_{jk}$, or, alternatively, to study the stability property in the space $\mathbb{R}^J$ of the market portfolio $1_j$. Once again the market portfolio has remarkable properties: we have already argued that if the agents are unanimous about the production policy of the firms, then this policy must satisfy the FOC of efficient internalization. The driving force in this result is the market clearing condition. We argue in the present section that, although market clearing does not ensure the first welfare theorem (because of the absence of markets for externalities), market clearing gives remarkable (relative) stability properties to the efficient internalization in the voting mechanism for the stakeholder democracy. The object of the sequel is to qualify this ‘political first welfare theorem’.

We propose two statistical approaches. The first one randomizes over endowments and builds on Theorem 3. The second approach randomizes over the direction of marginal external effects and builds on Theorem 4. Through both approaches, we show that the ‘score’ of perfect internalization converges to 0.5 as the economy becomes large.

Random initial endowments

In this approach, actions of firms $\bar{a} = (\bar{a}_j)_j$ and prices of shares $\bar{q} = (\bar{q}_j)_j$ are supposed to be fixed while the distribution of initial endowments and initial portfolios $(\omega_i, \delta_i)_i$ is supposed to be stochastic. However we only consider distributions of initial endowments and initial portfolios such that there exists a stock market equilibrium with fixed actions $(\bar{a}_j)_j$ where the prices of shares are $(\bar{q}_j)_j$.

Suppose that total resources are $r \in \mathbb{R}^{S+1}$ and $u \in C^1(X \times [0,1], \mathbb{R})$ such that $u(\cdot, t)$
satisfies (A.1) and (A.2). Then there exists a solution \((\bar{c}, \bar{a})\) where \(\bar{c} : [0, 1] \to \mathcal{X}\), to

\[
\max_{c, a} \int_t u(c(t), t) dt \quad \text{s.t.} \quad \int_t c(t) dt = r + \sum_j F_j(a).
\]

There exists a vector \(\bar{p} \in \mathbb{R}^{S+1}\) where \(\bar{p}^s > 0\) for all \(s\) such that \(D_x u(\bar{c}(t), t) = \bar{p}\) for all \(t\). Therefore for \(\bar{y} = (\bar{y}_j)_j\) where \(\bar{y}_j = F_j(\bar{a})\), let \(\bar{q}\) be defined by \(\bar{q}_j = (1/\bar{p}^0) \sum_s \bar{p}^s \bar{y}^s_j\).

For \(I \in \mathbb{N}\) let the allocation \(\bar{x} = (\bar{x}_i)_i\) be defined by

\[
\bar{x}_i = \int_{\frac{r}{1-1}}^{\frac{r}{2-1}} \bar{c}(t) dt,
\]

let the income distribution \(\bar{m} = (\bar{m}_i)_i\), be defined by \(\bar{m}_i = \sum_s \bar{p}^s \bar{x}^s_i\) and let \(u_i : \mathcal{X} \to \mathbb{R}\) be defined by

\[
u_i(x_i) = \sup \{ \int_{\frac{r}{i-1}}^{\frac{r}{j-1}} u(c(t), t) dt | \int_{\frac{r}{i-1}}^{\frac{r}{j-1}} c(t) dt = x_i \}.
\]

Then \((\bar{x}, \bar{a})\) is a Pareto optimal state.

The distribution of portfolios \(\bar{\theta} = (\bar{\theta}_i)_i\) depends on net-trades of the consumers \(\bar{x} - \omega = (\bar{x}_i - \omega_i)_i\), so for a given distribution of initial endowments and portfolios, only the distribution of initial endowments is relevant for the distribution of portfolios. Indeed if the initial distribution of initial endowments and initial portfolios is \((\omega, \delta)_i\), then the distribution of portfolios is \(\bar{\theta}\) where

\[
\begin{pmatrix}
\bar{\theta}_{i1} \\
\vdots \\
\bar{\theta}_{ij}
\end{pmatrix}
= \begin{pmatrix}
\bar{y}^1_1 & \cdots & \bar{y}^1_j \\
\vdots & \ddots & \vdots \\
\bar{y}^S_1 & \cdots & \bar{y}^S_j
\end{pmatrix}^{-1}
\begin{pmatrix}
\bar{x}^1_i - \omega^1_i \\
\vdots \\
\bar{x}^S_i - \omega^S_i
\end{pmatrix}.
\]

Let \(\pi_i : \mathbb{R}^{S+1} \to \mathbb{R}^J\) be defined by

\[
\pi_i(\omega_i) = \begin{pmatrix}
\bar{y}^1_1 & \cdots & \bar{y}^1_j \\
\vdots & \ddots & \vdots \\
\bar{y}^S_1 & \cdots & \bar{y}^S_j
\end{pmatrix}^{-1}
\begin{pmatrix}
\bar{x}^1_i - \omega^1_i \\
\vdots \\
\bar{x}^S_i - \omega^S_i
\end{pmatrix}.
\]

then \(\pi_i(\omega_i)\) is the portfolio of consumer \(i\). Therefore if \(\Omega \subset \mathbb{R}^{(S+1)J}\) is defined by

\[
\Omega = \{ (\omega_i)_i \in \mathbb{R}^{(S+1)J} | \sum_i \omega_i = r \}
\]

and if \(\Theta \subset \mathbb{R}^{IJ}\) is defined by

\[
\Theta = \{ (\theta_i)_i \in \mathbb{R}^{IJ} | \sum_i \theta_{ij} = 1 \text{ for all } j \}\]
then every probability measure $\Gamma$ on $\Omega$ induces a probability measure $\Phi$ on $\Theta$: let $\Phi$ be defined by

$$\Phi(A) = \Gamma(\{ (\omega_i)_i \in \Omega \mid (\pi_i(\omega_i))_i \in A \}).$$

Consequently probability measures on $\Theta$ rather than probability measures on $\Omega$ are considered.

Suppose that $\Psi$ is a probability measure with density $\psi$ on $\mathbb{R}$ such that the mean $E$ is positive, so

$$E = \int t \psi(t) dt > 0,$$

and the variance $V$ is finite, so

$$V = \int (t - E)^2 \psi(t) dt < \infty.$$

If $(\xi_{ij})_{ij}$ is the result of $IJ$ trials then let the associated distribution of portfolios $(\bar{\theta}_i)_i$ be defined by

$$\bar{\theta}_{ij} = \frac{\xi_{ij}}{\sum_k \xi_{kj}}.$$

Clearly $\sum_k \xi_{kj} \neq 0$ with probability 1 for all $j$, so $\bar{\theta}_{ij}$ is well-defined for all $i$ and $j$ and $\sum_i \bar{\theta}_{ij} = 1$ for all $j$.

**Theorem 3** For all $\rho > 1/2$ if $I$ tends to $\infty$, then the probability that $(\bar{q}, \bar{x}, \bar{\theta}, \bar{a})$ is a $\rho$-MSE converges to 1 for the stakeholder democracy $\mu^d$.

**Proof:** If the sequence $(\xi_{ij})_{ij}$ is the result of an infinite number of trials then let $((\theta_i^I)_i)_I$ be sequence of associated distributions of portfolios. Let the unit interval $[0,1]$ with the Lebesgue measure be the set of consumers, so for each $I$ the interval $[(i-1)/I, i/I]$ is consumer $i$, then $((\theta_i^I)_i)_I$ induces a signed vector-valued probability measure $\Lambda_I$ on $[0,1]$. Indeed the vector-valued density $\lambda_I : [0,1] \to \mathbb{R}^J$ of $\Lambda_I$ on the interval $[(i-1)/I, i/I]$ is $I\theta_i$, because then

$$I \int_{\frac{i-1}{I}}^{\frac{i}{I}} \theta_i dt = \theta_i.$$

Moreover $\Lambda_I$ induces a signed probability measure $\Upsilon_I$ on $\mathbb{R}^J$. Indeed for $A \subset \mathbb{R}^J$ let

$$\Upsilon_I(A) = \frac{\left| \{ i \in \{1, \ldots, I\} \mid I\theta_i \in A \} \right|}{I}.$$

According to Theorem 4.5.3 (Kolmogorov’s strong law of large numbers) in Ito (1984) the sequence $((\sum_i \xi_{ij})/I)_I$ converges to $E$ almost surely. Therefore the sequence of signed probability measures $(\Upsilon_I)_I$ converges almost surely to a signed probability measure $\Upsilon$ on $\mathbb{R}^J$ with density $\nu : \mathbb{R}^J \to \mathbb{R}$ defined by
\[ v(t) = \prod_j \frac{\psi(t_j)}{E}. \]

Clearly the distribution is symmetric with respect to the diagonal. Therefore for all \( p_j \) if \( p_j \neq \bar{p}_j \) then less than 50 pct. of the consumers support \( p_j \) against \( \bar{p}_j \). Hence for all \( \rho > 1/2 \) if \( I \) tends to \( \infty \) then the probability that \((\bar{q}, \bar{x}, \bar{\theta}, \bar{a})\) is a \( \rho \)-MSE converges to 1 for the stakeholder democracy \( \mu^d \).

**Q.E.D**

**Corollary 5** Suppose that \((\bar{p}_{jk})_k\) are linearly independent for at least one firm. Furthermore suppose that the measure of \( ]-\infty, 0[ \) is positive, so

\[ P(t < 0) = \int_{-\infty}^{0} \psi(t) \, dt > 0. \]

Then there exists \( \bar{\rho} > 1/2 \) such that for all \( \rho < \bar{\rho} \) if \( I \) tends to \( \infty \) then the probability that \((\bar{q}, \bar{x}, \bar{\theta}, \bar{a})\) is a \( \rho \)-MSE converges to 0 for the shareholder governance \( \theta^+ \).

**Proof:** According to the proof of Theorem 3 if \( I \) tends to infinity, then the sequence of distributions of portfolios converges to a distribution which is symmetric with respect to the diagonal. Therefore if

\[ P(t < 0) = \int_{-\infty}^{0} \psi(t) \, dt > 0 \]

then distribution for the shareholder governance \( \theta^+ \) is not symmetric with respect to the diagonal. Hence if \((\bar{p}_{jk})_k\) are linearly independent then there exists \( p_j \) such that more than 50 pct. of the shareholders supports \( p_j \) against \( \bar{p}_j \). Thus if \( I \) tends to \( \infty \) then the probability that \((\bar{q}, \bar{x}, \bar{\theta}, \bar{a})\) is a \( \rho \)-MSE converges to 0 for the shareholder governance \( \theta^+ \).

**Q.E.D**

**Random technology shocks**

The object of this section is to show that the conclusion of Theorem 3 can be kept even if the equilibrium distribution of portfolio is very ‘bad’ with respect to the political stakes. Let us consider a worst-case scenario as far as the equilibrium distribution of individual portfolio is concerned: suppose that at a SME (\(a\)) agents have shares in at most one firm. Without loss of generality, one can reduce the study to an economy with \( I = J \) agents (and identify \( I \) with \( J \)), each agent owning one firm and only one: this economy is called a sole proprietorship.

The collection of portfolios \((\theta_i)_{i \in J}\) is thus composed of the \( J \) vertices of a spherico-regular \((J - 1)\)-dimensional simplex, \( \Delta^{J-1} \). (It is regular since \( \|\theta_i - \theta_j\| > 0 \) has the
same value for all \( i \neq j \); moreover it is spherico-regular since the vertices lie of a sphere centered at the origin.) That case is also a worst-case scenario from a social choice perspective: all portfolios \( \theta \) in the positive orthant (the positive convex cone of the \( \theta_i \)'s), and thus also the market portfolio, has the same score under the stakeholder democracy \( \mu^d \):

\[
\rho_{\mu^d}(\theta; (\theta_i)_{i \in I}) = 1 - 1/J \quad \text{(the others have a score equal to one).}
\]

We know from Greenberg (1979) that \( 1 - 1/J \) is an upper bound to the min-max score in a very general spatial voting model. If we were to stick to the space of portfolios to study the political process within firms, we would get to the conclusion that no \( \rho \)-MSE\((\mu^d) \) exists for \( \rho < 1 - 1/J \), and thus no criterion does better than the mere Pareto criterion.

But it is still the case that this extremal collection of portfolios gives rise, within each firm \( j \in J \), to a \( n_j \)-dimensional social choice problem characterized by the collection of individual pricing vectors \((\bar{p}_{ij})_{i \in J}\), where \( \bar{p}_{ij} \) is the linear transformation of \( \theta_i \) through the matrix \( \bar{P}_j \) (as defined in the proof of Theorem 1). The operator \( \bar{P}_j \) projects the simplex of portfolios in a subspace of strictly smaller dimension if \( n_j < J \), and in that subspace it is not at all the case that the collection \((\bar{p}_{ij})_{i \in J}\) is distributed according to a worst-case scenario similar to that of portfolio (e.g., as vertices of a \((n_j - 1)\)-dimensional simplex).

On the contrary, we argue that if the image subspace of the operator \( \bar{P}_j \) is randomly chosen (in a sense made precise below), then up to an affine transformation, the resulting point set, \((\bar{p}_{ij})_{i \in J}\), coincides in distribution with a standard centered Gaussian sample in that subspace. Hence, if \( J \) is high enough compared to \( n_j \), the worst-case scenario in the space of portfolio gives rise, within each firm, to a best-case scenario: the Gaussian distribution being symmetric, one can hope for existence of \( \rho \)-MSE\((\mu^d) \) for \( \rho \) close to 0.5; and the sample being centered, statistically the min-max point is a vector collinear to the market portfolio, hence efficient internalization occurs at the min-max.

### A natural distribution of random points

More generally, when dealing with a \( n \)-dimensional spatial voting problem with, say, \( J \) voters, one is left with a combinatorial problem about \( J \)-tuples of (random) points in \( \mathbb{R}^n \). Many ‘natural’ distributions of these random points have been proposed in the mathematical literature (see Schneider (2004)). Among them, the one described above takes a central place: Every configuration of \( J > n \) numbered points in general position in \( \mathbb{R}^n \) is affinely equivalent to the orthogonal projection of the set of numbered vertices of a fixed spherico-regular \((J - 1)\)-dimensional simplex onto a unique \( n \)-dimensional linear subspace in \( \mathbb{R}^{J-1} \). This construction builds a one-to-one correspondence between the (orientation-preserving) affine equivalence classes of such point set configurations and an open dense subset of the Grassmanian \( G(J - 1, n) \) of oriented \( n \)-spaces in \( \mathbb{R}^{J-1} \). The so-called Grassmann approach (sometimes refered as the Goodman-Pollack model) considers the probability distribution on the set of affine equivalence classes of \( J \)-tuples in
general position in \( \mathbb{R}^n \) that stems from the unique rotation-invariant probability measure on \( G(J-1, n) \). Baryshnikov and Vitale (1994) (following an observation of Affentranger and Schneider (1992)) proved that under the Grassmann approach, the resulting point set coincides in distribution with a standard Gaussian sample in that subspace. As a consequence, an affine-invariant functional of \( J \)-tuples with this distribution is stochastically equivalent to the same functional taken at an i.i.d. \( J \)-tuple of standard normal points in \( \mathbb{R}^n \).

**Random production external effects**

Suppose \( n_j < J \) and consider the image subspace of the operator \( \bar{P}_j \), \( \text{Im} \bar{P}_j \): the subspace of \( \mathbb{R}^{J-1} \) generated by the pricing vectors \( (\bar{p}_{jk})_{k \in J} \) (which has dimension smaller or equal to \( n_j \); without loss of generality, we assume it has dimension \( n_j \) and take the first \( n_j \) vectors as a basis). Recall that the vector \( \bar{p}_{jk} \) measures the value of the marginal impact of firm \( j \)'s action on firm \( k \)'s output: it is the image of the unique state prices vector by the Jacobian matrix of the production function of firm \( k \), \( F_k \), with respect to action of firm \( j \) (see Equations 2). These marginal impacts are fixed by the exogenously fixed production function. Randomizing over the production function amounts to randomizing over \( \text{Im} \bar{P}_j \). Let us use the Grassmann approach.

The Grassmann approach amounts to randomly rotate the simplex \( \Delta^{J-1} \) and project it orthogonally on a fixed \( n_j \)-dimensional subspace. Without loss of generality, the spherically-regular simplex \( \Delta^{J-1} \) can be suitably translated so that it becomes centered: the sum of its vertices (the scaled-down market portfolio \( (1/J)1_J \), center of gravity under \( \mu^d \)) is translated at zero. Once more without loss of generality, the fixed \( n_j \)-dimensional subspace onto which the rotated simplex is projected can be taken as the first \( n_j \) coordinates of \( \mathbb{R}^J \). But our operator \( \bar{P}_j \) is not this orthogonal projection, but a linear (thus affine) transformation of the latter (one can pass from one to the other using the Gram-Schmidt procedure between the standard orthonormal basis and the basis defined by \( (\bar{p}_{jk})_{1 \leq k \leq n_j} \)). The composition of affine transformation being affine, we can directly apply Baryshnikov and Vitale (1994) to our problem, only exploiting the symmetry of the generated distribution.

**Theorem 4** Fix \( n_j \). When \( J \) tends toward infinity, then almost surely the min-max score converges to 0.5 for the stakeholder democracy, and the min-max set of directions of the \( (\bar{p}_{ij})_{i \in J} \) shrinks to the direction of the efficient price vector \( \bar{p}_j \).

**Proof:** Under the Grassman approach on production externalities, within each firm \( j \), the point set \( (\bar{p}_{ij})_{i \in J} \) coincides in distribution with a standard centered Gaussian sample. Therefore the convergence of the min-max of the sample to the min-max of the Gaussian distribution is a consequence of Theorem 3 in Caplin and Nalebuff (1988).
5 Efficient internalization as a permanent median voter

In this section, we benchmark the stakeholder democracy with respect to its ability to promote efficient internalization through 50%-majority voting. We design a governance which always results in perfect internalization. In this governance, voting weights are endogenously fixed at equilibrium and depend on the proposed challenger. This governance gives indications about the type and quantity of information that is needed to always get efficient internalization as the unique political outcome.

At a SME($a$), $E$, for a change of action $\Delta a_j$ within firm $j$, define endogenously the shareholders' voting weights by the following rule:

$$
\mu_{ij}^*(E, \Delta a_j) = \frac{\Delta a_j}{\| \Delta a_j \|} \cdot \bar{p}_{ij}.
$$

The major weakness of this governance is that shareholders’ voting weights depend on the alternative proposed to the status quo. Another (milder) one is that individual pricing devices, $\bar{p}_{ij}$, should be disclosed. But this problem seems not as serious as for ordinary public goods, since Equations (3) show that these price vectors are disclosed with the portfolio as soon as the price vectors for firms, $\bar{p}_{jk}$, are known. Arguably, portfolios and price vectors for firms are easier to disclose than individual willingness to pay for a public good. As the following proposition shows, this governance has the virtue of making the efficient internalization the only 0.5-MSE.

**Proposition 3** Under the governance defined by the mapping $\mu^*$, if a state $(\bar{x}, a)$ associated with a SME($a$), $E$, is Pareto optimal, then $E$ is 50%-majority stable. Inversely, any 0.5-MSE must satisfy the first order conditions of Pareto optimality as given by equations (4).

**Proof:** Consider a Pareto optimal SME($a$), $E$. Suppose it is not 0.5-majority stable with respect to $\mu^*$. By the minimum differentiation Lemma 1, it entails that there exists a firm, $j$, and an infinitesimal change of action, $da_j \in p_j^+$ such that

$$
\sum_{I_j(da_j)} \mu_{ij}^* > \sum_{I_j(-da_j) \cup I_j(da_j)} \mu_{ij}^* 
$$

where, for shorter notation, $I_j(da_j)$, resp. $I_j(-da_j)$, stands for $I_j(\bar{x}, \bar{\theta}, a, da_j) = \{i \in I \mid \bar{p}_{ij} \cdot da_j > 0\}$, resp. $I_j(\bar{x}, \bar{\theta}, a, -da_j) = \{i \in I \mid \bar{p}_{ij} \cdot da_j < 0\}$, and $I_j(da_j) = \{i \in I \mid \bar{p}_{ij} \cdot da_j = 0\}$. The latter inequality is equivalent to

$$
\sum_{I_j(da_j)} \bar{p}_{ij} \cdot da_j > \sum_{I_j(-da_j) \cup I_j(da_j)} -\bar{p}_{ij} \cdot da_j \iff \sum_{I} \bar{p}_{ij} \cdot da_j > 0.
$$
But $(\bar{x}, a)$ being Pareto optimal, one has that the vector $\sum_{\mathcal{I}} \vec{p}_{ij}$, which is equal to $\sum_{k \in \mathcal{J}} \vec{p}_{jk}$ thanks to financial market clearing, is collinear to $p_j$, a contradiction to the later inequality.

Reciprocally, consider a 0.5-MSE($\mu^*$). Suppose it is does not satisfy equations (4): there exists a firm, $j$, such that $p_j$ is not collinear to $\sum_{k \in \mathcal{J}} \vec{p}_{jk}$. Then there exists $d_{aj} \in p_j^\perp$ such that $d_{aj} \cdot \sum_{k \in \mathcal{J}} \vec{p}_{jk} > 0$. On the other hand, being at a 0.5-MSE($\mu^*$), one has

$$\sum_{\mathcal{I}(d_{aj})} \mu_{ij}^a \leq \sum_{\mathcal{I}(-d_{aj}) \cdot \mathcal{I}(d_{aj})} \mu_{ij}^a \iff [\sum_{\mathcal{I}} \vec{p}_{ij}] \cdot d_{aj} \leq 0 .$$

But financial market clearing give $\sum_{\mathcal{I}} \vec{p}_{ij} = \sum_{k \in \mathcal{J}} \vec{p}_{jk}$, hence a contradiction with the former strict inequality.

Q.E.D

References


