Stability in Supply Chain Networks

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Abstract

This paper presents a theory of matching in vertical networks, generalizing the theory of matching in two-sided markets introduced by Gale and Shapley. Under natural restrictions, stable networks are guaranteed to exist. The set of stable networks is a lattice, with side-optimal stable networks at the extremes. Several other key results on two-sided matching also extend naturally to the more general setting.

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1 Introduction

Two-sidedness has long been viewed as a critical condition for many of the results of matching theory, such as the existence of stable matchings and the special properties of some of them. The original paper on stability and matching (Gale and Shapley, 1962) shows by example that the "problem of the roommates," whose only difference from the "marriage problem" is the absence of two sides in the market, may fail to have a stable pairing. More generally, Abeledo and Isaak (1991) prove that to guarantee the existence of stable pairings under arbitrary preferences, it has to be the case that each agent belongs to one of two classes, and an agent in one class can match only with agents in the other class. Alkan (1988) shows that the "man-woman-child marriage problem," in which each match consists of agents of three different types, may also fail to have a stable matching. In their comprehensive survey of the two-sided matching literature, Roth and Sotomayor (1990, p. 186) state:

> In general, little is known about directions in which the two-sidedness of all the models we present here can be relaxed, and which of the results might be preserved.

In contrast, within the two-sided framework, significant progress has been made in the last four decades both in the theoretical literature and in empirical applications. Most recent extensions of the theoretical literature include matching with contracts, which unifies many of the matching results with those in the literature on private value auctions in a single framework (Hatfield and Milgrom, 2005), schedule matching, in which agents decide not only with whom to match, but also how much time to spend with them (Roth, Rothblum, and Vande Vate, 1993; Baiou and Balinski, 2002; Alkan and Gale, 2003), many-to-many matching, in which agents are allowed to match with multiple partners on the other side of the market (Echenique and Oviedo, 2004b), and other two-sided settings. Empirical applications include the redesign of the matching market for American physicians (Roth, 1984a; Roth and Peranson, 1999), the design of the matching market for graduating medical students in Scotland (Roth, 1991; Irving, 1998), the new high school admissions system in New York City (Abdulkadiroğlu, Pathak, and Roth, 2005), and several others.

This paper generalizes the results and techniques of two-sided matching to a much broader setting—supply chain networks. Consider an industry, which includes a number of agents: workers, producers, distributors, retailers, and so on. Some agents supply basic inputs for the industry, and do not consume any of the outputs (e.g., wheat farmers are the suppliers of basic inputs in the farmer–miller–baker–retailer supply chain). Some agents purchase the final outputs of the industry (e.g., car manufacturers are the consumers of final goods in the iron ore supplier–steel producer–steel consumer supply chain). The rest are intermediate agents, who get their inputs from some agents in the industry, convert them into outputs at a cost, and sell the outputs to some other agents (millers, bakers, and steel producers are intermediate agents in the above examples). There is a pre-determined upstream-downstream partial ordering on the set of agents: for a pair of agents A and B, either A is a potential supplier for B, or B is a potential supplier for A, but not both; it may also be the case that neither is a potential supplier for the other. The partial ordering can be complicated, with several alternative paths from one point to another, with chains of different lengths going to the same node, and so on. However, by transitivity, it can not have cycles. Note that if there are no intermediate agents, this setting reduces to a two-sided market.

Agents can trade discrete quantities of goods, with the smallest tradeable quantity (the *unit* of quantity) defined ex ante. For example, one unit may correspond to one million tons of steel, one hour of work, or one loaf of bread. In the Gale-Shapley two-sided marriage market, one unit corresponds to marriage, and each person can "trade" at most one unit. Units traded in the market are represented by contracts, following Hatfield and Milgrom (2005). Each contract specifies the buyer, the seller, the price (if monetary transfers are involved), and the serial number of the sold unit (if multiple units can be traded). A *network* is a set of contracts: it specifies who sells what to whom and at what price. Each agent has preferences over sets of contracts involving it: e.g., an intermediate agent's payoff from such a set depends on the payments it makes for its inputs (specified in its *upstream* contracts) and receives for its outputs (specified in its *downstream* contracts), as well as on the cost of converting the inputs into the outputs. For a consumer of final goods, the payoff depends on the utility from the goods it purchases and the payments it makes for these goods. A network is *chain stable* if there is no upstream-downstream sequence of agents (not necessarily going all the way to the suppliers of basic inputs and the consumers of final outputs) who could become better off by forming new contracts among themselves, and possibly dropping some of their current contracts. This condition is parallel to pairwise stability in two-sided markets, and is tautologically equivalent if there are no intermediate agents in the industry.

The concept of stability in networks is not strategic—I do not study the dynamics of network formation or "what-if" scenarios analyzed by agents who may be considering temporarily dropping or adding contracts in the hopes of affecting the entire network in a way beneficial to them, although these considerations are undoubtedly important in many settings. The concept is closer in spirit to general equilibrium models, where agents perceive conditions surrounding them as given, and optimize given those conditions. Under chain stability, agents also perceive conditions surrounding them as given (i.e., which other agents are willing to form contracts with them, and what those contracts are), and optimize given these conditions.

Without restrictions on preferences, the set of stable matchings may be empty even in the two-sided one-to-many setting. One standard restriction that is sufficient to guarantee the existence of stable matchings in that setting is the substitutes condition of Kelso and Crawford (1982). In the supply chain setting, I place an analogous pair of restrictions on preferences; these restrictions become tautologically equivalent to the substitutes condition if there are no intermediate agents in the industry. The restrictions are *same-side substitutability* and *cross-side complementarity*. Same-side substitutability says that when the set of available downstream contracts of a firm expands (i.e., there are more potential customers, or the potential customers' willingness to pay goes up), while the set of available upstream contracts remains unchanged, the set of downstream contracts that the firm rejects also (weakly) expands, and symmetrically, when the set of available upstream contracts expands and the set of available downstream contracts remains unchanged, the set of rejected upstream contracts also expands. Cross-side complementarity is a parallel restriction on how the firm's optimal set of downstream contracts depends on its set of available upstream contracts, and vice versa. It says that when the set of available downstream contracts of a firm expands, while the set of available upstream contracts remains unchanged, the set of upstream contracts that the firm forms also (weakly) expands, and symmetrically, when the set of available upstream contracts expands and the set of available downstream contracts remains unchanged, the set of downstream contracts that the firm forms also expands. Section 2 gives formal definitions of these conditions and discusses the restrictions they place on agents' production functions and utilities.

Section 3 states and constructively proves the main result of this paper: under same-side substitutability and cross-side complementarity, there exists a chain stable network. Section 4 studies properties of chain stable networks and shows that many key results from the theory of two-sided matching still hold in the more general setting. The chain stable network formed in the constructive proof of the existence theorem is upstream-optimal: it is the most preferred chain stable network for all suppliers of basic inputs and the least preferred chain stable network for all consumers of final outputs. A symmetric algorithm would produce the downstream-optimal chain stable network. In fact, just like in the two-sided setting, the set of chain stable networks is a lattice with upstream- and downstream-optimal chain stable networks as extreme elements. Also, adding a new supplier of basic inputs to the industry makes other such suppliers weakly worse off and makes the consumers of final outputs weakly better off at both upstream- and downstream-optimal chain stable networks. Symmetrically, adding a new consumer of final outputs makes other such consumers worse off and makes the suppliers of basic inputs better off. The section also presents results on the equivalence of chain stability and other solution concepts: tree stability (under same-side substitutability and cross-side complementarity) and weak core (under an additional, very restrictive condition: each node can form at most one upstream and at most one downstream contract). Section 5 concludes.

2 The Model of Matching in Supply Chains

This section introduces a model of matching in supply chain networks. The model can accommodate prices, quantities, multiple traded goods, and very general network configurations. A particular case that can be accommodated by the model, which is perhaps the easiest to keep in mind while going through the setup and the proofs, is a discrete analogue of a classical Walrasian equilibrium setup, with homogeneous goods, price-taking behavior, quasilinear utilities, decreasing marginal benefits of consumption, and increasing marginal costs of production.

Consider a market, consisting of a set of nodes (firms, countries, agents, workers, etc.), A, with a partial ordering "$\succeq$", where $a \succ b$ stands for b being a downstream node for a. The interpretation of this partial ordering is that if $a \succ b$, then, in principle, a could sell something to b, while if $a \not\succ b$ and $b \not\succ a$, then there can be no relationship between a and b. By transitivity, there are no loops in the market.

Relationships between pairs of nodes are represented by "contracts." Each contract c represents one unit of a good sold by one node to another: it is a vector, $c = (s, b, l, p)$, where $s \in A$ and $b \in A$ are the "seller" and the "buyer" involved in the contract, $s \succ b$; $l \in \mathbb{N}$ is the "serial number" of the unit of the good represented by the contract; and $p \in \mathbb{R}$ is the price that the buyer pays to the seller for that unit. The seller involved in contract c is denoted by s_c, the buyer is denoted by b_c, and so on.

Multiple contracts between a seller and a buyer can represent multiple units of the same good or service, units of different types of goods or services, or both. For example, if the unit is one ton, and a farmer sells 5 tons of wheat and 10 tons of rye to a miller, then this relationship will be represented by 15 contracts with 15 different serial numbers.

The set of possible contracts, C, is finite and is given exogenously. In the simplest case, it can include all possible contracts between nodes in A, with all possible serial numbers from some finite set, and all possible prices from some finite set. It can also be more complicated: for example, the U.S. trade embargo on Cuba can be incorporated simply by removing all contracts between the nodes in these countries from set C. If brewers are not allowed to sell beer directly to retailers and have to use the services of an intermediary (Asker, 2004), then contracts between brewers and retailers are excluded from set C.

Note that this model, restricted to one "tier" of sellers and one "tier" of buyers, encompasses various two-sided matching settings considered in the literature. Setting $l \equiv constant$ and $p \equiv 0$ turns this model into the marriage model of Gale and Shapley (1962) if each agent is allowed to have at most one partner and the college admissions model if agents on one side of the market are allowed to have multiple links. Setting $l \equiv constant$ turns it into the setup of Kelso and Crawford (1982) if agents on one side are restricted to having at most one link and into the many-to-many matching model of Roth (1984b, 1985) and Blair (1988) if agents on both sides are allowed to have multiple links. Setting $p \equiv 0$ and assuming that all links connecting two nodes are identical turns the model into a discrete version of the schedule matching problem of Baiou and Balinski (2002) and Alkan and Gale (2003).

Each node can be involved in several contracts, some as a seller, some as a buyer, but it cannot be involved in two contracts that differ only in price p, i.e., it cannot buy or sell the same unit twice. Nodes have preferences over sets of contracts that involve them as the buyer or the seller. For example, in the simplest case of quasilinear utilities and profits, the utility of node a involved in a set of contracts X is

$$V_a(X) = W_a\left(\{(s_c, b_c, l_c) | c \in X\}\right) + \sum_{c \in D} p_c - \sum_{c \in U} p_c,$$

where $D = \{c \in X | a = s_c\}$ and $U = \{c \in X | a = b_c\}$, i.e., D is the set of contracts in X in which a is involved as a seller and U is the set of contracts in which a is involved as a buyer. $W_a(\cdot)$ represents the utility from the purchased contracts for the consumers at the downstream end of the chain, the cost of producing the sold contracts for the suppliers at the upstream end of the chain, and the cost of converting inputs into outputs for the intermediate nodes.

For an agent $a \in A$ and a set of contracts X, let $Ch_a(X)$ be a's most preferred (possibly empty) subset of X, let $U_a(X)$ be the set of contracts in X in which a is the buyer (i.e., upstream contracts), and let $D_a(X)$ be the set of contracts in X in which a is the seller (i.e., downstream contracts). Subscript a will be omitted when it is clear from the context which agent's preferences are being considered. Preferences are strict, i.e., function $Ch_a(X)$ is single-valued. In the settings in which it is natural to assume that several different sets of contracts should result in identical payoffs (e.g., when two nodes can trade several identical units of a good), I assume that ties are broken in a consistent manner, e.g., lexicographically: in the case of several identical units of a good, that would imply that seller a prefers contract $(a, b, 1, p)$ to contract $(a, b, 2, p)$, but would prefer $(a, b, 2, p')$ to $(a, b, 1, p)$ for any $p' > p$.

Preferences of agent a are *same-side substitutable* if for any two sets of contracts X and Y such that $D(X) = D(Y)$ and $U(X) \subset U(Y)$, $U(X) \backslash U(Ch(X)) \subset U(Y) \backslash U(Ch(Y))$ and for any two sets X and Y such that $U(X) = U(Y)$ and $D(X) \subset D(Y)$, $D(X) \backslash D(Ch(X)) \subset D(Y) \backslash D(Ch(Y))$. That is, preferences are same-side substitutable if, choosing from a bigger set of contracts on one side, the agent does not accept any contracts *on that side* that he rejected when he was choosing from the smaller set.

Preferences of agent a are *cross-side complementary* if for any two sets of contracts X and Y such that $D(X) = D(Y)$ and $U(X) \subset U(Y)$, $D(Ch(X)) \subset D(Ch(Y))$ and for any two sets X and Y such that $D(X) \subset D(Y)$ and $U(X) = U(Y)$, $U(Ch(X)) \subset U(Ch(Y))$. That is, preferences are cross-side complementary if, when presented with a bigger set of contracts on one side, an agent does not reject any contract *on the other side* that he accepted before.

Same-side substitutability is a generalization of the gross substitutes condition introduced by Kelso and Crawford (1982) and used widely in the matching literature. If there are only two sides in the supply chain market, then these two conditions become tautologically equivalent. Cross-side complementarity can be viewed as a mirror image of same-side substitutability. It is automatically satisfied in any two-sided market.

It is important to highlight what is allowed and what is not allowed by this pair of assumptions. Two possibilities that they rule out are scale economies and production functions with fixed costs, because in those cases a firm may decide not to produce one unit of a good at a certain price, while being willing to produce ten units at the same price, violating same-side substitutability. In addition, complementary inputs (or outputs) are ruled out. In contrast, with substitutable inputs and outputs and decreasing returns to scale, many production and utility functions can be accommodated. The simplest example is a firm that can take one kind of input and produce one kind of output at a cost, with the marginal cost of production increasing or staying constant in quantity. The input good can come from several different nodes, and the output good may go to several different nodes, with different transportation costs. Much more general cases are possible as well: preferences and production functions with quotas and tariffs, several different inputs and outputs with discrete choice demands and production functions, capacity constraints and increasing transportation costs, etc. The interdependencies between different inputs or outputs can be rather complex as well. Consider the following example. A firm has two plants in the same location. Each plant's capacity is equal to one unit. The first plant can convert one unit of iron ore into one unit of steel for c_o^1 or it can convert one unit of steel scrap into one unit of steel for c_s^1. The second plant can convert one unit of iron ore into one unit of steel for c_o^2 or it can convert one unit of steel scrap into one unit of steel for c_s^2. Then, for a generic choice of costs and prices, the preferences of this firm will be same-side substitutable and cross-side complementary, even though the firm's preferences over iron ore and scrap are not trivial (they cannot be expressed by simply saying that two alternative inputs are perfect substitutes, with one being better than the other by a certain amount x). This example is an analogue of "endowed assignment valuations" in two-sided matching markets, introduced by Hatfield and Milgrom (2005), in which each firm has several unit-capacity jobs, each worker has a certain productivity at each job, and each firm has an initial endowment of workers. Even in the two-sided setting, it is an open question whether endowed assignment valuations exhaust the set of utility functions with substitutable preferences, and so it is an open question in the supply chain setting as well. For the remainder of this paper, all preferences are assumed to be same-side substitutable and cross-side complementary, and these restrictions will usually be omitted from the statements of results to avoid repetition.

A *network* is a collection of contracts that does not contain any two contracts differing only in price. Let $\mu(a)$ denote the set of contracts involving a in network μ. Network μ is *individually rational* if for any agent a, $Ch_a(\mu(a)) = \mu(a)$, i.e., no agent would like to unilaterally drop any of his contracts.

The most widely used solution concept in the two-sided matching literature is pairwise stability. Its analogue in the supply chain setting is chain stability, defined as follows. A *chain* is a sequence of contracts, $\{c_1, \ldots, c_n\}$, $n \geq 1$, such that for any $i < n$, $b_{c_i} = s_{c_{i+1}}$, i.e., the buyer in contract c_i is the same node as the seller in contract c_{i+1}. Note that the chain does not have to go all the way from one of the most upstream nodes in the market to one of the most downstream nodes; it can connect several nodes in the middle of the market. For notational convenience, let $b_i \equiv b_{c_i}$ and $s_i \equiv s_{c_i}$. For a network μ, a *chain block* is a chain $\{c_1, \ldots, c_n\}$ such that

- $\forall i \leq n$, $c_i \notin \mu$,
- $c_1 \in Ch_{s_1}(\mu(s_1) \cup c_1)$,
- $c_n \in Ch_{b_n}(\mu(b_n) \cup c_n)$, and

- $\forall i < n$, $\{c_i, c_{i+1}\} \subset Ch_{b_i = s_{i+1}}(\mu(b_i) \cup c_i \cup c_{i+1})$.

In other words, a chain block of network μ is a downstream sequence of contracts not belonging to μ, in which the buyer in one contract is the seller in the next one, such that each node involved in these contracts is willing to add all of its contracts in the sequence to its contracts in μ, possibly dropping some of its contracts in μ. A network is *chain stable* if it is individually rational and has no chain blocks.

Note that chain stability is not a strategic concept—each node views the set of contracts available to it as exogenously given, and maximizes its payoff given that set, analogously to how consumers in the Walrasian equilibrium setting choose quantities taking prices as given. Hence, each node treats its contracts independently of one another, ignoring the effect of forming one contract on other nodes' willingness to pay for other contracts. For example, if there are only two nodes in the market, the seller (whose marginal cost is increasing in quantity) and the buyer (whose marginal benefit is decreasing in quantity), who can trade multiple units of a good, then chain stability implies that the quantity traded between those two nodes is determined by the intersection of the agents' marginal cost and marginal benefit curves. In more general networks, nodes also ignore various sorts of externalities they may impose on others (e.g., limiting the supply of inputs available to competitors by buying too much and thus reducing the competition in the market for outputs). Hence, the model is not directly applicable to cases in which there are several large players manipulating the market; it is better suited to describing competitive markets with many small players, or markets in which nodes represent countries or regions rather than firms.

3 Stable Networks: Existence

This section shows that the set of chain stable networks is isomorphic to the non-empty set of fixed points of a certain isotone operator, and presents a family of algorithms for finding two special stable networks. These algorithms generalize the fixed-point algorithms of Adachi (2000), Echenique and Oviedo (2004a, 2004b), and Hatfield and Milgrom (2005), all of which are descendants of the Deferred Acceptance Algorithm (Gale and Shapley, 1962) and apply only to two-sided matching problems.

Let me first introduce some definitions and notation.

A *pre-network* is a set of arrows from nodes in A to other nodes in A, with the following properties. Each arrow r is a tuple (o_r, d_r, c_r), where o_r ("origin of arrow r") and d_r ("destination of arrow r") are two different nodes and c_r ("contract attached to arrow r") is a contract involving both o_r and d_r. If o_r is the seller and d_r is the buyer of contract c_r, the arrow is "downstream". Otherwise, o_r is the buyer and d_r is the seller of contract c_r, and the arrow is "upstream". For a pre-network ν and a node a, $\nu(a)$ is the set of contracts attached to arrows pointing <u>to</u> a, i.e., $\nu(a) = \{r \in \nu | d_r = a\}$.

There can be multiple arrows going from o_r to d_r, but any two arrows going from o_r to d_r must have different contracts attached to them (these contracts may differ in serial numbers, prices, or both). Arrows going in opposite directions (from node a to node b and from node b to node a) can have identical contracts attached to them.

Let R be the set of all possible arrows, and let $R' \subset R$ be an arbitrary set of arrows. Let ν be a pre-network. Define operator $T_{R'}$ on the set of pre-networks as

$$T_{R'}(\nu) = (\nu \backslash R') \cup \{r \in R' | c_r \in Ch_{o_r}(\nu(o_r) \cup c_r)\}.$$

In other words, operator $T_{R'}$ considers each arrow r in set R', and keeps or adds it to pre-network ν if the agent at r's origin, o_r, would like to keep or add the attached contract c_r to the set of contracts attached to arrows pointing to o_r; otherwise, it removes (or does not add) the arrow. For notational convenience, let T without a subscript denote T_R.

Pre-network ν^* is a *fixed point* of operator T if $T\nu^* = \nu^*$. Note that this is equivalent to saying that for any set of arrows R', $T_{R'}\nu^* = \nu^*$.

In the two-sided many-to-many matching setting, the set of fixed points of operator T is isomorphic to the set of pairwise stable matchings (Echenique and Oviedo, 2004b). Analogously, in the supply chain setting, there exists a natural one-to-one mapping from the set of fixed points of operator T to the set of chain stable networks. Specifically, let M be the set of all networks, M^* be the set of chain stable networks, N be the set of all pre-networks, and N^* be the set of fixed points of operator T. Define the following mappings:

- $F : N \to M$
  - Contract c belongs to $\mu = F(\nu)$ if and only if ν contains both arrows with contract c attached.
- $G : M \to N$
  - Arrow r belongs to $\nu = G(\mu)$ if and only if contract c_r belongs to μ.
- $H_k : M \to N$
  - $H_k(\mu) = T^k(G(\mu))$, i.e., $H_0(\mu) = G(\mu)$ and $H_k(\mu) = T(H_{k-1}(\mu))$ for $k > 0$.

Lemma 3.1 *For any $\nu \in N^*$, $F(\nu) \in M^*$. For any $\mu \in M^*$, there exists k such that $H_k(\mu) = H_{k+1}(\mu) \equiv H(\mu)$; $H(\mu) \in N^*$. For any $\nu \in N^*$, $H(F(\nu)) = \nu$, and* *for any $\mu \in M^*$, $F(H(\mu)) = \mu$. In other words, removing one-directional links from any fixed-point pre-network results in the corresponding chain stable network, and iterating operator T starting with a chain stable network results in the corresponding fixed-point pre-network.*

Proof. See Appendix. ■

To demonstrate the existence of chain stable networks, it is now sufficient to prove that operator T has a fixed point. To do that, I introduce a partial ordering on the set of pre-networks. Let ν_1 and ν_2 be two pre-networks. Then $\nu_1 \leq \nu_2$ if the set of downstream arrows in ν_1 is a subset of the set of downstream arrows in ν_2, and the set of upstream arrows in ν_1 is a superset of the set of upstream arrows in ν_2. Also, let ν_{min} be the pre-network that includes all possible upstream arrows and no downstream arrows, and let ν_{max} be the pre-network that includes no upstream arrows and all possible downstream arrows. By construction, for any pre-network ν, $\nu_{min} \leq \nu \leq \nu_{max}$.

Let $R_1, R_2, \ldots, R_k$ be an arbitrary sequence of sets of arrows. Let $\nu_0 = \nu_{min}$, and let $\nu_i = T_{R_i}(\nu_{i-1})$ for $i > 0$.

Lemma 3.2 *For any $i < k$, $\nu_i \leq \nu_{i+1}$.*

Proof. See Appendix. ■

For a given pre-network ν_0, call a sequence of sets of arrows *regular* if it satisfies the following condition: for $i \geq 1$, if $\nu_i \equiv T_{R_i}\nu_{i-1} = \nu_{i-1}$ and $R_i \neq R$, then $R_{i+1} \supsetneq R_i$. In particular, the sequence $R_i \equiv R$ is regular.

Theorem 3.1 *There exists a chain stable network.*

Proof. Fix $\nu_0 = \nu_{min}$ and take any infinite regular sequence of sets of arrows. By Lemma 3.2, the corresponding sequence of pre-networks is monotonically increasing.

The set of pre-networks is finite, and therefore this sequence of pre-networks converges, in finite time, to a fixed point, ν^*; $T_{R'}\nu^* = \nu^*$ for any set of arrows R'. By Lemma 3.1, $F(\nu^*)$ is a chain stable network. ■

Note that the proof of Theorem 3.1 is constructive: it gives an algorithm for computing a chain stable network. In fact, it produces a family of such algorithms: a different algorithm for each regular sequence of sets. It turns out that all these algorithms generate the same fixed point of operator T, and therefore the same chain stable network. To show that, I first state an auxiliary lemma. Its proof is completely analogous to the proof of Lemma 3.2, and is therefore omitted.

Lemma 3.3 *For any set of arrows R' and any pre-networks $\nu_1 \leq \nu_2$, $T_{R'}(\nu_1) \leq T_{R'}(\nu_2)$.*

Theorem 3.2 *Let $\nu_0 = \nu_{min}$. Then for all regular sequences of sets of arrows, the corresponding sequences of pre-networks converge to the same fixed point ν^*_{min}.*

Proof. Consider any two regular sequences of sets of arrows $\{R^1_1, R^1_2, \dots\}$ and $\{R^2_1, R^2_2, \dots\}$ and corresponding fixed points ν^1 and ν^2. Now,

$$\begin{aligned}
\nu^1 &\geq \nu_{min} = \nu^2_0 \\
T_{R^2_1}(\nu^1) = \nu^1 &\geq T_{R^2_1}(\nu_{min}) = \nu^2_1 \\
T_{R^2_2}(T_{R^2_1}(\nu^1)) = \nu^1 &\geq T_{R^2_2}(T_{R^2_1}(\nu_{min})) = \nu^2_2 \\
&\vdots \\
\nu^1 &\geq \nu^2_k.
\end{aligned}$$

Since ν^2_k converges to ν^2, we know that $\nu^1 \geq \nu^2$. Symmetrically, $\nu^2 \geq \nu^1$, and so $\nu^1 = \nu^2$. ■

Therefore, it does not matter in what order various parts of the pre-network are updated—the outcome will be the same. This flexibility may simplify and speed up computational implementations of the algorithm. This family of algorithms also corresponds to various two-sided matching algorithms in a natural way. The T-algorithm of Adachi (2000) and Echenique and Oviedo (2004a, b) and the generalized Gale-Shapley algorithm of Hatfield and Milgrom (2005) are equivalent to my algorithm with $R_k \equiv R$. Men-proposing deferred acceptance algorithm of Gale and Shapley (1962) is equivalent to my algorithm with ν_0 containing all arrows from women to men and no arrows from men to women, R_k containing all arrows from men to women when k is odd, and R_k containing all arrows from women to men when k is even.

Note also that all proofs would remain essentially unchanged if ν_0 was set equal to ν_{max} rather than ν_{min} everywhere. The algorithms would then converge to a possibly different fixed point, ν^*_{max}. The next section explores the properties of the set of fixed points of operator T, and the special role that ν^*_{min} and ν^*_{max} play in this set.

4 Stable Networks: Properties

This section discusses several properties of the set of chain stable networks (lattice structure, side optimality, and comparative statics) and connections between chain stability and other solution concepts (tree stability and the core).

4.1 Lattice Structure and Comparative Statics

The first result shows that ν^*_{min} and ν^*_{max} are the extreme fixed points of operator T, and that the set of fixed points of operator T is a lattice. This is a generalization of similar results for matching with contracts (Hatfield and Milgrom, 2005), many-to-many matching (Echenique and Oviedo, 2004b), schedule matching (Alkan and

Gale, 2003), and other two-sided matching settings. This result is a corollary of Tarski's fixed point theorem for isotone operators on lattices, but I also give a simple alternative proof for completeness.

Theorem 4.1 *The set of fixed points of operator T is a lattice with extreme elements ν^*_{min} and ν^*_{max}.*

Proof. See Appendix. ■

Now, let $\overline{A} = \{a \in A : U_a(A) = \emptyset\}$ and $\underline{A} = \{a \in A : D_a(A) = \emptyset\}$, i.e., $\overline{A}$ and $\underline{A}$ are the sets of suppliers of initial inputs for the market ("suppliers") and consumers of final outputs ("consumers"), respectively. In the two-sided matching setup, one side of the market is $\overline{A}$ and the other side is $\underline{A}$; in more general networks, there is also a set of "intermediate" points, $A \backslash (\overline{A} \cup \underline{A})$.

The following theorem generalizes another standard result from the two-sided matching literature, which says that the extreme elements of the lattice of stable matchings are side-optimal (Gale and Shapley, 1962).

Theorem 4.2 *Let $\mu_{min} = F(\nu^*_{min})$, $\mu_{max} = F(\nu^*_{max})$, and let μ be a chain stable network. Then any $a \in \overline{A}$ (weakly) prefers μ_{min} to μ and μ to μ_{max}, and any $a \in \underline{A}$ (weakly) prefers μ_{max} to μ and μ to μ_{min}.*

Proof. See Appendix. ■

Note that the theorem does not say anything about intermediate nodes. In fact, one can construct examples to show that an intermediate agent's most preferred chain stable network may be neither μ_{min} nor μ_{max} and, moreover, different intermediate agents may have different most preferred chain stable networks.

Finally, in two-sided one-to-many matching markets, adding a worker makes other workers (weakly) worse off at the firm- and worker-optimal stable matchings, and makes firms (weakly) better off; symmetrically, the opposite is true when a firm is added to the market (Kelso and Crawford, 1982; Gale and Sotomayor, 1985). The following theorem extends this result to the supply chain setting.

Theorem 4.3 *Let $A' = A \cup a'$, where $U_{a'}(A) = \emptyset$, and preferences of all nodes in the larger market remain same-side substitutable and cross-side complementary. Let μ'_{min} and μ'_{max} be the smallest and the largest chain stable matchings in A'. Then each $a \in \overline{A}$ is* at least *as well off in μ_{max} as in μ'_{max}, and* at least *as well off in μ_{min} as in μ'_{min}; each $a \in \underline{A}$ is* at most *as well off in μ_{max} as in μ'_{max}, and* at most *as well off in μ_{min} as in μ'_{min}. The opposite is true if a' is added to the other end of the market, i.e., $D_{a'}(A) = \emptyset$.*

Proof. See Appendix. ■

Again, the change in the welfare of intermediate agents is ambiguous—it can go either way. Adding new intermediate nodes can also have opposite effects on different extreme nodes (e.g., some $a \in \overline{A}$ may become better off, while others may become worse off), as well as on other intermediate nodes.

4.2 Chain Stability, Tree Stability, and the Core

In two-sided one-to-one matching markets, the set of pairwise stable matchings coincides with the core. In more general models, this is no longer true, even when preferences are substitutable: in one-to-many matching markets, the set of pairwise stable matchings is equal to the weak core but not to the strict core, and in many-to-many matching markets, even that result no longer holds. Nevertheless, pairwise stability is a natural solution concept even in many-to-many matching markets, since, as Roth and Sotomayor (1990, p. 156) argue, “identifying and organizing large coalitions may be more difficult than making private arrangements between two parties, and the experience of those regional [many-to-many matching] markets in the United Kingdom that are built around stable mechanisms suggests that pairwise stability is still of primary importance in these markets."

Since the model of this paper nests two-sided many-to-many matching models, different solution concepts can result in different predictions. However, just like pairwise stability in the two-sided setting, chain stability is a natural solution concept in the supply chain environment. The reason for that is that chain blocks are particularly easy to identify and organize: A customer just needs to pick up the phone and call a potential supplier asking him whether he would like to form a contract; the potential supplier, after receiving that phone call, in turn calls one of his potential suppliers, and so on. If there is a chain block, it can be easily identified in this way, and subsequently the contracts can be formed. In contrast, larger coalitions require much more coordination and information exchange between the agents, and may even violate antitrust laws if they require communication between competing firms. Of course, the similarity between the arguments behind pairwise stability in the two-sided case and chain stability in the more general case is not a coincidence: Chain stability reduces to pairwise stability if there are no intermediate agents.

Still, it is important to understand the differences and similarities between various solution concepts in matching markets. Several papers address these issues in two-sided markets (see, e.g., the recent papers by Echenique and Oviedo, 2004b, and Konishi and Ünver, 2005, and references in those papers). The following two results provide a starting point for the analysis of the relationship between chain stability and other solution concepts in supply chain networks.

The first result shows that under same-side substitutability and cross-side complementarity, blocking by "trees" is equivalent to blocking by chains. More formally, a sequence of contracts $c_1, \ldots, c_i$ is a *path* from node a to node b if: (i) node a is involved in contract c_1 and not involved in any contract c_j for $j > 1$; (ii) node b is involved in contract c_i and not involved in any contract c_j for $j < i$; and (iii) any other node x involved in one of the contracts c_j for $1 \leq j \leq i$ is involved in exactly two such contracts, and these two contracts are adjacent in the sequence (i.e., if one of the contracts is c_k, then the other is either c_{k-1} or c_{k+1}). Note that while each chain is a path, there are paths that are not chains: e.g., a pair of contracts with the same buyer and two different sellers is a path connecting the two sellers, but is not a chain. A *tree* is a set of contracts such that for any two nodes involved in these contracts, there exists exactly one path in this set connecting the two nodes. Note that every chain is a tree. A network, μ, is blocked by a tree, τ, if $\tau \cap \mu = \emptyset$ and for every node a involved in τ, $\tau(a) \subset Ch_a(\mu(a) \cup \tau(a))$. A network is tree stable if it is not blocked by any tree.

Theorem 4.4 *Under same-side substitutability and cross-side complementarity, the set of tree stable networks is equal to the set of chain stable networks.*

Proof. See Appendix. ■

The final result of this section shows that in a special case, in which each node is restricted to having at most one upstream contract and at most one downstream contract, the set of chain stable networks coincides with the weak core of the matching game. Network μ is in the weak core of the matching game if and only if there is no other network μ' and set M of nodes such that (i) for every node $a \in M$, for every contract c involving a, the other node involved in c is also in set M; (ii) every node $a \in M$ weakly prefers the set of contracts in which it is involved in μ' to the set of contracts in which it is involved in μ; and (iii) at least one node $a \in M$ strictly prefers the set of contracts in which it is involved in μ' to the set of contracts in which it is involved in μ.

Theorem 4.5 *If each node $a \in A$ can have at most one upstream contract and at most one downstream contract, and preferences are same-side substitutable and cross-side complementary, then the set of chain stable networks is equal to the weak core of the matching game.*

Proof. See Appendix. ■

5 Conclusion

This paper shows that two-sidedness is not a necessary condition for many key results of matching theory. Under same-side substitutability and cross-side complementarity, chain stable networks are guaranteed to exist. The set of chain stable networks is a lattice with two extreme elements: the optimal chain stable network for the suppliers of basic inputs, and the optimal chain stable network for the consumers of final outputs. Adding a supplier of basic inputs makes other suppliers weakly worse off at the side-optimal stable networks, and makes the consumers of final outputs weakly better off; adding a consumer of final outputs has the opposite effect.

There are several open questions and promising directions for future work related to the theory of matching in supply chains. One of them is figuring out how to model strategic interactions in this setting. With one strategic player, this is straightforward: the strategic player simply chooses the contracts that maximize his payoff, given the non-strategic behavior of the rest of the market. With more than one player, however, it is hard to model strategic behavior even in two-sided markets, and so it is unclear how to do it in the broader supply chain setting.

Another important question is whether cross-side complementarity or same-side substitutability restrictions can be partially relaxed, while preserving some of the properties of the model. In general, same-side complementarities present difficulties for the analysis of matching markets. However, it is possible that in some cases, complementary inputs or economies of scale can be incorporated into the standard models. For example, if a two-sided market includes one type of buyers ($\{x_1, x_2, \dots\}$) and two types of sellers ($\{y_1, y_2, \dots\}$ and $\{z_1, z_2, \dots\}$), so that the buyers are substitutes (from the point of view of the sellers), the sellers of the same type are also substitutes (from the point of view of the buyers), and the sellers of different types are complements (also from the point of view of the buyers), then stable matchings are guaranteed to exist (because this matching market can be viewed as the $Y \succ X \succ Z$ "supply chain market").

Finally, the model readily lends itself to empirical applications. In Ostrovsky (2005) it is used to incorporate heterogeneity of bilateral transportation costs into a model of international trade flows in the steel supply chain. Unlike conventional continuous equilibrium models that incorporate heterogeneity, this framework does not rely on any specific functional form assumptions and can accommodate arbitrary distributions of transportation costs, production functions, and utilities. There may be other interesting settings to which the theory can be applied. For instance, contracts in the model can specify different times at which goods can be shipped, making it possible to analyze various intertemporal settings in the matching framework. Further extending the theoretical model of matching in supply chains and using it to answer new empirical questions are likely to be exciting areas for future research.

Appendix

Proof of Lemma 3.1

Step 1. Let us show that for any pre-network ν such that $T\nu = \nu$, $\mu = F(\nu)$ is a chain stable network.

First, we need to make sure that μ is indeed a network, i.e., there are no contracts in μ that differ only in price. To see this, note that if there are two contracts that differ only in price in μ, that implies that each of the two agents involved in these contracts would choose both contracts when selecting from some larger set containing them. But this is impossible, because each agent, by definition, chooses only one contract with a given partner and a particular serial number—the one with the most favorable price.

Second, we need to show that network μ is individually rational. To see that, note that for any agent a, $\mu(a) = Ch_a(\nu(a))$, and so a does not want to drop any of its contract in μ (because that would imply that $\mu(a) \neq Ch_a(\mu(a)) = Ch_a(Ch_a(\nu(a))) = Ch_a(\nu(a)) = \mu(a)$).

Finally, we need to show that there are no chain blocks. Suppose $(c_1, c_2, \ldots, c_n)$ is a chain block of μ, and let s_i and b_i denote the seller and the buyer involved in contract i. Since $c_1 \in Ch_{s_1}(\mu(s_1) \cup c_1)$, it has to be the case that $c_1 \in Ch_{s_1}(\nu(s_1) \cup c_1)$ (otherwise, $Ch_{s_1}(\nu(s_1) \cup c_1) = Ch_{s_1}(\nu(s_1)) = \mu(s_1)$, and hence no subset of $\mu(s_1) \cup c_1 \subset \nu(s_1) \cup c_1$ can be better for s_1 than $\mu(s_1)$), and so the arrow r_1 from s_1 to b_1 with c_1 attached must be in $T\nu = \nu$. Now, by assumption, s_2 would like to sign contracts c_1 and c_2, i.e., $\{c_1, c_2\} \subset Ch_{s_2}(\mu(s_2) \cup c_1 \cup c_2)$. If neither c_1 nor c_2 are in $Ch_{s_2}(\nu(s_2) \cup c_1 \cup c_2)$, then $Ch_{s_2}(\nu(s_2) \cup c_1 \cup c_2) = Ch_{s_2}(\nu(s_2)) = \mu(s_2)$, and so $\{c_1, c_2\} \not\subset Ch_{s_2}(\mu(s_2) \cup c_1 \cup c_2)$, which would contradict our assumptions. Suppose $c_2 \notin Ch_{s_2}(\nu(s_2) \cup c_1 \cup c_2)$. Then $c_1 \in Ch_{s_2}(\nu(s_2) \cup c_1 \cup c_2) = Ch_{s_2}(\nu(s_2) \cup c_1)$, and so there must be an arrow from s_2 to

s_1 with contract c_1 attached in $T\nu = \nu$, which together with the fact that there is an arrow from s_1 to s_2 with c_1 attached in ν would imply that $c_1 \in \mu$, which would also contradict our assumptions. Hence, it must be the case that $c_2 \in Ch_{s_2}(\nu(s_2) \cup c_1 \cup c_2)$. Proceeding by induction, there is an arrow from s_i to s_{i+1} with c_i attached in ν for any $i < n$. Similarly, we could have started from node b_n, and so there must be an arrow going from b_n to $b_{n-1} = s_n$ with c_n attached in ν, which implies that $c_n \in \mu$—contradiction. Therefore, for any $\nu = T\nu$, $F(\nu)$ is a chain stable network.

Step 2. Let us now show that for any chain stable network μ, for some n, $H_n(\mu) = H_{n+1}(\mu)$ and moreover, $F(H_n(\mu)) = \mu$. For convenience, let $\nu^n = H_n(\mu)$, $\nu^0 = G(\mu)$. We will show by induction on k that: (i) $F(\nu^k) = \mu$ and (ii) $\nu^k \supset \nu^{k-1}$. For $k = 1$, (ii) follows from the individual rationality of μ and (i) follows from the absence of chain blocks of length 1 (i.e., pairwise blocks) of μ. Suppose (i) and (ii) hold up to $k-1$. Let us show that they hold for k.

(i) Suppose arrows r and r' with contract c_r attached are in ν^k, but $c_r \notin \mu$. We will now "grow" a chain block of μ from this contract c_r.

Consider arrow r first; without loss of generality, assume it is upstream. Let $c_0 = c_r$ and $r_0 = r$. If $c_r \in Ch_{o_r}(\mu(o_r) \cup c_r)$, stop. Otherwise, since $c_r \in Ch_{o_r}(\nu^{k-1}(o_r) \cup c_r)$, by same-side substitutability $c_r \in Ch_{o_r}(D_{o_r}(\nu^{k-1}(o_r)) \cup U_{o_r}(\mu(o_r)) \cup c_r)$. Let $x_1, x_2, \ldots, x_m$ be the contracts in $D_{o_r}(\nu^{k-1}(o_r)) \backslash D_{o_r}(\mu(o_r))$. Then for some j, $x_j \in Ch_{o_r}(\mu(o_r) \cup x_j \cup c_r)$ (otherwise, by same-side substitutability, for any j, $x_j \notin Ch_{o_r}(\mu(o_r) \cup c_r \cup x_1 \cup x_2 \cup \cdots \cup x_m)$, which is then equal to $Ch_{o_r}(\mu(o_r) \cup c_r)$, which contradicts our assumption that $c_r \notin Ch_{o_r}(\mu(o_r) \cup c_r)$). It must also be the case that $c_r \in Ch_{o_r}(\mu(o_r) \cup x_j \cup c_r)$, because otherwise $x_j \in Ch_{o_r}(\mu(o_r) \cup x_j)$ and so the downstream arrow with x_j attached is in ν^1, and is therefore in ν^{k-1} (by statement (ii) in the step of induction, $\nu^1 \subset \nu^{k-1}$). But then both arrows with x_j attached are in ν^{k-1}, which contradicts assumption (i) of induction for $k-1$. Hence,

$\{c_r, x_j\} \subset Ch_{o_r}(\mu(o_r) \cup x_j \cup c_r)$.

Let $c_1 = x_j$. By construction, the upstream arrow r_1 with c_1 attached is in ν^{k-1}, but $c_1 \notin \mu$. Let o_1 denote the origin of arrow r_1. If $c_1 \in Ch_{o_1}(\mu(o_1) \cup c_1)$, stop; otherwise, following the procedure above, generate $c_2 \in D_{o_1}(\nu^{k-2}(o_1)) \backslash D_{o_1}(\mu(o_1))$, and so on. At some point, this procedure will have to stop (since we keep going downstream). Now, "grow" c_r in the other direction, starting with arrow r'. We end up with a chain $c_x, c_{x+1}, \ldots, c_0, \ldots, c_{y-1}, c_y$, which, by construction, is a chain block of μ—contradiction.

(ii) Suppose some upstream arrow r is in ν^{k-1}, but not in ν^k, i.e., $c_r \in Ch_{o_r}(\nu^{k-2}(o_r) \cup c_r)$, but $c_r \notin Ch_{o_r}(\nu^{k-1}(o_r) \cup c_r)$. Then by (i), $Ch_{o_r}(\nu^{k-1}(o_r) \cup c_r) = \mu(o_r) = Ch_{o_r}(D_{o_r}(\nu^{k-1}(o_r)) \cup U_{o_r}(\mu(o_r)) \cup c_r)$. From $c_r \notin Ch_{o_r}(D_{o_r}(\nu^{k-1}(o_r)) \cup U_{o_r}(\mu(o_r)) \cup c_r)$, by same-side substitutability and by assumptions of induction for $k-2$, we get $c_r \notin Ch_{o_r}(D_{o_r}(\nu^{k-1}(o_r)) \cup U_{o_r}(\nu^{k-2}(o_r)) \cup c_r)$, and from that, by cross-side complementarity and assumption (ii) of induction for $k-1$ (i.e., $\nu^{k-2} \subset \nu^{k-1}$), we get $c_r \notin Ch_{o_r}(D_{o_r}(\nu^{k-2}(o_r)) \cup U_{o_r}(\nu^{k-2}(o_r)) \cup c_r) = Ch_{o_r}(\nu^{k-2}(o_r) \cup c_r)$, and so r is not in ν^{k-1}—contradiction. The proof for a downstream arrow r' is completely analogous. This completes the proof of statements (i) and (ii) of induction.

Now, since $G(\mu) \subset H_1(\mu) \subset H_2(\mu) \subset \ldots$ is an increasing sequence and the set of possible arrows is finite, this sequence has to converge, i.e., for some n, $H_n(\mu) = H_{n+1}(\mu) = H(\mu)$. By (ii), all arrows in $G(\mu)$ are also present in $H(\mu)$, and by (i), any pair of arrows with the same contract attached in $H(\mu)$ is also present in $G(\mu)$. Therefore, $F(H(\mu)) = \mu$.

Finally, we need to show that for two fixed points of operator T, ν_1^* and ν_2^*, $F(\nu_1^*) \neq F(\nu_2^*)$. Suppose $\nu_1^* \neq \nu_2^*$ and $F(\nu_1^*) = F(\nu_2^*) = \mu$. Consider the set of agents for whom the upstream arrows originating from them are not the same in ν_1^* and ν_2^*. Take one of the "most downstream" agents in this set (i.e., such an agent o that there is nobody downstream from him in this set), and take an upstream arrow r originating from o such that it is in only one of the two pre-networks. Without loss of generality, $r \in \nu_1^*$ and $r \notin \nu_2^*$. $r \notin \nu_2^* \Rightarrow c_r \notin Ch_o(\nu_2^*(o) \cup c_r) = Ch_o(\nu_2^*(o)) = \mu(o) = Ch_o(D_o(\nu_2^*(o)) \cup U_o(\mu(o)) \cup c_r)$. By the assumption that o is the "most downstream" agent whose upstream arrows differ in the two pre-networks, $D_o(\nu_2^*(o)) = D_o(\nu_1^*(o))$, and hence $c_r \notin Ch_o(D_o(\nu_1^*(o)) \cup U_o(\mu(o)) \cup c_r)$. Now, since $F(\nu_1^*) = \mu$, $U_o(\nu_1^*(o)) \supset U_o(\mu(o))$, and so by same-side substitutability, $c_r \notin Ch_o(D_o(\nu_1^*(o)) \cup U_o(\nu_1^*(o)) \cup c_r) = Ch_o(\nu_1^*(o) \cup c_r)$, and therefore $r \notin \nu_1^*$—contradiction.

Proof of Lemma 3.2

By induction on i. For $i = 0$, the statement is true by construction. Consider some i such that the statement is true for $0, \ldots, i-1$. We need to check that all downstream arrows in ν_i belong to ν_{i+1}, and that all upstream arrows in ν_{i+1} belong to ν_i. Since only arrows belonging to R_{i+1} are affected by operator $T_{R_{i+1}}$, we can restrict our attention to them.

Consider a downstream arrow r in $\nu_i \cap R_{i+1}$. Since there are no downstream arrows in ν_0, there must be at least one t such that $r \in R_t$. Let $j = max_{\{0<t<i+1\}}\{t | r \in R_t\}$. Since $r \in \nu_i$ and j was the last time this arrow was updated, r must belong to ν_j, and so $c_r \in Ch_{o_r}(\nu_{t-1}(o_r) \cup c_r)$. By induction and transitivity, $\nu_{j-1} \leq \nu_i$, and so $U(\nu_{j-1}(o_r)) \subset U(\nu_i(o_r))$ (i.e., the set of arrows pointing to o_r from upstream nodes is smaller in ν_{j-1} than it is in ν_i) and $D(\nu_{j-1}(o_r)) \supset D(\nu_i(o_r))$. Thus, $(D(\nu_{j-1}(o_r)) \cup c_r) \supset (D(\nu_i(o_r)) \cup c_r)$, and by same-side substitutability, $c_r \in Ch_{o_r}((D(\nu_i(o_r)) \cup c_r) \cup U(\nu_{j-1}(o_r)))$. Now, by cross-side complementarity, $c_r \in Ch_{o_r}((D(\nu_i(o_r)) \cup c_r) \cup U(\nu_i(o_r)))$, i.e., $c_r \in Ch_{o_r}(\nu_i(o_r) \cup c_r)$, and so $r \in \nu_{i+1}$.

For upstream arrows, the argument is completely symmetric: if an upstream arrow

$r' \in R_{i+1}$ is *not* in ν_i, consider the last time j this arrow was updated, and then show by same-side substitutability and cross-side complementarity that this arrow is not in ν_{i+1}.

Proof of Theorem 4.1

Take any two fixed points of operator T, ν_1^* and ν_2^*. Let ν_{12} be the least upper bound of these two pre-networks in the original lattice. $\nu_{12} \geq \nu_1^*$, $\nu_{12} \geq \nu_2^* \Rightarrow T\nu_{12} \geq T\nu_1^* = \nu_1^*$, $T\nu_{12} \geq T\nu_2^* = \nu_2^* \Rightarrow T\nu_{12} \geq \nu_{12}$, and so for some n, $\nu_{12} \leq T\nu_{12} \leq T^2\nu_{12} \leq \cdots \leq T^n\nu_{12} = T^{n+1}\nu_{12} = \nu_{12}^*$. By construction, $\nu_{12}^* \geq \nu_1^*$ and $\nu_{12}^* \geq \nu_2^*$. To see that any other upper bound of ν_1^* and ν_2^* (say, ν_3^*) has to be greater than ν_{12}^*, note that $\nu_3^* \geq \nu_1^*$, $\nu_3^* \geq \nu_2^*$ implies $\nu_3^* \geq \nu_{12} \Rightarrow T\nu_3^* = \nu_3^* \geq T\nu_{12} \Rightarrow \cdots \Rightarrow \nu_3^* \geq \nu_{12}^*$. The greatest lower bound of ν_1^* and ν_2^* can be constructed in an analogous way.

To show that ν_{min}^* is the lowest fixed point, consider another fixed point ν^*, and note that $\nu^* \geq \nu_{min} \Rightarrow T\nu^* = \nu^* \geq T\nu_{min} \Rightarrow \cdots \Rightarrow \nu^* \geq \nu_{min}^*$. Analogously, ν_{max}^* is the highest fixed point of operator T.

Proof of Theorem 4.2

Let $\nu = H(\mu)$. Since ν_{min}^* and ν_{max}^* are the extreme fixed points of operator T, $\nu_{min}^* \leq \nu \leq \nu_{max}^*$. Take any $a \in \overline{A}$ (the proof for the symmetric case $a \in \underline{A}$ is completely analogous). By definition of $\overline{A}$, a can only be connected with downstream nodes. Therefore, the set of arrows pointing to a in ν_{min}^* is a superset of arrows pointing to a in ν, which in turn is a superset of arrows pointing to a in ν_{max}^*. But that implies that $Ch_a(\nu_{min}^*(a)) = \mu_{min}(a)$ is at least as good for a as $Ch_a(\nu(a)) = \mu(a)$, which in turn is at least as good for a as $Ch_a(\nu_{max}^*(a)) = \mu_{max}(a)$.

Proof of Theorem 4.3

The proof consists of two independent steps—one compares μ_{max} with μ'_{max} and the other compares μ_{min} with μ'_{min}.

Step 1. Consider ν^*_{max}. Add node a' to market A, so that $U_{a'}(A) = \emptyset$. Let $\nu_+ = \nu^*_{max} \cup \{r : a' = d_r \ \& \ c_r \in Ch_a(\nu^*_{max}(a) \cup c_r)\}$; that is, ν_+ contains all arrows in ν^*_{max} plus all such arrows r from nodes $a \in A$ to the new node a' that a would like to add the attached contract c_r to its list of contracts (and possibly drop some of its other contracts). Now, note that $T\nu_+ \geq \nu_+$ (for any $a \in A$, $\nu_+(a) = \nu^*_{max}(a)$, and so all arrows in $T\nu_+$ originating from points in A are exactly the same as in ν_+; all new arrows originate from a' and thus necessarily point downstream). But then $\nu_+ \leq T\nu_+ \leq \cdots \leq T^n\nu_+ = T^{n+1}\nu_+ \leq \nu'_{max}$, where $\nu'_{max} = H(\mu'_{max})$ is the highest fixed point in market A'. This, in turn, implies that for any $a \in \overline{A}$, $\nu^*_{max}(a) = \nu_+(a) \supset \nu'_{max}(a)$, and so a is at least as well off in $\mu_{max} = Ch_a(\nu^*_{max}(a))$ as in $\mu'_{max} = Ch_a(\nu'_{max}(a))$. Similarly, for any $a \in \underline{A}$, $\nu^*_{max}(a) = \nu_+(a) \subset \nu'_{max}(a)$, and so a is at most as well off in $\mu_{max} = Ch_a(\nu^*_{max}(a))$ as in $\mu'_{max} = Ch_a(\nu'_{max}(a))$.

Step 2. Now start with the larger market A' and consider the lowest fixed point of T, ν'_{min}. Exclude node $a' \in \overline{A'}$ with all the arrows going to and from a'. Denote the resulting pre-network on A by ν_-. Note that $T\nu_- \leq \nu_-$ (for any node $a \in A$, $U_a(\nu_-(a)) \subset U_a(\nu'_{min}(a))$ and $D_a(\nu_-(a)) = D_a(\nu'_{min}(a))$; thus (i) by same-side substitutability, the set of upstream arrows originating at a in $T\nu_-$ is a superset of the set of upstream arrows originating at a in $T\nu'_{min}$ and (ii) by cross-side complementarity, the set of downstream arrows originating at a in $T\nu_-$ is a subset of the set of downstream arrows originating at a in $T\nu'_{min}$). Therefore, $\nu_- \geq T\nu_- \geq \ldots T^n\nu_- = T^{n+1}\nu_- \geq \nu^*_{min}$. This, in turn, implies that for any $a \in \overline{A}$, $\nu'_{min}(a) = \nu_-(a) \subset \nu^*_{min}(s)$, and so a is at least as well off in $\mu_{min} = Ch_a(\nu^*_{min}(a))$ as in $\mu'_{min} = Ch_a(\nu'_{min}(a))$. Similarly, for any $a \in \underline{A}$, $\nu'_{min}(a) \supset \nu_{-}(a) \supset \nu^*_{min}(a)$, and so a is at most as well off in $\mu_{min} = Ch_a(\nu^*_{min}(a))$ as in $\mu'_{min} = Ch_a(\nu'_{min}(a))$.

The case where a' is added to the other end of the market is completely symmetric.

Proof of Theorem 4.4

Since every chain is a tree, the set of tree stable networks is a subset of the set of chain stable networks. Let us now show that any chain stable network is also tree stable.

Consider a network, μ, that is chain stable but not tree stable. Let τ be a tree with the smallest possible number of contracts blocking μ. Since, by assumption, τ is not a chain, there must exist a node, a, that is involved in at least two contracts in τ as a seller or in at least two contracts in τ as a buyer. Assume that a is involved in contracts $\{c_1, \ldots, c_k\} \subset \tau$ as a seller, $k \geq 2$; the case in which a is involved in two or more contracts as a buyer is completely symmetric and is therefore omitted. Let $\upsilon = Ch_a[\mu(a) \cup c_1 \cup U_a(\tau(a))] \cap U_a(\tau(a))$, that is, the set of upstream contracts in blocking tree τ that a would choose to add to μ if the only additional downstream contract it had was c_1. Note that, by same-side substitutability, $c_1 \in Ch_a(\mu(a) \cup c_1 \cup U_a(\tau(a)))$, and so $(c_1 \cup \upsilon) \subset Ch_a(\mu(a) \cup (c_1 \cup \upsilon))$. Set υ can, of course, be empty. Let τ' be the subset of τ which consists of contracts that involve only the nodes that have paths connecting them to a and containing either c_1 or a contract from υ. In other words, τ' is obtained by cutting off the branches of tree τ (viewing a as the root) that do not start with contracts in $c_1 \cup \upsilon$. By construction, τ' is a tree, $\tau'(a) = (c_1 \cup \upsilon) \subset Ch_a(\mu(a) \cup \tau'(a))$, and for any other node b involved in τ', $\tau'(b) = \tau(b)$, and so $\tau'(b) \subset Ch_b(\mu(b) \cup \tau'(b))$. Therefore, τ' is a tree block of μ, and contains fewer contracts than τ does, which contradicts the assumption that τ is a tree with the smallest possible number of contracts blocking μ.

Proof of Theorem 4.5

Suppose network μ is in the weak core, but has a chain block, $(c_1, \ldots, c_k)$. Let $(x_1, \ldots, x_m) \subset \mu$ be the longest chain in μ such that the seller in contract c_1 is the buyer in contract x_m, and let $(y_1, \ldots, y_n) \subset \mu$ be the longest chain in μ such that the buyer in contract c_k is the seller in contract y_1. Let $\mu' = \{x_1, \ldots, x_m, c_1, \ldots, c_k, y_1, \ldots, y_n\}$, and let M be the set of nodes involved in μ'. Then μ' weakly dominates μ via coalition M, and hence μ could not be in the weak core. The proof of the fact that any network in the weak core is individually rational is very similar, and is therefore omitted.

Now consider any chain stable network μ that is not in the weak core, and consider a network μ' that weakly dominates it via some coalition M and has the smallest possible number of contracts in $(\mu' \backslash \mu)$ among such networks. Take a node $a \in M$ that strictly prefers its set of contracts in μ' to its set of contracts in μ and that doesn't have any upstream nodes that strictly prefer their sets of contracts in μ' to their sets of contracts in μ. $Ch_a(\mu(a) \cup \mu'(a)) \neq \mu(a)$. If $Ch_a(\mu(a) \cup \mu'(a)) \subset \mu(a)$, then μ is not individually rational, contradicting its chain stability. Otherwise, take contract $c_1 \in Ch_a(\mu(a) \cup \mu'(a)) \backslash \mu(a)$. Contract c_1 must be downstream for a, because a was chosen as one of the most upstream nodes that strictly benefit from a switch from μ to μ', and because all preferences are strict.

Let b be the buyer in contract c_1. Preferences of agent b are strict, $\mu'(b) \neq \mu(b)$, and therefore $Ch_b(\mu(b) \cup \mu'(b)) \neq \mu(b)$. $Ch_b(\mu(b) \cup \mu'(b)) \not\subset \mu(b)$ by individual rationality, and so set $Z = Ch_b(\mu(b) \cup \mu'(b)) \backslash \mu(b)$ is not empty. There are three possibilities: (i) Z contains only c_1, (ii) Z contains only some downstream contract c_2, and (iii) Z contains c_1 and a downstream contract c_2. Let us consider these possibilities one by one.

(i) In this case, (c_1) is a chain block of μ.

(ii) In this case, consider network μ'' that includes contract c_2, the longest possible chain in μ' that begins with c_2, and the longest possible chain in μ that ends at node b. Then μ'' weakly dominates μ via the coalition of all nodes involved in μ'', and $|\mu''\backslash\mu| < |\mu'\backslash\mu|$, contradicting the assumptions.

(iii) In this case, consider the buyer of contract c_2, and repeat the same operation with this buyer as what we did with buyer b.

Eventually, since we keep going downstream, we will have to end up at case (i) or (ii), and so will either find a chain block of μ, or a network μ'' that weakly dominates μ such that $|\mu''\backslash\mu| < |\mu'\backslash\mu|$, both of which are impossible by assumption.

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