Economics 204 Summer/Fall 2010
Final Exam

Answer all of the questions below. Be as complete, correct, and concise as possible. There are 6 questions for a total of 165 points possible; point values for each problem are in parentheses. For questions with subparts, each subpart is worth the same number of points. You have 180 minutes to complete the exam. Use the points as a guide to allocating your time. You may use any result from class with appropriate references unless you are specifically being asked to prove it.

1. (15) Define or state each of the following.
   (a) Brouwer's Fixed Point Theorem
   (b) contraction mapping
   (c) metric

2. (30) Prove that for every $n \in \mathbf{N}$,
$$\sum_{k=1}^{n} k^3 = \frac{1}{4}\left(n(n+1)\right)^2$$

3. (30) Let X and Y be vector spaces over the same field F, and $T \in L(X, Y)$, that is, $T : X \to Y$ is a linear transformation.
   (a) Show that $\ker T$ is a vector subspace of X and that $\operatorname{Im} T$ is a vector subspace of Y.
   (b) Suppose $\dim X = \dim Y$ and $\ker T = \{0\}$. Show that if $V = \{v_1, \ldots, v_n\}$ is a basis for X, then $\{T(v_1), \ldots, T(v_n)\}$ is a basis for Y.

4. (30) Consider $\mathbf{R}$ with the usual metric.

   (a) Let $C = \{\frac{n}{n^2+1} : n = 0, 1, 2, 3, \ldots\}$. Show directly from the definition that C is compact.

   (**Note:** An otherwise correct answer that does not use the open cover definition will receive 10 points.)

   (b) Let $C_1 = C \setminus \{0\} = \{\frac{n}{n^2+1} : n = 1, 2, 3, \ldots\}$. Is C_1 compact? Justify your answer.

   (**Note:** Answers with no justification will receive no points.)

5. (30) Let $f : A \to \mathbf{R}^m$ be continuous, where $A \subseteq \mathbf{R}^n$ is open and convex. Show that if f is differentiable on A and $\|df_x\|$ is bounded on A, then f is uniformly continuous on A.

6. (30) Suppose $\Psi_1, \Psi_2 : X \to 2^Y$ are closed-valued, upper hemicontinuous correspondences, where $X \subseteq \mathbf{R}^n, Y \subseteq \mathbf{R}^m$ for some n, m. Suppose that $\Psi_1(x) \cap \Psi_2(x) \neq \emptyset$ for each $x \in X$. Show that $\Psi_1 \cap \Psi_2$ is upper hemicontinuous, where $\Psi_1 \cap \Psi_2 : X \to 2^Y$ is defined by

$$(\Psi_1 \cap \Psi_2)(x) = \Psi_1(x) \cap \Psi_2(x) \quad \forall x \in X$$

   (**Note:** For full credit, the answer will have to directly use the definition of upper hemicontinuity. An otherwise correct answer that uses alternative characterizations of upper hemicontinuity will receive 50% credit, provided any necessary additional assumptions are clearly stated.)