Announcements

- PS1 due tomorrow in lecture

Econ 204 2011

Lecture 4

Outline

1. Open and Closed Sets
2. Continuity in Metric Spaces

Open and Closed Sets

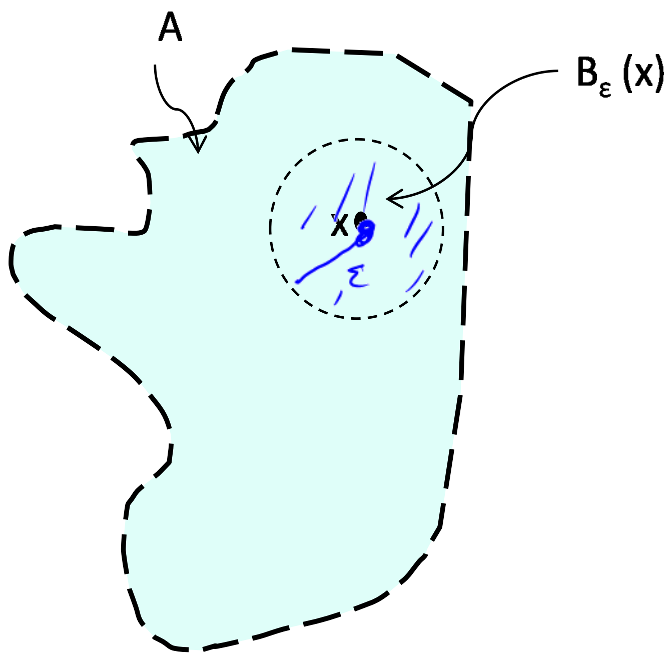
Definition 1. *Let* (X, d) *be a metric space. A set* $A \subseteq X$ *is* open *if*

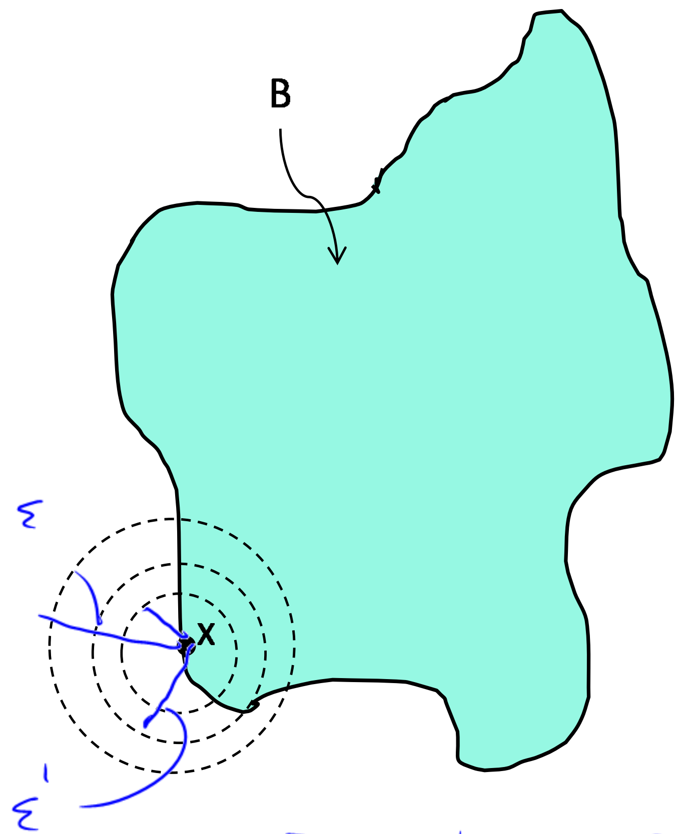
$$\underbrace{\forall x \in A}\ \underbrace{\exists \varepsilon > 0}\ \textit{s.t.}\ B_\varepsilon(x) \subseteq A$$

A set $C \subseteq X$ *is* closed *if* $X \setminus C$ *is open.*

$\exists x \in A$ s.t. $\nexists \varepsilon > 0$ s.t.

$B_\varepsilon(x) \subseteq A$





A open

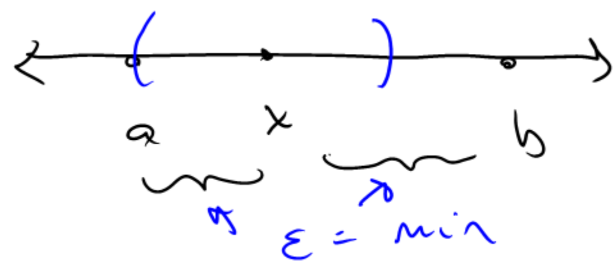
B not open

$\nexists\ \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq B$

Open and Closed Sets

Example: (a,b) is open in the metric space $\mathbf{E}^1$ ($\mathbf{R}$ with the usual Euclidean metric). Given $x \in (a,b)$, $a < x < b$. Let

$$\varepsilon = \min\{x-a, b-x\} > 0$$



Then

$-\varepsilon \geq -(x-a)$

$\varepsilon \leq b-x$

$$\begin{aligned} y \in B_\varepsilon(x) \ &\Rightarrow \ y \in (x-\varepsilon, x+\varepsilon) \\ &\subseteq \ (x-(x-a), x+(b-x)) \\ &= \ (a,b) \end{aligned}$$

so $B_\varepsilon(x) \subseteq (a,b)$, so (a,b) is open.

Notice that ε depends on x; in particular, ε gets smaller as x nears the boundary of the set.

Open and Closed Sets

Example: In $\mathbf{E}^1$, $[a, b]$ is closed. $\mathbf{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is a union of two open sets, which must be open.

$X =$ w/ standard metric

Example: In the metric space $[0, 1]$, $[0, 1]$ is open. With $[0, 1] = X$ as the underlying metric space,

$\varepsilon \in (0, 1)$

$$B_\varepsilon(0) = \{x \in [0, 1] : |x - 0| < \varepsilon\} = [0, \varepsilon) \subseteq [0, 1]$$

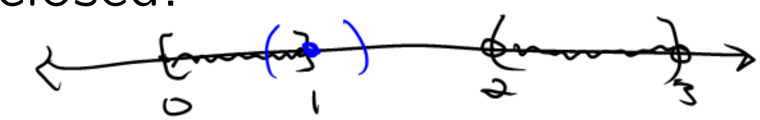
$$\{x \in X : d(x, 0) < \varepsilon\}$$

Thus, openness and closedness depend on the underlying metric space as well as on the set.

Open and Closed Sets

$(-\infty, 0) \cup (1, 2] \cup [3, +\infty)$

Example: Most sets are neither open nor closed. For example, in E^1, $[0, 1] \cup (2, 3)$ is neither open nor closed.



Example: An open set may consist of a single point. For example, if $X = \mathbf{N}$ and $d(m, n) = |m - n|$, then

$$B_{1/2}(1) = \{m \in \mathbf{N} : |m - 1| < 1/2\} = \{1\}$$

Since 1 is the only element of the set $\{1\}$ and $B_{1/2}(1) = \{1\} \subseteq \{1\}$, the set $\{1\}$ is open.

Open and Closed Sets

Example: In any metric space (X, d) both $\emptyset$ and X are open, and both $\emptyset$ and X are closed.

To see that $\emptyset$ is open, note that the statement

$$\underline{\forall x \in \emptyset}\ \exists \varepsilon > 0\ B_\varepsilon(x) \subseteq \emptyset$$

is vacuously true since there aren't any $x \in \emptyset$. To see that X is open, note that since $B_\varepsilon(x)$ is by definition $\{z \in X : d(z,x) < \varepsilon\}$, $\subseteq X$ it is trivially contained in X.

Since $\emptyset$ is open, X is closed; since X is open, $\emptyset$ is closed.

$X = X \setminus \emptyset$ $\qquad$ $\emptyset = X \setminus X$

Open and Closed Sets

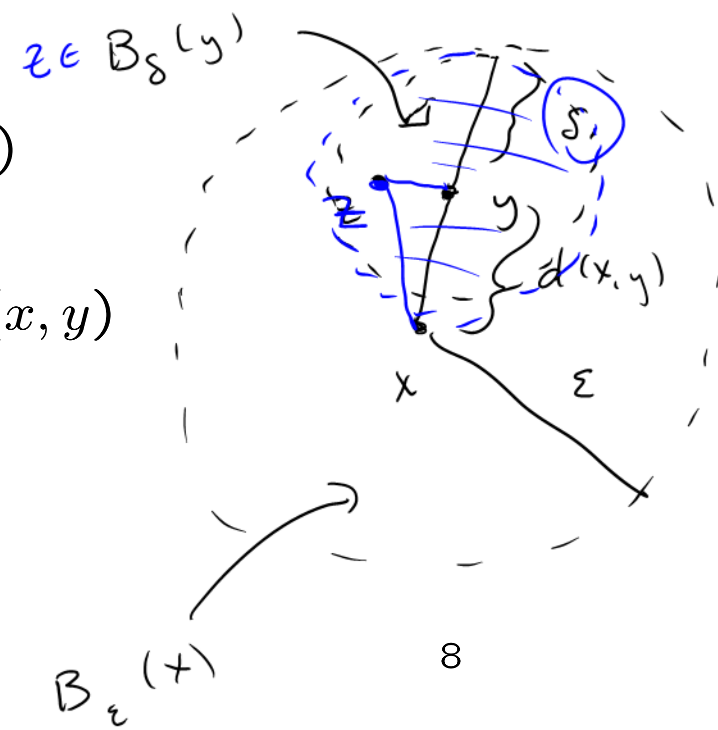
Example: Open balls are open sets. (X, d)

Fix $x \in X$, $\varepsilon > 0$

Suppose $y \in B_\varepsilon(x)$. Then $d(x, y) < \varepsilon$. Let $\delta = \varepsilon - d(x, y) > 0$. If $d(z, y) < \delta$, then

$$\begin{aligned} d(z, x) &\leq d(z, y) + d(y, x) \\ &< \delta + d(x, y) \\ &= \varepsilon - d(x, y) + d(x, y) \\ &= \varepsilon \end{aligned}$$

so $B_\delta(y) \subseteq B_\epsilon(x)$, so $B_\varepsilon(x)$ is open.



Open and Closed Sets

Theorem 1 (Thm. 4.2)**.** *Let* (X, d) *be a metric space. Then*

1. $\emptyset$ *and* X *are both open, and both closed.*

2. *The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open.*

3. *The intersection of a finite collection of open sets is open.*

Proof. 1. We have already shown this.

2. Suppose $\{A_\lambda\}_{\lambda \in \Lambda}$ is a collection of open sets.

$$x \in \bigcup_{\lambda \in \Lambda} A_\lambda \Rightarrow \exists \lambda_0 \in \Lambda \text{ s.t. } x \in A_{\lambda_0} \leftarrow \text{open}$$

$$\Rightarrow \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$$

so $\cup_{\lambda \in \Lambda} A_\lambda$ is open.

3. Suppose $A_1, \ldots, A_n \subseteq X$ are open sets. If $x \in \cap_{i=1}^n A_i$, then

$$x \in \underset{\text{open}}{\underset{\uparrow}{A_1}}, x \in \underset{\text{open}}{\underset{\uparrow}{A_2}}, \ldots, x \in \underset{\text{open}}{\underset{\uparrow}{A_n}}$$

so

$$\exists \varepsilon_1 > 0, \ldots, \varepsilon_n > 0 \text{ s.t. } B_{\varepsilon_1}(x) \subseteq A_1, \ldots, B_{\varepsilon_n}(x) \subseteq A_n$$

Let*

$$\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0$$

Then

$$B_\varepsilon(x) \subseteq B_{\varepsilon_1}(x) \subseteq A_1, \ldots, B_\varepsilon(x) \subseteq B_{\varepsilon_n}(x) \subseteq A_n$$

so

$$B_\varepsilon(x) \subseteq \bigcap_{i=1}^{n} A_i$$

which proves that $\cap_{i=1}^n A_i$ is open.

$\square$

*Note this is where we need the fact that we are taking a finite intersection. The infimum of an infinite set of positive numbers could be zero. And the intersection of an infinite collection of open sets need not be open.

(X, d), $A \subseteq X$

Interior, Closure, Exterior and Boundary

Definition 2. • *The* interior *of* A*, denoted* $\operatorname{int} A$*, is the largest open set contained in* A *(the union of all open sets contained in* A*).*

A not open $\iff$ $\operatorname{int} A \subsetneq A$

• *The* closure *of* A*, denoted* $\bar{A}$*, is the smallest closed set containing* A *(the intersection of all closed sets containing* A*)*

A not closed $\iff$ $A \subsetneq \bar{A}$

• *The* exterior *of* A*, denoted* $\operatorname{ext} A$*, is the largest open set contained in* $X \setminus A$*.* $\left(= \operatorname{int}(X \setminus A) \right)$

• *The* boundary *of* A*, denoted* $\partial A = \overline{(X \setminus A)} \cap \bar{A}$

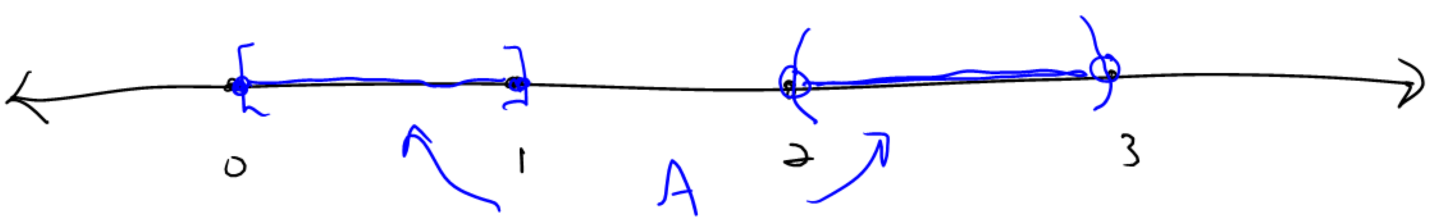
$A \setminus (\operatorname{int} A)$ closed? No.

$\mathbb{Q} \subseteq \mathbb{R}$

Interior, Closure, Exterior and Boundary

Example: Let $A = [0, 1] \cup (2, 3)$. Then

$$
\begin{aligned}
\operatorname{int} A &= (0,1) \cup (2,3) \\
\bar{A} &= [0,1] \cup [2,3] \\
\operatorname{ext} A &= \operatorname{int}(X \setminus A) = \operatorname{int}\big((-\infty,0) \cup (1,2] \cup [3,+\infty)\big) \\
&= (-\infty,0) \cup (1,2) \cup (3,+\infty) \\
\partial A &= \overline{(X \setminus A)} \cap \bar{A} \\
&= \big((-\infty,0] \cup [1,2] \cup [3,+\infty)\big) \cap \big([0,1] \cup [2,3]\big) \\
&= \{0,1,2,3\}
\end{aligned}
$$



Sequences and Closed Sets

Theorem 2 (Thm. 4.13)**.** *A set* A *in a metric space* (X, d) *is closed if and only if*

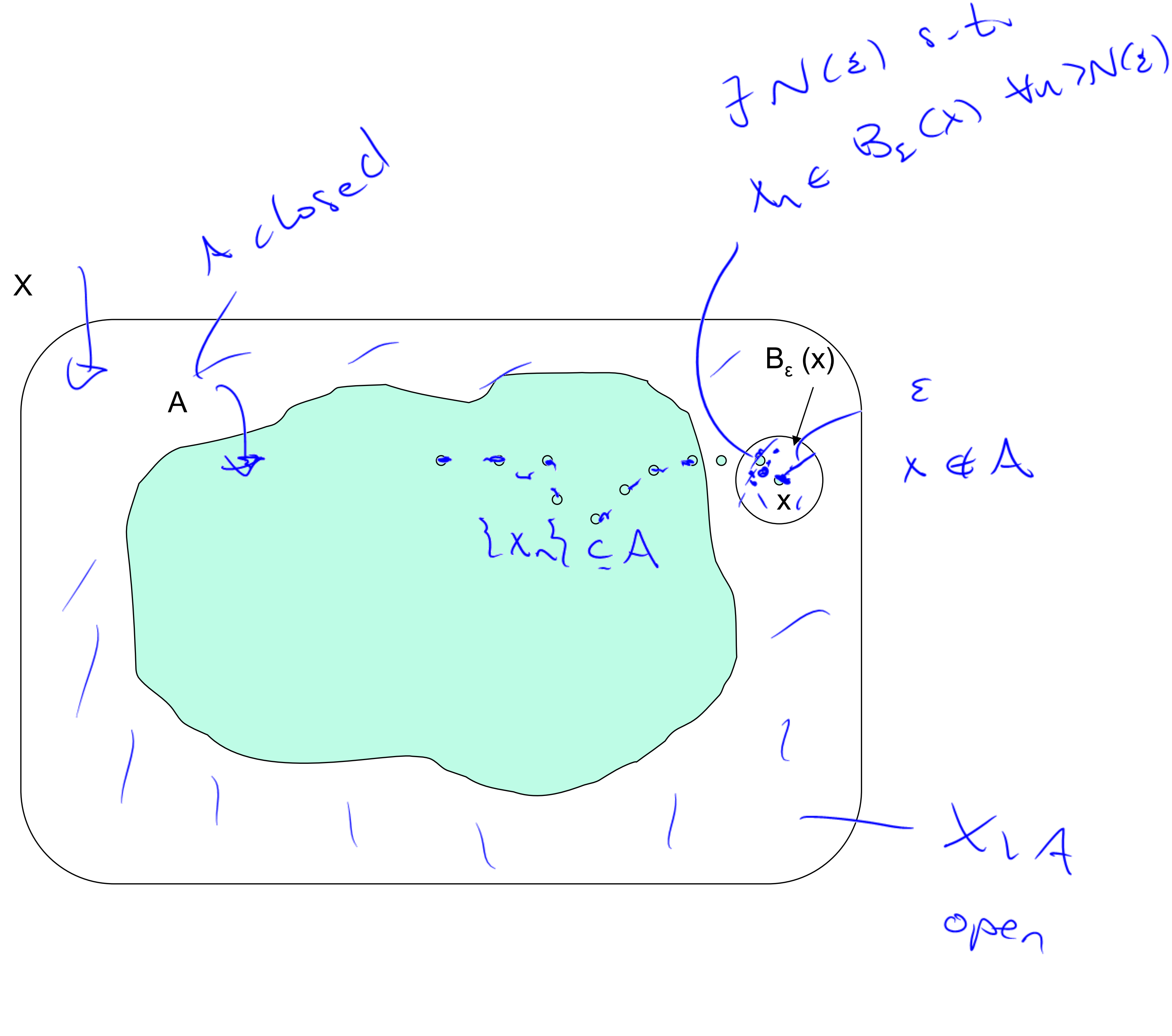
$$\{x_n\} \subset A, x_n \to x \in X \Rightarrow x \in A$$

Proof. Suppose A is closed. Then $X \setminus A$ is open. Consider a convergent sequence $x_n \to x \in X$, with $x_n \in A$ for all n. If $x \notin A$, $x \in X \setminus A$, so there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq X \setminus A$ (why?). Since $x_n \to x$, there exists $N(\varepsilon)$ such that

$$\begin{aligned} n > N(\varepsilon) &\Rightarrow x_n \in B_\varepsilon(x) \\ &\Rightarrow x_n \in X \setminus A \\ &\Rightarrow x_n \notin A \end{aligned}$$

contradiction. Therefore,

$$\{x_n\} \subset A, x_n \to x \in X \Rightarrow x \in A$$


X
A closed
A
$B_\varepsilon(x)$
$\exists N(\varepsilon)$ s.t. $x_n \in B_\varepsilon(x)$ $\forall n > N(\varepsilon)$
ε
$x \notin A$
x
$\{x_n\} \subseteq A$
$X \setminus A$
open

Conversely, suppose

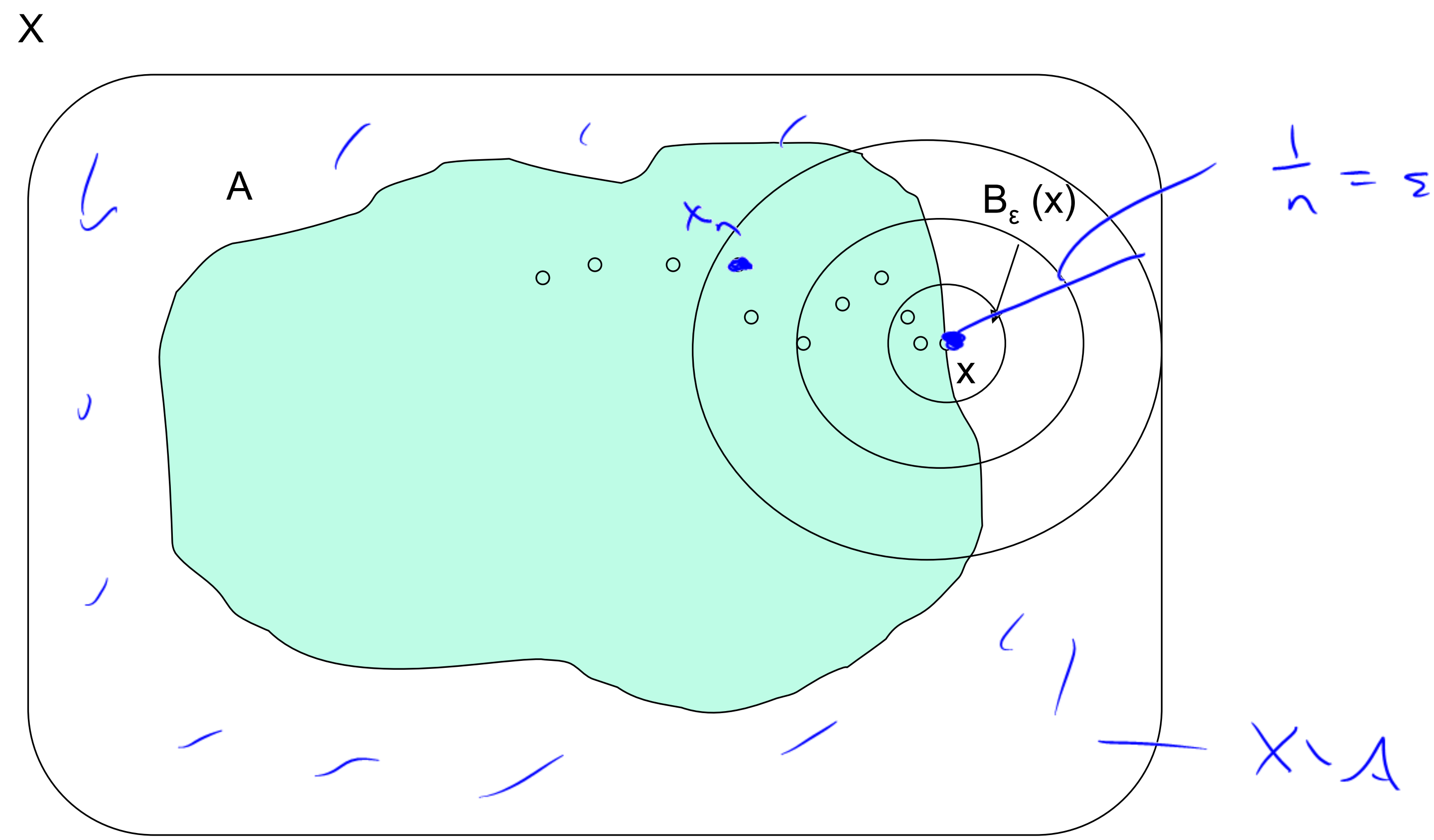
$$\{x_n\} \subset A, x_n \to x \in X \Rightarrow x \in A$$

We need to show that A is closed, i.e. $X \setminus A$ is open. Suppose not, so $X \setminus A$ is not open. Then there exists $x \in X \setminus A$ such that for every $\varepsilon > 0$,

$$B_\varepsilon(x) \not\subseteq X \setminus A$$

so there exists $y \in B_\varepsilon(x)$ such that $y \notin X \setminus A$. Then $y \in A$, hence

$$B_\varepsilon(x) \bigcap A \neq \emptyset$$



X
A
x_n
$B_\varepsilon(x)$
x
$\frac{1}{n} = \varepsilon$
$X \setminus A$

Construct a sequence $\{x_n\}$ as follows: for each n, choose

$$x_n \in B_{\frac{1}{n}}(x) \cap A$$

$\Rightarrow x_n \in B_{\frac{1}{n}}(x) \subseteq B_\varepsilon(x)$

Given $\varepsilon > 0$, we can find $N(\varepsilon)$ such that $N(\varepsilon) > \frac{1}{\varepsilon}$ by the Archimedean Property, so $n > N(\varepsilon) \Rightarrow \frac{1}{n} < \frac{1}{N(\varepsilon)} < \varepsilon$, so $x_n \to x$. Then $\{x_n\} \subseteq A$, $x_n \to x$, so $x \in A$, contradiction. Therefore, $X \setminus A$ is open, so A is closed. $\square$

Continuity in Metric Spaces

Definition 3. *Let* (X, d) *and* (Y, ρ) *be metric spaces. A function* $f : X \to Y$ *is* continuous at a point $x_0 \in X$ *if*

$$\forall \varepsilon > 0 \ \exists \delta(x_0, \varepsilon) > 0 \ \textit{s.t.} \ d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

f *is* continuous *if it is continuous at every element of its domain.*

Note that δ can depend on x_0 and ε.

Continuity in Metric Spaces

Continuity at x_0 requires:

- $f(x_0)$ is defined; and
- either
  - x_0 is an isolated point of X, i.e. $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x_0) = \{x_0\}$; or
  - $\lim_{x \to x_0} f(x)$ exists and equals $f(x_0)$

Continuity in Metric Spaces

Suppose $f : X \to Y$ and $A \subseteq Y$. Define

$$f^{-1}(A) = \{x \in X : f(x) \in A\}$$

Theorem 3 (Theorem 6.14). *Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$. Then f is continuous if and only if*

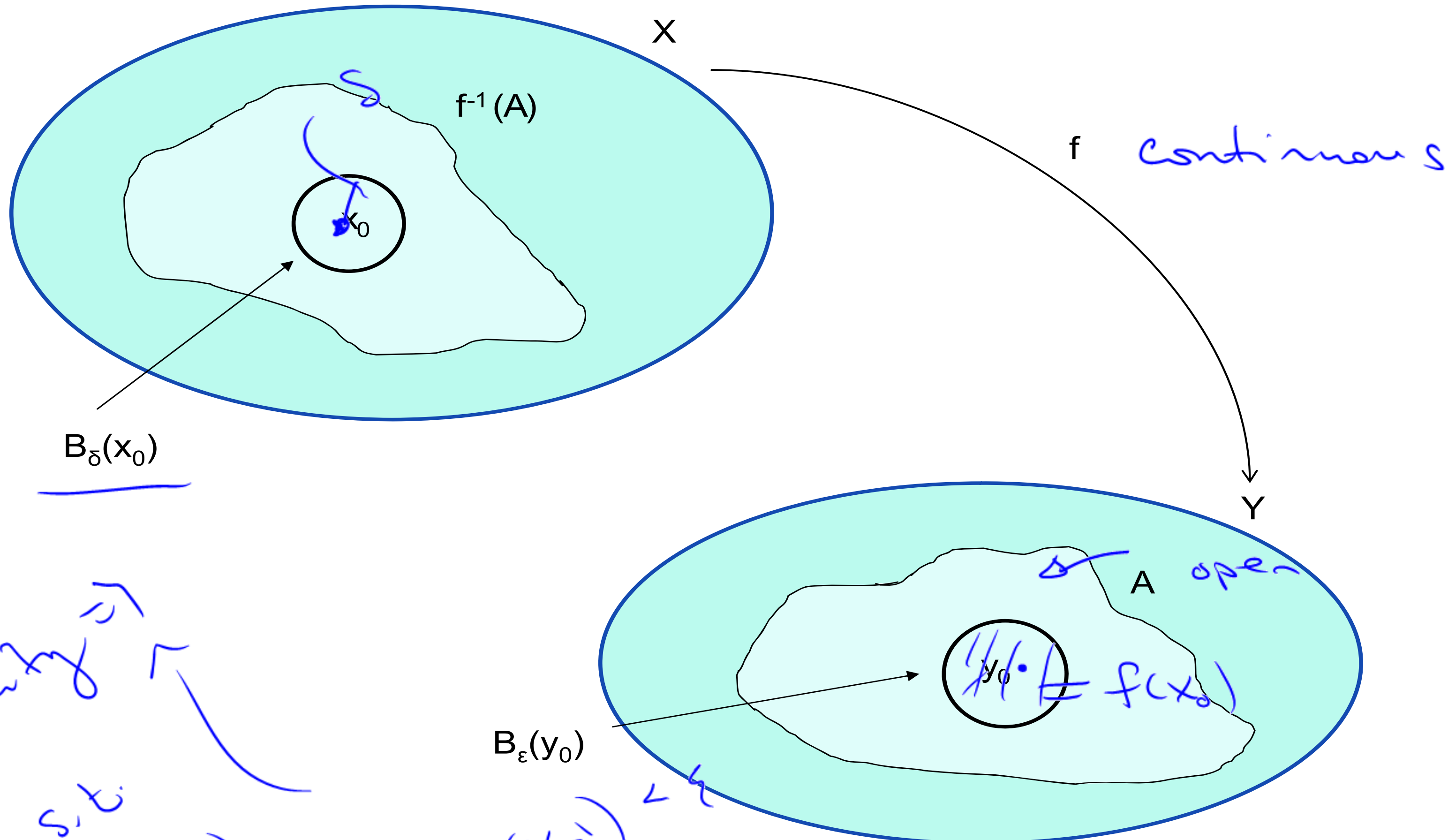
$$f^{-1}(A) \textit{ is open in } X \ \forall A \subseteq Y \textit{ s.t. } A \textit{ is open in } Y$$

Alternatively, f is continuous $\iff$ $f^{-1}(C)$ is closed in X for every closed $C \subseteq Y$.

Proof. Suppose f is continuous. Given $A \subseteq Y$, A open, we must show that $f^{-1}(A)$ is open in X. Suppose $x_0 \in f^{-1}(A)$. Let $y_0 = f(x_0) \in A$. Since A is open, we can find $\varepsilon > 0$ such that $B_\varepsilon(y_0) \subseteq A$. Since f is continuous, there exists $\delta > 0$ such that

$$
\begin{aligned}
d(x, x_0) < \delta \quad &\Rightarrow \quad \rho(f(x), f(x_0)) < \varepsilon \\
&\Rightarrow \quad f(x) \in B_\varepsilon(\overbrace{y_0}^{f(x_0)}) \subseteq A \\
&\Rightarrow \quad f(x) \in A \\
&\Rightarrow \quad x \in f^{-1}(A)
\end{aligned}
$$

so $B_\delta(x_0) \subseteq f^{-1}(A)$, so $f^{-1}(A)$ is open.



continuity ⇒ $\exists\, \delta > 0$ s.t. $x \in B_\delta(x_0) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$

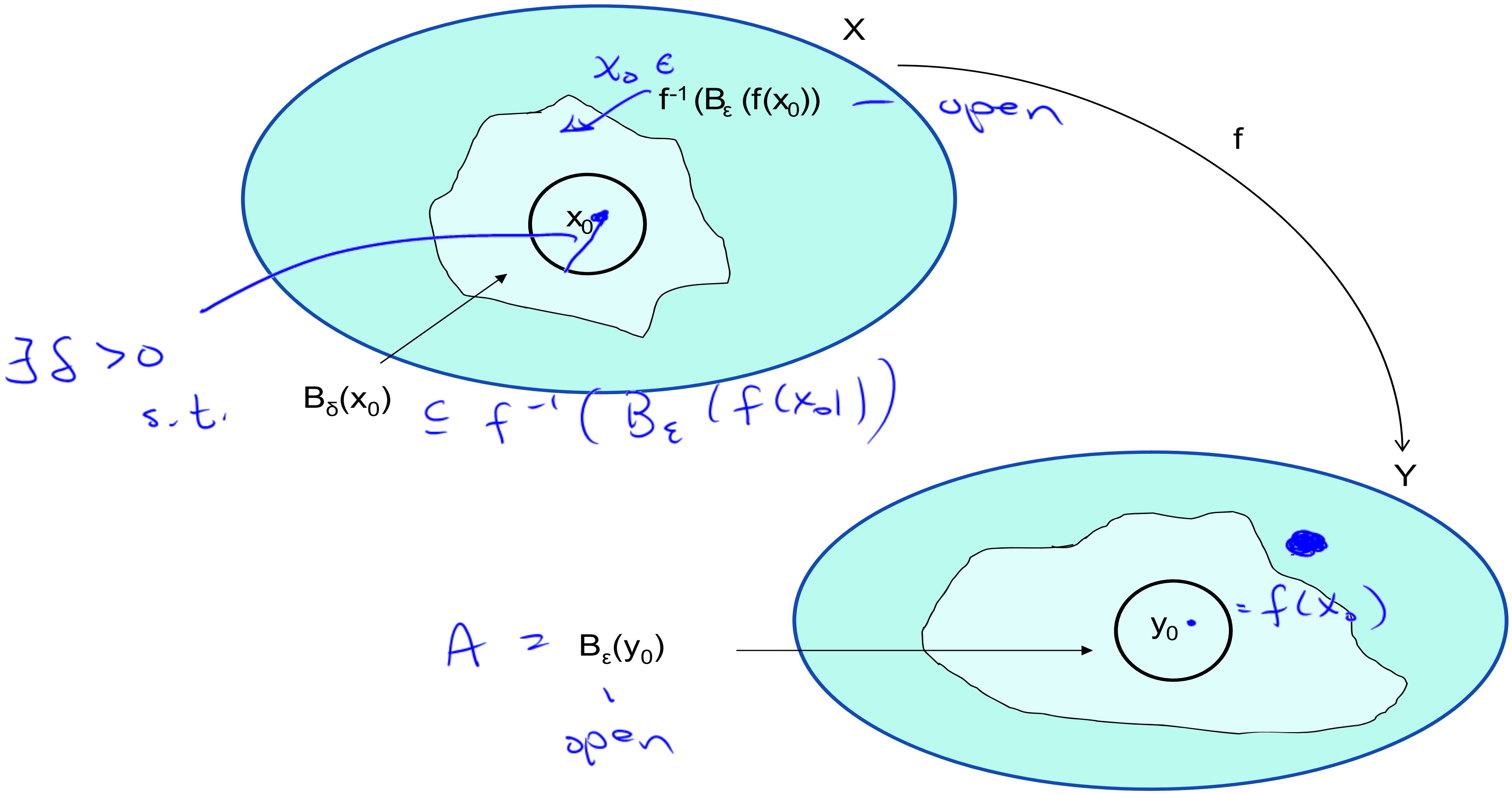
Conversely, suppose

$$f^{-1}(A) \text{ is open in } X \ \forall A \subseteq Y \text{ s.t. } A \text{ is open in } Y$$

We need to show that f is continuous. Let $x_0 \in X$, $\varepsilon > 0$. Let $A = B_\varepsilon(f(x_0))$. A is an open ball, hence an open set, so $f^{-1}(A)$ is open in X. $x_0 \in f^{-1}(A)$, so there exists $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(A)$.

open

$$\begin{aligned} d(x, x_0) < \delta \ &\Rightarrow\ x \in B_\delta(x_0) \subseteq f^{-1}(A) \\ &\Rightarrow\ x \in f^{-1}(A) \\ &\Rightarrow\ f(x) \in A(= B_\varepsilon(f(x_0))) \\ &\Rightarrow\ \rho(f(x), f(x_0)) < \varepsilon \end{aligned}$$


X
Y
f
$x_0 \in$
$f^{-1}(B_\varepsilon(f(x_0)))$ — open
x_0
$\exists \delta > 0$
s.t.
$B_\delta(x_0)$ $\subseteq f^{-1}(B_\varepsilon(f(x_0)))$
$A = B_\varepsilon(y_0)$
open
y_0
$= f(x_0)$

Thus, we have shown that f is continuous at x_0; since x_0 is an arbitrary point in X, f is continuous. $\square$

Continuity in Metric Spaces

The composition of continuous functions is continuous:

Theorem 4 (Slightly weaker version of Thm. 6.10)**.** *Let* (X, d_X), (Y, d_Y) *and* (Z, d_Z) *be metric spaces. If* $f : X \to Y$ *and* $g : Y \to Z$ *are continuous, then* $g \circ f : X \to Z$ *is continuous.*

$g(f(x))$

Proof. Suppose $A \subseteq Z$ is open. Since g is continuous, $\underline{g^{-1}(A)}$ is open in Y; since f is continuous, $f^{-1}(\underbrace{g^{-1}(A)}_{\text{open}})$ is open in X.

We claim that

$$f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$$

Observe

$$\begin{aligned} x \in f^{-1}(g^{-1}(A)) &\Leftrightarrow f(x) \in g^{-1}(A) \\ &\Leftrightarrow g(f(x)) \in A \\ &\Leftrightarrow (g \circ f)(x) \in A \\ &\Leftrightarrow x \in (g \circ f)^{-1}(A) \end{aligned}$$

which establishes the claim. This shows that $(g \circ f)^{-1}(A)$ is open in X, so $g \circ f$ is continuous. $\square$

Uniform Continuity

Definition 4 (Uniform Continuity). *Let* (X, d) *and* (Y, ρ) *be metric spaces. A function* $f : X \to Y$ *is* uniformly continuous *if*

$$\forall \varepsilon > 0 \; \exists \delta(\varepsilon) > 0 \text{ s.t. } \forall x_0 \in X, \; d(x, x_0) < \delta(\varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

Notice the important contrast with continuity: f is continuous means

$$\forall x_0 \in X, \varepsilon > 0 \; \exists \delta(x_0, \varepsilon) > 0 \text{ s.t. } d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

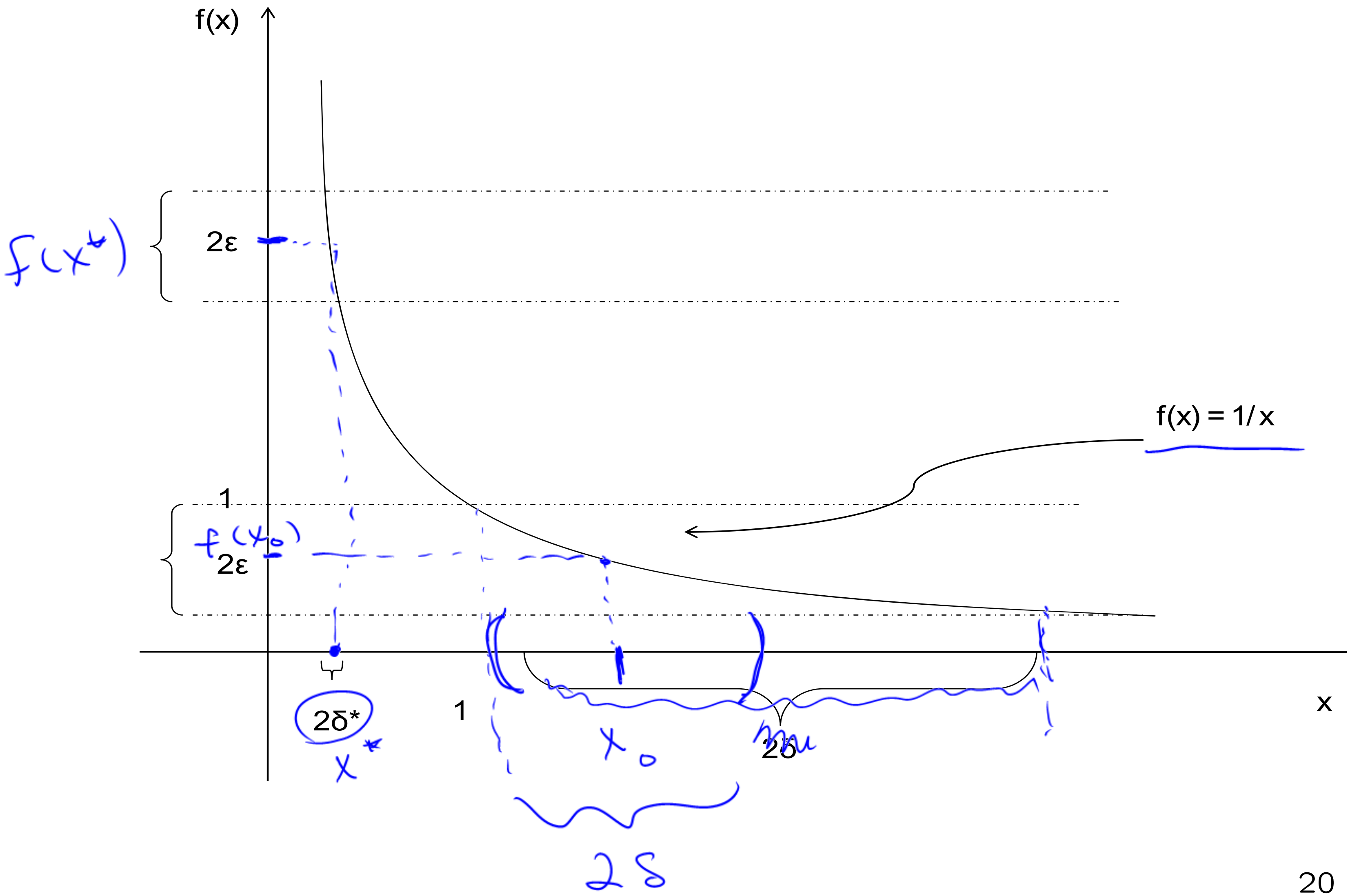
Uniform Continuity

Example: Consider $f : (0,1] \to \mathbb{R}$

$$f(x) = \frac{1}{x}, \;\; x \in (0,1]$$

f is continuous (why?). We will show that f is **not** uniformly continuous.

Fix $\varepsilon > 0$, x_0



Fix $\varepsilon > 0$ and $x_0 \in (0,1]$. If $x = \frac{x_0}{1+\varepsilon x_0}$, then

$$
\begin{aligned}
1 + \varepsilon x_0 &> 1 \\
x = \frac{x_0}{1+\varepsilon x_0} &< x_0 \\
\frac{1}{x} - \frac{1}{x_0} &> 0 \\
|f(x) - f(x_0)| &= \left|\frac{1}{x} - \frac{1}{x_0}\right| \\
&= \frac{1}{x} - \frac{1}{x_0} \\
&= \frac{1+\varepsilon x_0}{x_0} - \frac{1}{x_0} \\
&= \frac{\varepsilon x_0}{x_0} \\
&= \varepsilon
\end{aligned}
$$

An easier estimate:

Notice that $\frac{1}{x}$ is decreasing on $(0,1)$, so

$$x < x_0 \Rightarrow \frac{1}{x} - \frac{1}{x_0} > 0$$

Now look for the point $x < x_0$ such that

$$\begin{aligned}
\frac{1}{x} - \frac{1}{x_0} &= \varepsilon \\
\frac{1}{x} &= \frac{1}{x_0} + \varepsilon \\
&= \frac{1 + \varepsilon x_0}{x_0} \\
\Rightarrow \boxed{x = \frac{x_0}{1 + \varepsilon x_0}}
\end{aligned}$$

Note for $x' > 0$, $x' < x \Rightarrow f(x') - f(x_0) > \varepsilon$

Thus, $\delta(x_0, \varepsilon)$ must be chosen small enough so that

$$\left| \frac{x_0}{1 + \varepsilon x_0} - x_0 \right| \geq \delta(x_0, \varepsilon)$$

$$\begin{aligned} \delta(x_0, \varepsilon) &\leq x_0 - \frac{x_0}{1 + \varepsilon x_0} \\ &= \frac{\varepsilon (x_0)^2}{1 + \varepsilon x_0} \\ &< \underline{\varepsilon (x_0)^2} \end{aligned}$$

> 0

which converges to zero as $x_0 \to 0$. So there is no $\delta(\varepsilon)$ that will work for all $x_0 \in (0, 1]$.

$\exists\, C > 0$ s.t. $|f'(x)| \le C$ $\forall x \in [a,b]$

Uniform Continuity

Example: If $f : \mathbf{R} \to \mathbf{R}$ and $f'(x)$ is defined and ~~uniform~~ly bounded on an interval $[a, b]$, then f is uniformly continuous on $[a, b]$. However, even a function with an unbounded derivative may be uniformly continuous. Consider

$$f(x) = \sqrt{x}, \;\; x \in [0, 1]$$

f is continuous (why?). We will show that f is uniformly continuous. Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. Then given any $x_0 \in [0, 1]$,

$|x - x_0| < \delta$ implies by the Fundamental Theorem of Calculus

$$\begin{aligned} |f(x) - f(x_0)| &= \left| \int_{x_0}^{x} \frac{1}{2\sqrt{t}}\, dt \right| \\ &\leq \int_0^{|x-x_0|} \frac{1}{2\sqrt{t}}\, dt \\ &= \sqrt{|x - x_0|} \\ &< \sqrt{\delta} \\ &= \sqrt{\varepsilon^2} \\ &= \varepsilon \end{aligned}$$

Thus, f is uniformly continuous on $[0, 1]$, even though $f'(x) \to \infty$ as $x \to 0$.

Lipschitz Continuity

Definition 5. *Let X, Y be normed vector spaces, $E \subseteq X$. A function $f : X \to Y$ is* Lipschitz on *E if*

$$\exists K > 0 \text{ s.t. } \|f(x) - f(z)\|_Y \leq K\|x - z\|_X \quad \forall x, z \in E$$

f is locally Lipschitz *on E if*

$$\forall x_0 \in E \; \exists \varepsilon > 0 \text{ s.t. } f \text{ is Lipschitz on } B_\varepsilon(x_0) \cap E$$

$\exists K > 0$ s.t. $x \neq z \Rightarrow$

$$\frac{\|f(x) - f(z)\|_Y}{\|x - z\|_X} \leq K$$

Notions of Continuity

Lipschitz continuity is stronger than either continuity or uniform continuity:

$$\begin{aligned} \text{locally Lipschitz} \;&\Rightarrow\; \text{continuous} \\ \text{Lipschitz} \;&\Rightarrow\; \text{uniformly continuous} \end{aligned}$$

Every C^1 function is locally Lipschitz. (Recall that a function $f : \mathbf{R}^m \to \mathbf{R}^n$ is said to be C^1 if all its first partial derivatives exist and are continuous.)

Homeomorphisms

Definition 6. *Let* (X, d) *and* (Y, ρ) *be metric spaces. A function* $f : X \to Y$ *is called a* homeomorphism *if it is one-to-one, onto, continuous, and its inverse function is continuous.*

Topological properties are invariant under homeomorphism:

Homeomorphisms

Suppose that f is a homeomorphism and $U \subset X$. Let $g = f^{-1} : Y \to X$.

$$\begin{aligned} y \in g^{-1}(U) &\Leftrightarrow g(y) \in U \\ &\Leftrightarrow y \in f(U) \\ U \text{ open in } X &\Rightarrow g^{-1}(U) \text{ is open in } (f(X), \rho) \\ &\Rightarrow f(U) \text{ is open in } (f(X), \rho) \end{aligned}$$

This says that (X, d) and $\left(f(X), \rho|_{f(X)}\right)$ are identical in terms of properties that can be characterized solely in terms of open sets; such properties are called "topological properties."