# Econ 204 2011

Lecture 9

Outline

1. Quotient Vector Spaces
2. Matrix Representations of Linear Transformations
3. Change of Basis and Similarity
4. Eigenvalues and Eigenvectors
5. Diagonalization

# Quotient Vector Spaces

Given a vector space $X$ and a vector subspace $W$ of $X$, define an equivalence relation by

$$x \sim y \iff x - y \in W$$

Form a new vector space $X/W$: the set of vectors is

$$\{[x] : x \in X\}$$

where $[x]$ denotes the equivalence class of $x$ with respect to $\sim$.

$X/W$ is read "$X$ mod $W$".

Note that the vectors in $X/W$ are **sets** of vectors in $X$: for $x \in X$,

$$[x] = \{x + w : w \in W\}$$

# Quotient Vector Spaces

We claim that $X/W$ can be viewed as a vector space over $F$. Define the vector space operations $+, \cdot$ in $X/W$ as follows:

Define

$$\begin{aligned} [x] + [y] &= [x + y] \\ \alpha[x] &= [\alpha x] \end{aligned}$$

**Exercise:** Verify that $\sim$ is an equivalence relation and that vector addition and scalar multiplication are well-defined.

Then $X/W$ is a vector space over $F$ with these definitions for $+$ and $\cdot$

# Quotient Vector Spaces

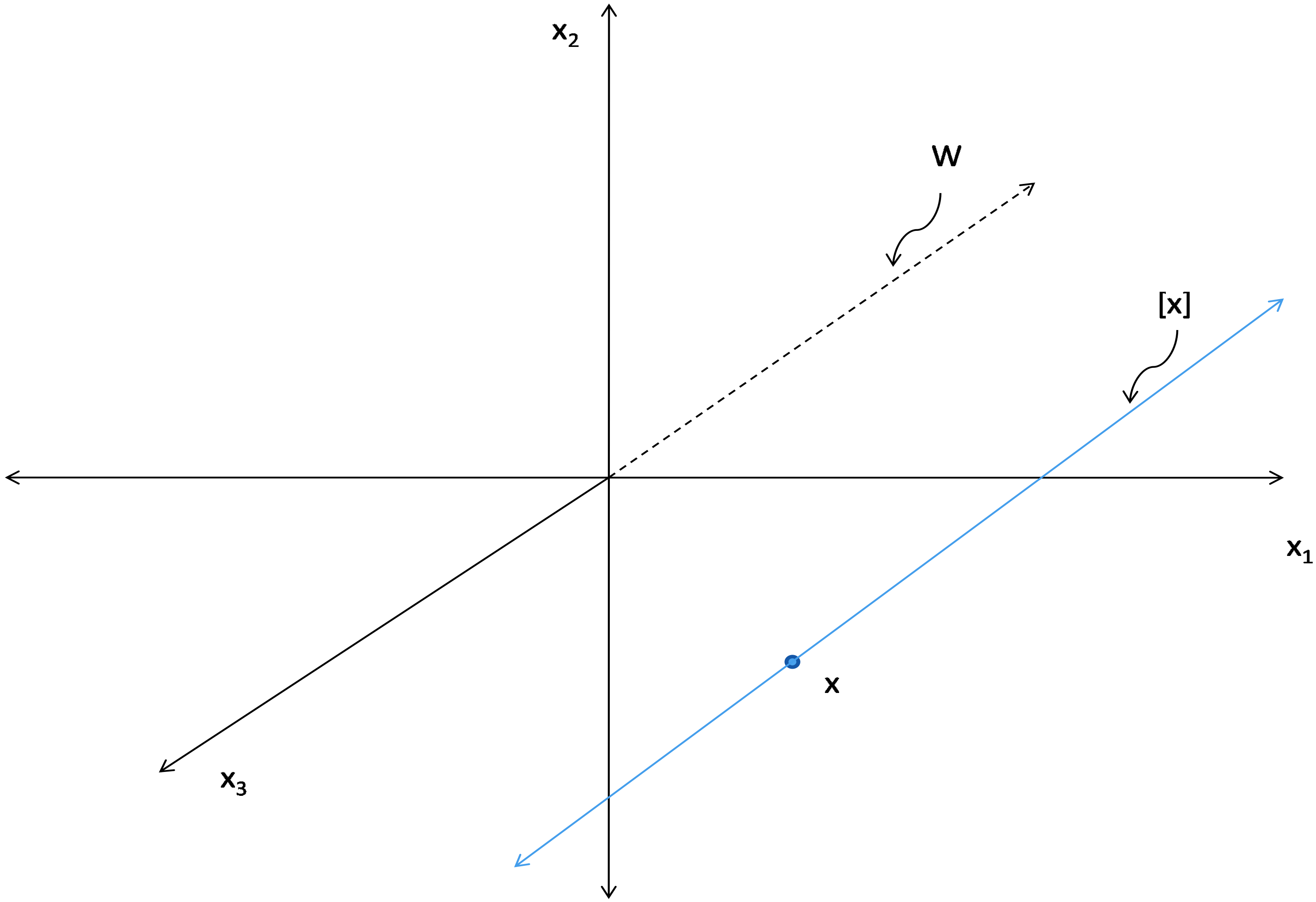
**Example:** Let $X = \mathbf{R}^3$ and let $W = \{x \in \mathbf{R}^3 : x_1 = x_2 = 0\}$. Then for $x, y \in \mathbf{R}^3$,

$$\begin{aligned} x \sim y \quad &\Longleftrightarrow \quad x - y \in W \\ &\Longleftrightarrow \quad x_1 - y_1 = 0, x_2 - y_2 = 0 \\ &\Longleftrightarrow \quad x_1 = y_1, x_2 = y_2 \end{aligned}$$

and

$$[x] = \{x + w : w \in W\} = \{(x_1, x_2, z) : z \in \mathbf{R}\}$$

So the equivalence class corresponding to $x$ is the line in $\mathbf{R}^3$ through $x$ parallel to the axis of the third coordinate.



$x_2$
W
[x]
$x_1$
x
$x_3$

**Example, cont.**

What is $X/W$? Intuitively this equivalence relation ignores the third coordinate, and we can identify the equivalence class $[x]$ with the vector $(x_1, x_2) \in \mathbf{R}^2$.

The next two results show how to formalize this connection.

# Quotient Vector Spaces

**Theorem 1.** *If $X$ is a vector space with* $\dim X = n$ *for some* $n \in \mathbf{N}$ *and $W$ is a vector subspace of $X$, then*

$$\dim(X/W) = \dim X - \dim W$$

*Proof.* (Sketch) Begin with a basis $\{w_1, \ldots, w_c\}$ for $W$, and a basis $\{[x_1], \ldots, [x_k]\}$ for $X/W$. Show that

$$\{w_1, \ldots, w_c\} \cup \{x_1, \ldots, x_k\}$$

is a basis for $X$.

Here is the proof we did in class, showing that $\{w_1, \ldots, w_c\} \cup \{x_1, \ldots, x_k\}$ is a linearly independent set in $X$. First, notice that $W = [0]$, that is, $x \sim 0 \iff x - 0 = x \in W$ by definition. Thus $[x] = [0] \iff x \in W$. Then suppose there exist $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_c \in F$ such that

$$\sum_{i=1}^{c} \beta_i w_i + \sum_{j=1}^{k} \alpha_j x_j = 0$$

Then

$$
\begin{aligned}
\sum_{i=1}^{c} \beta_i w_i + \sum_{j=1}^{k} \alpha_j x_j &= 0 \\
\Rightarrow \sum_{j=1}^{k} \alpha_j x_j &= -\sum_{i=1}^{c} \beta_i w_i \in W \\
\Rightarrow [\sum_{j=1}^{k} \alpha_j x_j] &= [0] \\
\Rightarrow [\sum_{j=1}^{k} \alpha_j x_j] = \sum_{j=1}^{k} \alpha_j [x_j] &= [0]
\end{aligned}
$$

Thus $\alpha_j = 0 \; \forall j$ since $\{[x_1], \ldots, [x_k]\}$ is linearly independent. Then

$$
\sum_{i=1}^{c} \beta_i w_i + \sum_{j=1}^{k} \alpha_j x_j = \sum_{i=1}^{c} \beta_i w_i = 0
$$

so $\beta_i = 0 \; \forall i$, since $\{w_1, \ldots, w_c\}$ is linearly independent. From this we conclude that $\{w_1, \ldots, w_c\} \cup \{x_1, \ldots, x_k\}$ is a linearly independent set in $X$.

To see that this set spans $X$, let $x \in X$ and consider $[x]$. Since $\{[x_1], \ldots, [x_k]\}$ spans $X/W$, there exist $\alpha_1, \ldots, \alpha_k$ such that

$$\begin{aligned} [x] &= \sum_{j=1}^{k} \alpha_j [x_j] \\ &= [\sum_{j=1}^{k} \alpha_j x_j] \end{aligned}$$

Thus $x \sim \sum_{j=1}^{k} \alpha_j x_j$, so $x - \sum_{j=1}^{k} \alpha_j x_j \in W$. Since $\{w_1, \ldots, w_c\}$

spans $W$, there exist $\beta_1, \ldots, \beta_c$ such that

$$\begin{aligned} x - \sum_{j=1}^{k} \alpha_j x_j &= \sum_{i=1}^{c} \beta_i w_i \\ \Rightarrow x &= \sum_{j=1}^{k} \alpha_j x_j + \sum_{i=1}^{c} \beta_i w_i \end{aligned}$$

Thus $\{w_1, \ldots, w_c\} \cup \{x_1, \ldots, x_k\}$ spans $X$ $\square$

# Quotient Vector Spaces

**Theorem 2.** *Let $X$ and $Y$ be vector spaces over the same field $F$ and $T \in L(X, Y)$. Then* $\operatorname{Im} T$ *is isomorphic to* $X/\ker T$.

*Proof.* Notice that if $X$ is finite-dimensional, then

$$\begin{aligned}
\dim(X/\ker T) &= \dim X - \dim\ker T \quad \text{(by the previous theorem)} \\
&= \operatorname{Rank} T \quad \text{(by the Rank-Nullity Theorem)} \\
&= \dim\operatorname{Im} T
\end{aligned}$$

so $X/\ker T$ is isomorphic to $\operatorname{Im} T$. (why??)

We prove that this is true in general, and that the isomorphism is natural.

Define $\tilde{T} : X/\ker T \to \operatorname{Im} T$ by

$$\tilde{T}([x]) = T(x)$$

We first need to check that this is well-defined, that is, that if $[x] = [x']$ then $\tilde{T}([x]) = \tilde{T}([x'])$.

$$\begin{aligned} [x] = [x'] &\Rightarrow x \sim x' \\ &\Rightarrow x - x' \in \ker T \\ &\Rightarrow T(x - x') = 0 \\ &\Rightarrow T(x) = T(x') \end{aligned}$$

so $\tilde{T}$ is well-defined.

Clearly, $\tilde{T} : X/\ker T \to \operatorname{Im} T$. It is easy to check that $\tilde{T}$ is linear, so $\tilde{T} \in L(X/\ker T, \operatorname{Im} T)$. Next we show that $\tilde{T}$ is an isomorphism.

$$\begin{aligned}\tilde{T}([x]) = \tilde{T}([y]) &\Rightarrow T(x) = T(y)\\ &\Rightarrow T(x-y) = 0\\ &\Rightarrow x - y \in \ker T\\ &\Rightarrow x \sim y\\ &\Rightarrow [x] = [y]\end{aligned}$$

so $\tilde{T}$ is one-to-one.

$$\begin{aligned}y \in \operatorname{Im} T &\Rightarrow \exists x \in X \text{ s.t. } T(x) = y\\ &\Rightarrow \tilde{T}([x]) = y\end{aligned}$$

so $\tilde{T}$ is onto, hence $\tilde{T}$ is an isomorphism. $\square$

**Example:** Consider $T \in L(\mathbf{R}^3, \mathbf{R}^2)$ defined by

$$T(x_1, x_2, x_3) = (x_1, x_2)$$

Then

$$\ker T = \{x \in \mathbf{R}^3 : x_1 = x_2 = 0\}$$

is the $x_3$-axis.

Given $x$, the equivalence class $[x]$ is just the line through $x$ parallel to the $x_3$-axis.

$$\tilde{T}([x]) = T(x_1, x_2, x_3) = (x_1, x_2)$$

and

$$\operatorname{Im} T = \mathbf{R}^2, \quad X/\ker T \cong \mathbf{R}^2 = \operatorname{Im} T$$

as we suggested intuitively above (here the symbol $\cong$ denotes isomorphism, that is, we write $Y \cong Z$ if $Y$ and $Z$ are isomorphic.)

# Coordinate Representations

Every real vector space $X$ with dimension $n$ is isomorphic to $\mathbf{R}^n$. What's the isomorphism?

Let $X$ be a finite-dimensional vector space over $\mathbf{R}$ with $\dim X = n$. Fix any Hamel basis $V = \{v_1, \ldots, v_n\}$ of $X$. Any $x \in X$ has a unique representation

$$x = \sum_{j=1}^{n} \beta_j v_j$$

(here, we allow $\beta_j = 0$).

$$crd_V(x) = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in \mathbf{R}^n$$

$crd_V(x)$ is the vector of coordinates of $x$ with respect to the basis $V$.

$$crd_V(v_1) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad crd_V(v_2) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad crd_V(v_n) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$crd_V$ is an isomorphism from $X$ to $\mathbf{R}^n$

# Matrix Representations of Linear Transformations

Suppose $T \in L(X, Y)$, $\dim X = n$, $\dim Y = m$. Fix bases

$$\begin{aligned} V &= \{v_1, \ldots, v_n\} \text{ of } X \\ W &= \{w_1, \ldots, w_m\} \text{ of } Y \end{aligned}$$

$T(v_j) \in Y$, so

$$T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i$$

Define

$$Mtx_{W,V}(T) = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}$$

# Matrix Representations of Linear Transformations

Notice that the columns are the coordinates (expressed with respect to $W$) of $T(v_1), \ldots, T(v_n)$.

Observe

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{m1} \end{pmatrix}$$

so

$$\begin{aligned} Mtx_{W,V}(T) \cdot crd_V(v_j) &= crd_W(T(v_j)) \\ Mtx_{W,V}(T) \cdot crd_V(x) &= crd_W(T(x)) \quad \forall x \in X \end{aligned}$$

# Matrix Representations

Multiplying a vector by a matrix does two things:

- Computes the action of $T$
- Accounts for the change in basis

**Example:** $X = Y = \mathrm{R}^2$, $V = \{(1,0),(0,1)\}$, $W = \{(1,1),(-1,1)\}$, $T = id$, that is, $T(x) = x$ for each $x$.

$$Mtx_{W,V}(T) \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$Mtx_{W,V}(T)$ is the matrix that *changes basis* from $V$ to $W$.

How do we compute it?

$$
\begin{aligned}
v_1 = (1,0) &= \alpha_{11}(1,1) + \alpha_{21}(-1,1) \\
\alpha_{11} - \alpha_{21} &= 1 \\
\alpha_{11} + \alpha_{21} &= 0 \\
2\alpha_{11} &= 1, \alpha_{11} = \frac{1}{2} \\
\alpha_{21} &= -\frac{1}{2} \\
v_2 = (0,1) &= \alpha_{12}(1,1) + \alpha_{22}(-1,1) \\
\alpha_{12} - \alpha_{22} &= 0 \\
\alpha_{12} + \alpha_{22} &= 1 \\
2\alpha_{12} &= 1, \alpha_{12} = \frac{1}{2} \\
\alpha_{22} &= \frac{1}{2}
\end{aligned}
$$

So

$$Mtx_{W,V}(id) = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

# Matrix Representations

**Theorem 3** (Thm. 3.5'). *Let $X$ and $Y$ be vector spaces over the same field $F$, with* $\dim X = n$, $\dim Y = m$. *Then $L(X, Y)$, the space of linear transformations from $X$ to $Y$, is isomorphic to $F_{m \times n}$, the vector space of $m \times n$ matrices over $F$. If $V = \{v_1, \ldots, v_n\}$ is a basis for $X$ and $W = \{w_1, \ldots, w_m\}$ is a basis for $Y$, then*

$$Mtx_{W,V} \in L(L(X, Y), F_{m \times n})$$

*and $Mtx_{W,V}$ is an isomorphism from $L(X, Y)$ to $F_{m \times n}$.*

# Matrix Representations

**Theorem 4** (From Handout). *Let $X, Y, Z$ be finite-dimensional vector spaces over the same field $F$ with bases $U, V, W$ respectively. Let $S \in L(X, Y)$ and $T \in L(Y, Z)$. Then*

$$Mtx_{W,V}(T) \cdot Mtx_{V,U}(S) = Mtx_{W,U}(T \circ S)$$

*i.e. matrix multiplication corresponds via the matrix representation isomorphism to composition of linear transformations.*

*Proof.* See handout. $\square$

Note that $Mtx_{W,V}$ is a function from $L(X, Y)$ to the space $F_{m \times n}$ of $m \times n$ matrices, while $Mtx_{W,V}(T)$ is an $m \times n$ matrix.

# Matrix Representations

The theorem can be summarized by the following "Commutative Diagram:"

$$
\begin{tikzcd}
X \arrow[r, "S"] \arrow[d, leftrightarrow, "crd_U"'] & Y \arrow[r, "T"] \arrow[d, leftrightarrow, "crd_V"] & Z \arrow[d, leftrightarrow, "crd_W"] \\
\mathbf{R}^n \arrow[r, "Mtx_{V,U}(S)"'] & \mathbf{R}^m \arrow[r, "Mtx_{W,V}(T)"'] & \mathbf{R}^r
\end{tikzcd}
$$

We say the diagram commutes because you get the same answer any way you go around the diagram (in directions allowed by the arrows). The $crd$ arrows go in both directions because $crd$ is an isomorphism.

# Change of Basis

Let $X$ be a finite-dimensional vector space with basis $V$. If $T \in L(X,X)$ it is customary to use the same basis in the domain and range. In this case, $Mtx_V(T)$ denotes $Mtx_{V,V}(T)$.

**Question:** If $W$ is another basis for $X$, how are $Mtx_V(T)$ and $Mtx_W(T)$ related?

$$
\begin{aligned}
Mtx_{V,W}(id) \cdot Mtx_W(T) \cdot Mtx_{W,V}(id) &= Mtx_{V,W}(id) \cdot Mtx_{W,V}(T \circ id) \\
&= Mtx_{V,V}(id \circ T \circ id) \\
&= Mtx_V(T)
\end{aligned}
$$

and

$$
\begin{aligned}
Mtx_{V,W}(id) \cdot Mtx_{W,V}(id) &= Mtx_{V,V}(id) \\
&= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}
\end{aligned}
$$

So this says that

$$Mtx_V(T) = P^{-1}Mtx_W(T)P$$

for the invertible matrix

$$P = Mtx_{W,V}(id)$$

that is the change of basis matrix.

On the other hand, if $P$ is any invertible matrix, then $P$ is also a change of basis matrix for appropriate corresponding bases (see handout).

# Similarity

**Definition 1.** *Square matrices $A$ and $B$ are* similar *if*

$$A = P^{-1}BP$$

*for some invertible matrix $P$.*

# Similarity

**Theorem 5.** *Suppose that $X$ is a finite-dimensional vector space.*

1. *If $T \in L(X, X)$ then any two matrix representations of $T$ are similar. That is, if $U, W$ are any two bases of $X$, then $Mtx_W(T)$ and $Mtx_U(T)$ are similar.*

2. *Conversely, two similar matrices represent the same linear transformation $T$, relative to suitable bases. That is, given similar matrices $A, B$ with $A = P^{-1}BP$ and any basis $U$, there is a basis $W$ and $T \in L(X, X)$ such that*

$$\begin{aligned} B &= Mtx_U(T) \\ A &= Mtx_W(T) \\ P &= Mtx_{U,W}(id) \\ P^{-1} &= Mtx_{W,U}(id) \end{aligned}$$

*Proof.* See Handout on Diagonalization and Quadratic Forms. ∎

# Eigenvalues and Eigenvectors

Here, we define eigenvalues and eigenvectors of a linear transformation and show that $\lambda$ is an eigenvalue of $T$ if and only if $\lambda$ is an eigenvalue for some matrix representation of $T$ if and only if $\lambda$ is an eigenvalue for every matrix representation of $T$.

**Definition 2.** *Let $X$ be a vector space and $T \in L(X, X)$. We say that $\lambda$ is an* eigenvalue *of $T$ and $v \neq 0$ is an* eigenvector corresponding to $\lambda$ *if $T(v) = \lambda v$.*

# Eigenvalues and Eigenvectors

**Theorem 6** (Theorem 4 in Handout)**.** *Let $X$ be a finite-dimensional vector space, and $U$ a basis. Then $\lambda$ is an eigenvalue of $T$ if and only if $\lambda$ is an eigenvalue of $Mtx_U(T)$. $v$ is an eigenvector of $T$ corresponding to $\lambda$ if and only if $crd_U(v)$ is an eigenvector of $Mtx_U(T)$ corresponding to $\lambda$.*

*Proof.* By the Commutative Diagram Theorem,

$$\begin{aligned} T(v) = \lambda v &\Leftrightarrow crd_U(T(v)) = crd_U(\lambda v) \\ &\Leftrightarrow Mtx_U(T)(crd_U(v)) = \lambda(crd_U(v)) \end{aligned}$$

$\square$

# Computing Eigenvalues and Eigenvectors

Suppose $\dim X = n$; let $I$ be the $n \times n$ identity matrix. Given $T \in L(X, X)$, fix a basis $U$ and let

$$A = Mtx_U(T)$$

Find the eigenvalues of $T$ by computing the eigenvalues of $A$:

$$\begin{aligned} Av = \lambda v \quad &\Longleftrightarrow \quad (A - \lambda I)v = 0 \\ &\Longleftrightarrow \quad (A - \lambda I) \text{ is not invertible} \\ &\Longleftrightarrow \quad \det(A - \lambda I) = 0 \end{aligned}$$

We have the following facts:

- If $A \in \mathbf{R}_{n \times n}$,

$$f(\lambda) = \det(A - \lambda I)$$

is an $n^{th}$ degree polynomial in $\lambda$ with real coefficients; it is called the *characteristic polynomial* of $A$.

- $f$ has $n$ roots in $\mathbf{C}$, counting multiplicity:

$$f(\lambda) = (\lambda - c_1)(\lambda - c_2) \cdots (\lambda - c_n)$$

where $c_1, \ldots, c_n \in \mathbf{C}$ are the eigenvalues; the $c_j$'s are not necessarily distinct. Notice that $f(\lambda) = 0$ if and only if $\lambda \in \{c_1, \ldots, c_n\}$, so the roots are the solutions of the equation $f(\lambda) = 0$.

- the roots that are not real come in conjugate pairs:

$$f(a + bi) = 0 \Leftrightarrow f(a - bi) = 0$$

- if $\lambda = c_j \in \mathbf{R}$, there is a corresponding eigenvector in $\mathbf{R}^n$.

- if $\lambda = c_j \notin \mathbf{R}$, the corresponding eigenvectors are in $\mathbf{C}^n \setminus \mathbf{R}^n$.

# Diagonalization

**Definition 3.** *Suppose $X$ is a finite-dimensional vector space with basis $U$. Given a linear transformation $T \in L(X,X)$, let*

$$A = Mtx_U(T)$$

*We say that $A$* can be diagonalized *if there is a basis $W$ for $X$ such that $Mtx_W(T)$ is a diagonal matrix, that is,*

$$Mtx_W(T) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

So

$$\begin{aligned} A \text{ can be diagonalized} \quad & \iff \quad A \text{ is similar to a diagonal matrix} \\ & \iff \quad A = P^{-1}BP \text{ where } B \text{ is diagonal} \end{aligned}$$

Suppose there is a basis $W$ such that

$$Mtx_W(T) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

Then the standard basis vectors of $\mathbf{R}^n$ are eigenvectors of $Mtx_W(T)$.

$z_j$ is an eigenvector of $T$ corresponding to $\lambda_j \iff crd_W(z_j)$ is an eigenvector of $Mtx_W(T)$ corresponding to $\lambda_j$.

So an eigenvector corresponding to $\lambda_j$ is $w_j$, since $crd_W(w_j) = e_j$, the $j^{th}$ standard basis vector in $\mathbf{R}^n$.

Thus $Mtx_W(T)$ is diagonal if and only if $W = \{w_1, \ldots, w_n\}$ where $w_j$ is an eigenvector of $T$ corresponding to $\lambda_j$ for each $j$.

Then the action of $T$ is clear: it stretches each basis element $w_i$ by the factor $\lambda_i$.

# Diagonalization

**Theorem 7** (Thm. 6.7'). *Let $X$ be an $n$-dimensional vector space, $T \in L(X, X)$, $U$ any basis of $X$, and $A = Mtx_U(T)$. Then the following are equivalent:*

1. *$A$ can be diagonalized*

2. *there is a basis $W$ for $X$ consisting of eigenvectors of $T$*

3. *there is a basis $V$ for $\mathbf{R}^n$ consisting of eigenvectors of $A$*

*Proof.* Follows from Theorem 6.7 in de la Fuente and Theorem 4 from the Handout. $\square$

# Diagonalization

**Theorem 8** (Thm. 6.8'). *Let $X$ be a vector space and $T \in L(X, X)$.*

1. *If $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of $T$ with corresponding eigenvectors $v_1, \ldots, v_m$, then $\{v_1, \ldots, v_m\}$ is linearly independent.*

2. *If $\dim X = n$ and $T$ has $n$ distinct eigenvalues, then $X$ has a basis consisting of eigenvectors of $T$; consequently, if $U$ is any basis of $X$, then $Mtx_U(T)$ is diagonalizable.*

*Proof.* This is an adaptation of the proof of Theorem 6.8 in de la Fuente. $\square$