Economics 204
Fall 2011
Problem Set 1 Suggested Solutions

1. Suppose k is a positive integer. Use induction to prove the following two statements.

   (a) For all $n \in \mathbb{N}_0$, the inequality $(k^2+n)! \geq k^{2n}$ holds.

   (b) For all natural $n \geq 2k^2$, the inequality $n! \geq k^n$ holds.

   (Recall that the factorial of a non-negative integer n is defined by $n! = \prod_{m=1}^{n} m$ with the convention $0! = 1$.)

   Solution:

   (a) **Base step $n = 0$:** The RHS is $k^0 = 1$, while the LHS equals $(k^2)!$, which is clearly greater or equal to 1 since k is a positive integer.
   Induction step: Assume $(k^2+n)! \geq k^{2n}$ holds for some $n \in \mathbb{N}$. Now consider the corresponding inequality for $n+1$. Starting from the LHS, we have:

   $$\begin{aligned}(k^2+n+1)! &= (k^2+n)!(k^2+n+1)\\ &\geq k^{2n}(k^2+n+1)\\ &= k^{2n+2}+k^{2n}(n+1)\\ &\geq k^{2(n+1)},\end{aligned}$$

   where the first inequality follows from the inductions assumption. So by mathematical induction, $(k^2+n)! \geq k^{2n}$ for all $n \in \mathbb{N}_0$.

   (b) **Base step $n = 2k^2$:** The LHS can now be expressed as:

   $$(2k^2)! = (k^2+k^2)!,$$

   while the RHS equals k^{2k^2}. The inequality $(2k^2)! \geq k^{2k^2}$ then follows from part (a) of this problem for $n = k^2$.

Induction step: Assume $n! \geq k^n$ holds for some $n \geq 2k^2$. Now for $n+1$, we have:

$$\begin{aligned}(n+1)! &= n!(n+1) \\ &\geq k^n(n+1) \\ &> k^n(2k^2) \\ &> k^{n+1},\end{aligned}$$

where the first inequality follows from the induction assumption and the second one from the fact that $n+1 > n \geq 2k^2$. So by mathematical induction, $n! \geq k^n$ for all $n \geq 2k^2$.

2. Let n be a positive integer, and suppose that n chords are drawn in a circle, cutting the circle into a number of regions. Prove that the regions can be colored with two colors in such a way that adjacent regions (that is, regions that share an edge) are different colors.

   Solution: We will prove this using mathematical induction.

   The **base step n=1** is trivial.

   Induction step: Assume that the hypothesis holds for all circles with n chords. Now consider a circle with $n+1$ chords. Number these chords from 1 to $n+1$. Now we color the circle in the following manner:

   - Take the first n chords (those numbered 1 to n) and assume that they divide the circle into K regions. Denote them $A_1, A_2, \ldots, A_K$ and call their collection A:

     $$A = \{A_i : i \in \mathbb{N}, i \leq K\}$$

     Color the regions in A so that they satisfy the desired property (this coloring is possible by the induction assumption).

   - Now consider the regions defined by all $n+1$ chords, preserving the coloring from the previous step. Similarly to above, let the $n+1$ chords divide the circle into $K' \geq K$ regions. Call the collections of these regions B:

     $$B = \{B_i : i \in \mathbb{N}, i \leq K'\}.$$

Notice that for each $B_i \in B$ we have $B_i \subseteq A_j$ for some j. Unless the $n+1$-st chord coincides with one of the other n chords, the coloring now does not satisfy the desired property since adjacent regions in B whose border lies on the $n+1$-st chord are part of the same region in A and thus have the same color.

- On its own (i.e. without any of the other chords), the $n+1$-st chord partitions the circle into two parts. Similarly, it also partitions B into two parts: call them U and V.
- Flip all the colors for all regions in U: if the two colors are fuchsia and lime for example, change the color of all fuchsia regions to lime and vice versa. Keep the colors of all regions in V unchanged.

We now need to show that this coloring satisfies the desired properties. To do that we just need to consider any pair of adjacent regions in B: without loss of generality, call them B_1 and B_2. There are two cases to consider.

- B_1 and B_2's common edge is not the $n+1$-st chord: in this case, there are $A_i, A_j \in A$ such that $B_1 = A_i$ and $B_2 = A_j$. Furthermore, either $\{B_1, B_2\} \subseteq V$ or $\{B_1, B_2\} \subseteq U$. Since the initial coloring satisfied the desired property for the regions in A, the two regions ($B_1 = A_i$ and $B_2 = A_j$) must be colored differently (even if they are in U and had their colors reversed).
- B_1 and B_2's common edge *is* the $n+1$-st chord: in this case, there exists $A_i \in A$ such that B_1 and B_2 partition A_i. This means that with the original coloring, B_1 and B_2 had the same color. Furthermore, we also know that exactly one from B_1 and B_2 is a member of U, while the other one is a member of V. Therefore, exactly one from B_1 and B_2 had its color changed. Thus the two regions are now colored differently.

Please see Figure 1 for an illustration of the construction of the coloring in the induction step for the case $n = 3$. The first panel shows a coloring for the regions defined by three of the four chords. The second panel adds the fourth chord; in the third panel, the colors in the regions "above" that chord are reversed, creating a coloring that satisfies the requirements.

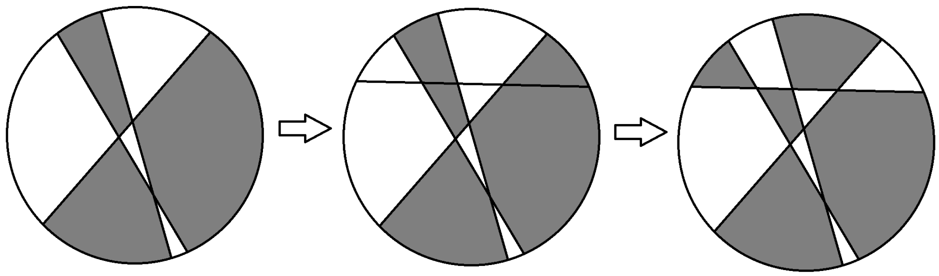

Figure 1: Coloring Construction in the Induction Step

3. Define the relation $\sim$ on the space of all sets in the following manner: $A \sim B$ iff there exists a bijection $f : A \to B$. Show that $\sim$ is an equivalence relation. (Notice that this is just the definition of numerical equivalence. The problem is asking you to prove that numerical equivalence is indeed an equivalence relation, as the name suggests.)

   Solution: We need to show that the $\sim$ relation is reflexive, symmetric, and transitive. Let X, Y, and Z in the following denote sets.

   - Reflexive: Consider the identity function $f : X \to X$ defined by $f(x) = x$. It is obviously both 1-1 and onto. Hence $X \sim X$.
   - Symmetric: Let $X \sim Y$. Then there exists an 1-1 and onto function $f : X \to Y$. Now consider its inverse $f^{-1} : Y \to 2^X$, defined as usual by $f^{-1}(y) = \{x : f(x) = y\}$. This is a well-defined function from Y to X since f is 1-1 (so $f^{-1}(y)$ is at most a singleton set for all $y \in Y$) and onto (so f^{-1} is non-empty for all $y \in Y$).

     Furthermore, it is 1-1 and onto. If it weren't 1-1, we would have $y \neq y'$, such that $f^{-1}(y) = f^{-1}(y')$, which implies two different values for some $f(x)$, which is impossible. The inverse function is onto since f's domain is all of X. Since f^{-1} is a bijection from Y to X, the two sets are numerically equivalent and $Y \sim X$.
   - Transitive: Let $X \sim Y$ and $Y \sim Z$ with the corresponding bijections $f : X \to Y$ and $g : Y \to Z$. Consider the composition of f and g, $g \circ f : X \to Z$, defined by $(g \circ f)(x) = g(f(x))$. Since f and g are both 1-1, we have:

     $$x \neq x' \Rightarrow f(x) \neq f(x') \Rightarrow g(f(x)) \neq g(f(x')).$$

Therefore, $g \circ f$ is 1-1. Further, since f and g are both onto (and hence $f(X) = Y$ and $f(Y) = Z$), we have:

$$(g \circ f)(X) = g(f(X)) = g(Y) = Z.$$

So $g \circ f$ is also onto. Hence $g \circ f$ is a bijection and $X \sim Z$.

4. Prove that the countable union of countable sets is countable.

Solution:

Since we have countably many sets, we can enumerate them using the space of integers $\mathbb{Z}$ as an index. In other words, let the countable collection of sets be: $\{A_i\}_{i \in \mathbb{Z}}$. Similarly, we can enumerate the elements in each of the sets A_i: $A_i = \{a^i_j\}_{j \in \mathbb{Z} \setminus \{0\}}$ (notice that now we are using the integers without zero as the index set for the elements of A_i, which is permissible since $\mathbb{Z} \setminus \{0\}$ is also countable). Using this notation, the union is:

$$U = \bigcup \{A_i\}_{i \in \mathbb{Z}} = \{a^i_j\}_{i,j \in \mathbb{Z}, j \neq 0}.$$

But U then clearly has the same cardinality as the set

$$\mathbb{Q} = \{\frac{i}{j} : i, j \in \mathbb{Z}, j \neq 0\},$$

which is the set of all rationals and is countable.

You cannot use induction to prove this statement since the principle of induction can be used to prove only statements of the form "all natural numbers larger than or equal to n_0 have the property P". It is relatively easy to come up with examples of properties that hold for any collection of finitely many sets but fail for countably many sets: for example, think about the intersection of open sets and whether it is open.

Aside: This proof used the fact that $\mathbb{Q}$ is countable. A direct (but more ungainly) proof could enumerate the sets A_i and the elements a^i_j in each set using $\mathbb{N}$ (i.e. $U = \{a^i_j\}_{i,j \in \mathbb{N}}$) and then we can define $f : \mathbb{N} \to U$ by $f(n) = \{a^i_j : i + j = m_n + 2; i = n - \frac{(m_n+1)m_n}{2}\}$, where $m_n = \max\{m \in \mathbb{N}_0 : \frac{(m+1)m}{2} < n\}$. Then one can verify that $f(n)$ is always a singleton and hence f is a function, and that f is a bijection.

This function simply adapts and formalizes the "picture" argument presented for the countability of $\mathbb{Q}$.

Bigger aside: This problem originally also asked you to prove that the uncountable union of countable sets is uncountable, which is incorrect without an additional assumption. To see that, we can just take uncountably many copies of $\mathbb{N}$; naturally, their union ($\mathbb{N}$) is countable. Apologies for the confusion!

An additional sufficient condition is that the sets are pairwise disjoint. Then, similarly to the previous proof, we can let:

$$U = \{a_j^i\}_{i \in I; j \in \mathbb{N}},$$

where I is some uncountable set. Importantly, due to the additional assumption, we have: $a_j^i \neq a_{j'}^{i'}$ whenever $(i, j) \neq (i', j')$.

Let also:

$$V = \{a_1^i\}_{i \in I}.$$

It is clear that V and I have the same cardinality (i.e. V has uncountably many distinct elements) but also $V \subset U$. Thus U is a superset of an uncountable set and is therefore itself uncountable.

Notice that the sets being pairwise unequal does not suffice to guarantee that the union will be uncountable. To see that, let $\mathcal{P}^\infty(\mathbb{N})$ be the collection of all infinite subsets of $\mathbb{N}$. It is clear that all elements of $\mathcal{P}^\infty(\mathbb{N})$ are countable sets and pairwise unequal. However, we can show that $\mathcal{P}^\infty(\mathbb{N})$ itself is uncountable[1], while the union of the elements of $\mathcal{P}^\infty(\mathbb{N})$ equals $\mathbb{N}$.

5. Suppose $A \subseteq \mathbb{R}_+$, $b \in \mathbb{R}_+$, and for every list $a_1, a_2, \ldots, a_n$ of finitely many distinct elements of A, $a_1 + a_2 + \cdots + a_n \leq b$. Prove that A is at most countable (i.e. either finite or countable). (Hint: Consider the sets $A_n = \{x \in A | x \geq 1/n\}$. Feel free to use problem 4.)

   Solution: If A is finite, we are done. Assume instead that A is an infinite set. Let $A_n = \{x \in A | x \geq 1/n\}$ for all $n \in \mathbb{N}$.

[1] Showing this is somewhat challenging. If you want to try, you can think about a mapping from $\mathcal{P}^\infty(\mathbb{N})$ into $[0, 1]$; in particular, think about the binary expansion of the elements in $[0, 1]$. Equivalently, you can prove that there are countably many finite subsets of $\mathbb{N}$ (i.e. $\mathcal{P}(\mathbb{N}) \setminus \mathcal{P}^\infty(\mathbb{N})$ is countable): show that the collection of subsets of $\mathbb{N}$ with k elements is countable for all $k \in \mathbb{N}$ (try mapping it into $\mathbb{N}^k$) and then use this problem.

Notice that an arbitrary set A_n cannot have more than nb elements. Otherwise, for the sum of $nb+1$ elements that all belong to A_n, we would have:

$$a_1 + \cdots + a_{nb+1} \geq (nb+1)(1/n) = b + 1/n > b,$$

which is impossible by the assumptions on A. Therefore A_n is finite for all n.

By the Archimedean property, however, we have:

$$A \subseteq \bigcup_{n=0}^{\infty} A_n,$$

where $A_0 = \{0\}$. Notice that the set inclusion follows from the fact that for all $a \in A$: either $a = 0 \in A_0$, or $a > 0$ and $\exists \bar{n} \in \mathbb{N} : 1/\bar{n} < a$ so $a \in A_{\bar{n}}$.

So A is a subset of a countable union of finite sets, which implies that A is at most countable by Problem 4.

6. Consider the space $\mathbb{R}^\infty$ of all sequences $x = \{x_1, x_2, ...\}$ of real numbers. Define the function $d : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}$ by:

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

   (a) Show that d is well-defined (i.e. $d(x,y) < \infty$ for all $x, y \in \mathbb{R}^\infty$).

   (b) Show that d is a metric on $\mathbb{R}^\infty$.

   (c) A metric d is said to be *induced by a norm* ϕ if

$$d(x,y) = \phi(x-y),$$

   where $x - y = (x_1 - y_1, x_2 - y_2, \ldots)$. Show that d is not induced by any norm on $\mathbb{R}^\infty$.

Solution:

(a) For any $x, y \in \mathbb{R}^\infty$, we have:

$$\begin{aligned} d(x,y) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|} \\ &< \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= 1, \end{aligned}$$

where the inequality follows from the fact that for any $\alpha \geq 0$: $\frac{\alpha}{1+\alpha} < 1$.

(b) Since $\frac{1}{2^n} \frac{|x_n - y_n|}{1+|x_n - y_n|} \geq 0$ for any $x_n, y_n \in \mathbb{R}$, $d(x,y) \geq 0$ as the (infinite) sum of non-negative reals. The expression holds with an equality iff $x_n = y_n$. Therefore $d(x,y) = 0$ iff $x_n = y_n$ for all $n \in \mathbb{N}$. Since $|a-b| = |b-a|$, it is easy to see that $d(x,y) = d(y,x)$. Notice that to show that $d(x,z) \leq d(x,y) + d(y,z)$, it suffices to show:

$$\frac{|x_n - z_n|}{1 + |x_n - z_n|} \leq \frac{|x_n - y_n|}{1 + |x_n - y_n|} + \frac{|y_n - z_n|}{1 + |y_n - z_n|}$$

for all n. Now for notational simplicity, fix n and let:

$$\begin{aligned} |x_n - z_n| &= \alpha \\ |x_n - y_n| &= \beta \\ |y_n - z_n| &= \gamma, \end{aligned}$$

where by the triangle inequality we have: $\alpha \leq \beta + \gamma$. Modifying the expression that we need to show:

$$\begin{aligned} \frac{\alpha}{1+\alpha} &\leq \frac{\beta}{1+\beta} + \frac{\gamma}{1+\gamma} \\ \Leftrightarrow \alpha(1+\beta)(1+\gamma) &\leq \beta(1+\alpha)(1+\gamma) + \gamma(1+\alpha)(1+\beta) \\ \Leftrightarrow \alpha + \alpha\beta + \alpha\gamma + \alpha\beta\gamma &\leq \beta + \beta\alpha + \beta\gamma + \alpha\beta\gamma + \gamma + \gamma\alpha + \gamma\beta + \alpha\beta\gamma \\ \Leftrightarrow \alpha &\leq \beta + \gamma + 2\beta\gamma + \alpha\beta\gamma \end{aligned}$$

Since $\alpha, \beta, \gamma \geq 0$, the last inequality is implied by the triangle inequality $\alpha \leq \beta + \gamma$.

(c) Assume toward contradiction that d is induced by some norm ϕ. Then for some $x \neq y$, we have: $d(x,y) = \phi(x-y) > 0$. Let $\phi(x-y) = c > 0$. Then, since ϕ is a norm, we have:

$$\begin{aligned}
2 &= \frac{2}{c}c \\
&= \frac{2}{c}\phi(x-y) \\
&= \phi(\frac{2}{c}(x-y)) \\
&= \phi(\frac{2}{c}x - \frac{2}{c}y) \\
&= d(\frac{2}{c}x, \frac{2}{c}y),
\end{aligned}$$

but in part (a) we showed that $d(\cdot,\cdot) < 1$. This is a contradiction and, therefore, d is not induced by a norm on $\mathbb{R}^\infty$.

A norm's property that $\phi(ax) = |\alpha|\phi(x)$ suggests that the range of a norm $\phi : X \to \mathbb{R}_+$ must be unbounded. Therefore, this proof can be adapted to show that any metric that is bounded (i.e. $\rho(x,y) < M$ for all $x, y \in X$ and some $M > 0$) is not induced by a norm.

7. Suppose that $\{a_n\}$ is a sequence of real numbers and $\{b_n\}$ is a sequence obtained by some rearrangement of the terms of $\{a_n\}$ (in other words, $\{a_n\}$ and $\{b_n\}$ have exactly the same terms, and repeated terms appear the same number of times in $\{b_n\}$ as in $\{a_n\}$). Prove that $\{a_n\} \to x$ iff $\{b_n\} \to x$.

Solution: Assume $\{a_n\} \to x$. Now we need to show $\{b_n\} \to x$. We will show this directly using the definition of convergent sequences. Fix $\varepsilon > 0$. We know:

$$\exists N \in \mathbb{N} : \forall m \geq N, a_m \in B_\varepsilon(x).$$

Thus, there are finitely many (at most $N-1$) elements of $\{a_n\}$ (and hence of $\{b_n\}$) that are *not* in $B_\varepsilon(x)$. Let the highest index of those elements in $\{b_n\}$ be M. Then $\forall m > M : b_m \in B_\varepsilon(x)$. But since ε was arbitrary, this implies $\{b_n\} \to x$. The proof that $\{b_n\} \to x$ implies $\{a_n\} \to x$ is identical.