Economics 204
Fall 2011
Problem Set 1
Due Friday, July 29 in Lecture

1. Suppose k is a positive integer. Use induction to prove the following two statements.

   (a) For all $n \in \mathbb{N}_0$, the inequality $(k^2 + n)! \geq k^{2n}$ holds.

   (b) For all natural $n \geq 2k^2$, the inequality $n! \geq k^n$ holds.

   (Recall that the factorial of a non-negative integer n is defined by $n! = \prod_{m=1}^{n} m$ with the convention $0! = 1$.)

2. Let n be a positive integer, and suppose that n chords are drawn in a circle, cutting the circle into a number of regions. Prove that the regions can be colored with two colors in such a way that adjacent regions (that is, regions that share an edge) are different colors.

3. Define the relation $\sim$ on the space of all sets in the following manner: $A \sim B$ iff there exists a bijection $f : A \to B$. Show that $\sim$ is an equivalence relation. (Notice that this is just the definition of numerical equivalence. The problem is asking you to prove that numerical equivalence is indeed an equivalence relation, as the name suggests.)

4. Prove that the countable union of countable sets is countable.

5. Suppose $A \subseteq \mathbb{R}_+$, $b \in \mathbb{R}_+$, and for every list $a_1, a_2, \ldots, a_n$ of finitely many distinct elements of A, $a_1 + a_2 + \cdots + a_n \leq b$. Prove that A is at most countable (i.e. either finite or countable). (Hint: Consider the sets $A_n = \{x \in A | x \geq 1/n\}$. Feel free to use problem 4.)

6. Consider the space $\mathbb{R}^\infty$ of all sequences $x = \{x_1, x_2, ...\}$ of real numbers. Define the function $d : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}$ by:

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

   (a) Show that d is well-defined (i.e. $d(x, y) < \infty$ for all $x, y \in \mathbb{R}^\infty$).

(b) Show that d is a metric on $\mathbb{R}^\infty$.

(c) A metric d is said to be *induced by a norm* ϕ if

$$d(x, y) = \phi(x - y),$$

where $x - y = (x_1 - y_1, x_2 - y_2, \ldots)$. Show that d is not induced by any norm on $\mathbb{R}^\infty$.

7. Suppose that $\{a_n\}$ is a sequence of real numbers and $\{b_n\}$ is a sequence obtained by some rearrangement of the terms of $\{a_n\}$ (in other words, $\{a_n\}$ and $\{b_n\}$ have exactly the same terms, and repeated terms appear the same number of times in $\{b_n\}$ as in $\{a_n\}$). Prove that $\{a_n\} \to x$ iff $\{b_n\} \to x$.