1. Determine whether the following sets are open, closed, both or neither under the topology induced by the usual metric. (Hint: think about limit points of those sets.)

(a) The interval $(0, 1)$ as a subset of $\mathbb{R}$

Solution. Open, not closed. Any point is an interior point. Also note that 1 is a limit point.

(b) The interval $(0, 1)$ as a subset of $\mathbb{R}^2$, that is $\{(x, 0) \in \mathbb{R}^2 \mid x \in (0, 1)\}$

Solution. Not open, not closed — none of its points are interior points (remember though, only need to find one for it to be not open) and $(1, 0)$ is a limit point not in the set.

(c) $\mathbb{R}$ as the subset of $\mathbb{R}$

Solution. Open and closed.

(d) $\mathbb{R}$ imbedded as a subset $\{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ of $\mathbb{R}^2$

Solution. Not open, as none of its points are interior points. It is closed.

(e) $\{1/n \mid n \in \mathbb{N}\}$ as a subset of $\mathbb{R}$

Solution. Not open, not closed (note that zero is a limit point).

(f) $\{1/n \mid n \in \mathbb{N}\}$ as a subset of the interval $(0, \infty)$.

Solution. Not open, closed.

2. Consider the following two sets:

\[ A = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 \leq 3\} \]
\[ B = \{(x, y) \in \mathbb{R}^2 \mid y > \sqrt{|x|}\} \]

(a) Using the “pre-image of a closed set is closed (under a continuous function)” definition, determine whether sets $A$ and $B$ are open, closed, both or neither

Solution.

Set $A$. Closed, not open. First, note that if a function $g : \mathbb{R} \to \mathbb{R}$ is continuous, then the functions $H, G : \mathbb{R}^2 \to \mathbb{R}$ defined by:

\[ G(x, y) = g(x) \quad \text{and} \quad H(x, y) = g(y) \]
are continuous. Then for some open $A \subset \mathbb{R}$ we have:

$$G^{-1}(A, \mathbb{R}) = g^{-1}(A) \times \mathbb{R}$$

$$H^{-1}(A, \mathbb{R}) = \mathbb{R} \times g^{-1}(A).$$

If $g$ is continuous, $g^{-1}(A)$ is open. Since $\mathbb{R}$ is open, $G^{-1}(A)$ and $H^{-1}(A)$ are also open and, hence, $G$ and $H$ are continuous.

We can then use this to show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 - y^2$ is continuous. Letting $H(x, y) = y^2$ and $G(x, y) = x^2$, $H(x, y)$ and $G(x, y)$ are continuous on $\mathbb{R}^2$ by the argument above and hence $f = G - H$ is continuous as well. Now consider the set we are interested in:

$$A = \{(x, y) \in \mathbb{R}^2 | x^2 - y^2 \leq 3\}$$

$$= \{(x, y) \in \mathbb{R}^2 | f(x, y) \leq 3\}$$

$$= f^{-1}((-\infty, 3])$$

$f$ is continuous and $(-\infty, 3]$ is closed; therefore, the set under consideration must be closed as it is a continuous inverse image of a closed set.

The set is not open. We see that $(2, 1)$ is an element of it and any open ball around $(2, 1)$ contains some $(x', y')$ with $x' > 2$ and $y' > 1$. Clearly though, $(x', y')$ is not an element of our set.

Set $B$. Open, not closed. Observe that Theorem 7 in Lecture 4 assures us that composition of continuous functions is a continuous function, i.e. if $g(x) = |x|$ and $h(x) = \sqrt{x}$, then the function $r(x) = h \circ g$ is continuous as well. Now, as above, letting $H(x, y) = r$ and $G(x, y) = y$, $H(x, y)$ and $G(x, y)$ are continuous on $\mathbb{R}^2$ and hence $f = G - H$ is continuous as well. Now consider the set we are interested in:

$$B = \{(x, y) \in \mathbb{R}^2 | y > \sqrt{|x|}\}$$

$$= \{(x, y) \in \mathbb{R}^2 | f(x, y) > 0\}$$

$$= f^{-1}((0, \infty))$$

$f$ is continuous and $(0, \infty)$ is open; therefore, the set under consideration must be open as it is a continuous inverse image of a closed set.

The set is not closed. We see that, note $(0, 0)$ is a limit point of $B$ but it is not an element of our set.

(b) Find their closure, exterior and boundary.

Solution.

Set $A$.

$$\partial A = \{(x, y) \in \mathbb{R}^2 | x^2 - y^2 = 3\}$$

$$\text{ext } A = \{(x, y) \in \mathbb{R}^2 | x^2 - y^2 > 3\}$$

$$\bar{A} = \{(x, y) \in \mathbb{R}^2 | x^2 - y^2 \leq 3\}$$
Set $B$.

$$
\partial B = \{(x,y) \in \mathbb{R}^2 \mid y = \sqrt{|x|}\}
$$

$$
ext B = \{(x,y) \in \mathbb{R}^2 \mid y < \sqrt{|x|}\}
$$

$$
\bar{B} = \{(x,y) \in \mathbb{R}^2 \mid y \geq \sqrt{|x|}\}
$$

3. Let $A, B \subset X$. Suppose that $\text{int } A = \text{int } B = \emptyset$.

(a) Prove that if $A$ is closed in $X$, then $\text{int } (A \cup B) = \emptyset$.

**Solution.** Let $\text{int } A = \text{int } B = \emptyset$ and $A$ is closed in $X$, but assume to contradiction that $\text{int } (A \cup B) \neq \emptyset$. Then, there exists some point $x \in X$ with its open neighborhood $U_x$ that is contained entirely in $\text{int } A \cup B$. Let consider the following two cases:

Case A. Let suppose that $U_x \setminus A \neq \emptyset$. Notice that $U_x \setminus A = U_x \cap A^c$ and because intersection of two open sets is open, $U_x \setminus A$ is open. Now, if $y \in U_x \setminus A$ then $y \in \text{int } B$, but if this is so, we get a contradiction because $U_x \setminus A \subset B$ and $B$ has an empty interior.

Case B. Let suppose now that $U_x \setminus A = \emptyset$, which means if $x \in U_x \subset A$ then $x \in \text{int } A$. As before we get a contradiction that proves us the result we seek.

(b) Give an example with $\text{int } (A \cup B) \neq \emptyset$ if $A$ isn’t necessarily closed in $X$.

**Solution.** Let $X = \mathbb{R}$, $A = \mathbb{Q}$, and $B = \mathbb{R} \setminus \mathbb{Q}$. Notice that $\text{int } A = \text{int } B = \emptyset$ because $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense sets and for any $x \in \mathbb{R}$ and any open $U_x$ containing $x$, both rational and irrational points are contained in $U_x$. Also, observe that neither $\mathbb{Q}$ nor $\mathbb{R} \setminus \mathbb{Q}$ are closed in $\mathbb{R}$. Finally, we have $A \cup B = \mathbb{R}$ and, definitely, $\text{int } \mathbb{R} \neq \emptyset$.

4. Let $f$ be a monotonic, increasing function from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $f(0) > 0$ and $f(100) < 100$. Prove that $f(x) = x$ for some $x \in (0, 100)$.

**Solution.** We provide two solutions, with the first one being more intuitive and the second one being more rigorous. Suppose that we have $f(0) > 0$ and $f(100) < 100$. Set $a_0 = 0$ and $b_0 = 100$. If $f(50) = 50$ then we stop, otherwise either $f(50) < 50$ or $f(50) > 50$. If $f(50) < 50$, set $a_1 = a_0$ and $b_1 = 50$; if $f(50) > 50$, set $a_1 = 50$ and $b_1 = b_0$. Now, let's repeat this process on $(a_1, b_1)$, $(a_2, b_2)$, ... . This yields a sequence of open intervals $\{(a_n, b_n)\}$ such that for all $n \in \mathbb{N}$ we have

$$
f(a_n) > a_n
$$

$$
f(b_n) < b_n
$$
and 
\[ b_n - a_n = \frac{1}{2^n} \cdot 100. \]

Also, we can immediately see that \( \{a_n\} \) is an increasing sequence and \( \{b_n\} \) is an decreasing sequence. By monotonicity of \( f(x) \) we get that \( \{f(a_n)\} \) is an increasing sequence and \( \{f(b_n)\} \) is a decreasing one. Moreover, for all \( n \in \mathbb{N} \) we must have \( f(a_n) \leq f(b_n) \).

We claim that 
\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = x^* \]

and that 
\[ f(x^*) = x^*. \]

To see this, notice that both \( \{a_n\} \) and \( \{b_n\} \) are bounded and monotonic, thus, both \( \lim_{n \to \infty} a_n = a \) and \( \lim_{n \to \infty} b_n = b \) exist. Moreover, we must have \( a = b \), because otherwise \( b - a > 0 \) and \( \exists N \) such that for all \( n > N \) we have

\[ |a_n - b_n| < \frac{1}{2} \cdot 100 < b - a \]

But

\[ |a_n - a| < \frac{1}{2} \cdot (b - a) \quad \text{and} \quad |b_n - b| < \frac{1}{2} \cdot (b - a) \]

which is a contradiction and we must have \( a = b = x^* \).

Also, let

\[ \lim_{n \to \infty} f(a_n) = y \]
\[ \lim_{n \to \infty} f(b_n) = z \]

and \( f(a_n) < f(x^*) < f(b_n) \) for all \( n \in \mathbb{N} \), which implies that \( y \leq f(x^*) \leq z \).

Notice that by construction for all \( n \in \mathbb{N} \) we have \( f(a_n) > a_n \) and \( f(b_n) < b_n \). Thus, we get that \( z \leq x^* \leq y \), or that

\[ x^* \leq y \leq f(x^*) \leq z \leq x^*. \]

This means that \( y = z = x^* = f(x^*) \).

Now, we provide our second proof that does not rely on the iterative process of “shrinking intervals.” Let’s consider a set

\[ A = \{ x \in (0, 100) \mid f(x) \geq x \} \]

Observe that \( A \) is non-empty and bounded, thus, it has a finite supremum which we will denote by \( x^* \). We claim that \( f(x^*) \geq x^* \). To see this, note that for all \( x \in A \) we have \( x \leq x^* \) and that by monotonicity of \( f \) we must have

\[ x \leq f(x) \leq f(x^*) \quad \text{for all} \quad x \in A, \]
from which it follows that $x^* \leq f(x^*)$ because $f(x^*)$ is also an upper bound for $A$ and $x^* = \sup A$.

Now, if we can show that $x^* \geq f(x^*)$ we would be done. To demonstrate that, we will use again the fact that $f(x)$ is monotonically increasing. So, suppose to contradiction that $x^* < f(x^*)$. Take $x^* < x < f(x^*)$, which implies that $f(x^*) \leq f(x) \leq (f(f^*))$ since $f$ is monotone. Putting this together, yields

$$x < f(x^*) \leq f(x) \leq f(f(x^*))$$

This, in turn, implies that $x \in A$ but $x > x^*$, contradicting our definition of $x^*$ being a supremum of $A$. /par

then for all $x$ such that $x^* < x < x^* + f(x^*) - x^*$ we have $f(x) - x \geq f(x^*) - x > f(x^*) - x^* - f(x^*) + x^* = 0$, because $f(x)$ is monotonically increasing. Thus, $f(x) > x > x^*$, but this contradicts the definition of $x^*$ as the supremum of all $x$ such that $x \leq f(x)$.

We are done.

5. Call a mapping of $X$ into $Y$ open if $f(V)$ is an open set in $Y$ whenever $V$ is an open set in $X$. Prove that every continuous open mapping of $\mathbb{R}$ into $\mathbb{R}$ is monotonic.

**Solution.** We prove it by contradiction. Without any loss of generality, assume there are two points $x, z \in X$ with $x < z$ such that $f(x) = f(z)$ but $f$ is not constant on $(x, z)$. If it would be constant, we immediately get a contradiction as the pre-image of a closed set (point) would be open.

Now, since $f$ is not constant and it is continuous, it achieves on $(x, z)$ its maximum, $M$, where for all $y \in [x, z] : f(y) \leq M$, its minimum, $m$, where for all $y \in [x, z] : f(y) \geq m$, or both. Lets suppose it achieves its maximum at some $y \in (x, z)$. So, take an open neighborhood $U_y \in (x, y)$. By our assumption $f(U_y)$ is open, but this means that $M + \epsilon \in f(U_y)$ for some arbitrarily small $\epsilon > 0$. Thus, we arrive at contradiction as $M$ can’t be maximum, or, in other words, $M$ can’t be an interior point of $U_y$. The case of minimum is handled similarly.

6. Suppose that $\{x_n\}$ is a convergent sequence of points while lies, together with its limit $x$, in a set $X \subset \mathbb{R}^n$. Suppose that $\{f_n\}$ converges on $X$ to the function $f$.

(a) Is it true that $f(x) = \lim f_n(x_n)$? Prove if true or provide counterexample.

**Solution.** Consider following counterexample: $X = [0, 1]$, $f_n(x) = x^n$ and $x_n = 1 - 1/n$. It is easy to see that $x_n \to 1$ as $n \to \infty$,

$$f_n(x) \to \begin{cases} 1, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$
and that
\[ \lim_{n \to \infty} \left( 1 - \frac{1}{n} \right)^n = \frac{1}{e}. \]
But clearly, we have \( 1/e \neq f(1) = 1 \). We definitely need stronger assumptions.

(b) Would you change your answer if you know that convergence of \( \{f_n\} \) is uniform on \( X \)? Again, prove if true or provide counterexample.

**Solution.** No. Even uniform convergence does not guarantee us this result. To see this consider the following example with \( f(x) \) being a Heaviside step function:

\[ f(x) = \begin{cases} 
1, & x \geq 0 \\
0, & x < 0 
\end{cases} \]

with \( x_n = -1/n \) and \( f_n(x) = f(x) + 1/n \). Clearly, \( x_n \to 0 \) and \( f_n \) converge uniformly to \( f \) but \( \lim_{n \to \infty} f_n(x_n) = 0 \neq 1 = f(0) \). Actually, to get the result we seek, we need a uniform convergence to a continuous function \( f \).

7. Some practice with contraction maps

(a) Let \( (X, d) \) be a space of continuous function on a closed interval \([0, \beta]\) with a supremum norm, i.e. \( X = C([0, \beta]) \), and \( d(f, g) = \max_t |f(t) - g(t)| \), where \( \beta < 1 \). Define \( T : X \to X \) by

\[ (Tf)(t) = \int_0^t f(s) \, ds + g(t), \]

for some continuous function \( g(t) \).

Show that \( T \) has a unique fixed point.\(^1\)

**Solution.** Lets first show that \( C([0, \beta]) \) is a complete metric space, i.e. that an arbitrary Cauchy sequence \( \{f_n(x)\}_{n=1}^\infty \) in \( C([0, \beta]) \) converges. In particular, we need to demonstrate that the limit of a Cauchy sequence of continuous functions on \([0, \beta]\), firstly, exists in the sense that for each \( x \in [0, 1] \), lim \( f_n(x) \) exists, and secondly, pointwise limit function, \( f(x) \), is continuous on \([0, \beta]\).

The first part follows immediately from the fact that Cauchy sequences of real numbers converge (to real numbers) by the completeness of \( \mathbb{R} \) argument. Hence, we are guaranteed the existence of our pointwise limit function.

\(^1\)Note that to invoke Contraction Mapping Theorem here, you need first to show that \( (X, d) \) is a complete metric space. Completeness of \( \mathbb{R} \) will come to your rescue here.
Now, we just need to show continuity of \( f(x) \). Fix \( \epsilon > 0 \) and consider \( s, t \in [0, 1] \). We have

\[
|f(s) - f(t)| = |f(s) - f_n(s) + f_n(s) - f(t)|
\leq |f(s) - f_n(s)| + |f_n(s) - f(t)|
= |f(s) - f_n(s)| + |f_n(s) - f_n(t) + f_n(t) - f(t)|
\leq |f(s) - f_n(s)| + |f_n(s) - f_n(t)| + |f_n(t) - f(t)| \tag{1}
\]

From the definition of \( d(f_n, f) \) and the fact that \( d(f_n, f) \to 0 \) as \( n \to \infty \), there exists an \( N(\epsilon) \) such that whenever \( n > N(\epsilon) \), the first and last terms on the RHS of (1) are less than \( \epsilon / 3 \). Furthermore, for any \( n \), \( f_n \in C([0, 1]) \). Fix some \( n > N(\epsilon) \). (Uniform) Continuity implies that there is a \( \delta_n > 0 \) such that whenever \( |s - t| < \delta_n \Rightarrow |f_n(s) - f_n(t)| < \epsilon / 3 \). Therefore, all three terms of the RHS of (*) are less than \( \epsilon / 3 \). So we get that

\[|f(s) - f(t)| < \epsilon, \text{ whenever } |s - t| < \delta_n\]

This proves that \( f \) is (uniformly) continuous.

Now, we show that \( T \) is a contraction.

\[
d(Tf, Tg) = \max_{t \in [0, \beta]} |Tf(t) - Tg(t)|
= \max_{t \in [0, \beta]} |\int_0^t f(s) \, ds - \int_0^t g(s) \, ds|
\leq \max_{t \in [0, \beta]} \int_0^t |f(s) - g(s)| \, ds
\leq \int_0^\beta \max_{t \in [0, \beta]} \{|f(t) - g(t)|\} \, ds
= \beta d(f, g)
\]

(b) Now suppose \( (X, d) \) is an arbitrary complete metric space, but \( T : X \to X \) is an \textit{expansion}, i.e. there exists \( \beta > 1 \) such that \( d(Tx, Ty) \geq \beta d(x, y) \) for all \( x, y \) in \( X \), and that \( T(X) = X \). Show that \( T \) has a fixed point.

**Solution.** We will do it in a sequence of steps. Firstly, we will verify that \( T \) is one-to-one and, thus, since it is also a surjection, the inverse must exists. Secondly, we will argue that \( T^{-1} \) is a contraction. Finally, we will invoke Contraction Mapping Theorem and show that the fixed point of \( T^{-1} \) is also a fixed point of \( T \).

Step 1: verifying \( T \) is one-to-one, i.e. if \( x_1 \neq x_2 \) then \( T(x_1) \neq T(x_2) \). But this is immediate since \( x_1 \neq x_2 \) implies directly that \( d(T(x_1), T(x_2)) \geq \)
\[ \beta d(x_1, x_2) > 0. \] Therefore, \( T \) is one-to-one. By our assumption, \( T \) is onto, thus, there exists \( T^{-1} : X \to X \) and \( T \circ T^{-1} = id. \)

**Step 2:** Show that \( T^{-1} \) is a contraction, i.e. there exists \( \alpha < 1 \) such that \( d(T^{-1}(x_1), T^{-1}(x_2)) \leq \alpha d(x_1, x_2) \) for all \( x_1, x_2 \in X \).

So, if \( T \) is one-to-one and onto, \( T^{-1} : X \to X \) is also one-to-one and onto. Therefore, we can apply expansion \( T \) to any arbitrary image of \( T^{-1} \) since for any \( x \in X : T^{-1}(x) \in X \)

\[
\begin{align*}
d(T(T^{-1}(x_1)), T(T^{-1}(x_2))) &\geq \beta d(T^{-1}(x_1), T^{-1}(x_2)) &\iff &
d(x_1, x_2) &\geq \beta d(T^{-1}(x_1), T^{-1}(x_2)) \iff \\
d(T^{-1}(x_1), T^{-1}(x_2)) &\leq \frac{1}{\beta} d(x_1, x_2)
\end{align*}
\]

We have shown that there exists \( \alpha = \frac{1}{\beta} < 1 \) such that \( d((T^{-1}(x_1), T^{-1}(x_2)) \leq \alpha d(x_1, x_2) \) for all \( x_1, x_2 \in X \), i.e. \( T^{-1} \) is a contraction.

**Step 3:** By the Contraction Mapping Theorem, we can conclude that there exists \( x^* \), a fixed point, such that \( T^{-1}(x^*) = x^* \). Now, we show that \( x^* \) is also a fixed point of the expansion \( T \).

Since \( x^* = T^{-1}(x^*) \) is an identity, we can apply to each side any arbitrary transformation that preserve the equality. In particular, let’s choose \( T \)

\[ T(x^*) = T(T^{-1}(x^*)) \]

which implies

\[ T(x^*) = x^* \]

and we get the result we seek.