Economics 204
Fall 2011
Problem Set 2 Suggested Solutions

1. Determine whether the following sets are open, closed, both or neither under the topology induced by the usual metric. (Hint: think about limit points of those sets.)

   (a) The interval $(0,1)$ as a subset of $\mathbf{R}$

   Solution. Open, not closed. Any point is an interior point. Also note that 1 is a limit point.

   (b) The interval $(0,1)$ as a subset of $\mathbf{R}^2$, that is $\{(x,0) \in \mathbf{R}^2 \mid x \in (0,1)\}$

   Solution. Not open, not closed — none of its points are interior points (remember though, only need to find one for it to be not open) and $(1,0)$ is a limit point not in the set.

   (c) $\mathbf{R}$ as the subset of $\mathbf{R}$

   Solution. Open and closed.

   (d) $\mathbf{R}$ imbedded as a subset $\{(x,0) \in \mathbf{R}^2 \mid x \in \mathbf{R}\}$ of $\mathbf{R}^2$

   Solution. Not open, as none of its points are interior points. It is closed.

   (e) $\{1/n \mid n \in \mathbf{N}\}$ as a subset of $\mathbf{R}$

   Solution. Not open, not closed (note that zero is a limit point).

   (f) $\{1/n \mid n \in \mathbf{N}\}$ as a subset of the interval $(0,\infty)$.

   Solution. Not open, closed.

2. Consider the following two sets:

$$A = \{(x,y) \in \mathbf{R}^2 \mid x^2 - y^2 \leq 3\}$$
$$B = \{(x,y) \in \mathbf{R}^2 \mid y > \sqrt{|x|}\}$$

   (a) Using the "pre-image of a closed set is closed (under a continuous function)" definition, determine whether sets A and B are open, closed, both or neither

   Solution.
   Set A. Closed, not open. First, note that if a function $g : \mathbf{R} \to \mathbf{R}$ is continuous, then the functions H, $G : \mathbf{R}^2 \to \mathbf{R}$ defined by:

$$G(x,y) = g(x) \quad \text{and} \quad H(x,y) = g(y)$$

are continuous. Then for some open $A \subset \mathbf{R}$ we have:

$$\begin{aligned} G^{-1}(A, \mathbf{R}) &= g^{-1}(A) \times \mathbf{R} \\ H^{-1}(A, \mathbf{R}) &= \mathbf{R} \times g^{-1}(A). \end{aligned}$$

If g is continuous, $g^{-1}(A)$ is open. Since $\mathbf{R}$ is open, $G^{-1}(A)$ and $H^{-1}(A)$ are also open and, hence, G and H are continuous.
We can then use this to show that the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) = x^2 - y^2$ is continuous. Letting $H(x, y) = y^2$ and $G(x, y) = x^2$, $H(x, y)$ and $G(x, y)$ are continuous on $\mathbf{R}^2$ by the argument above and hence $f = G - H$ is continuous as well. Now consider the set we are interested in:

$$\begin{aligned} A &= \{(x, y) \in \mathbf{R}^2 \,|\, x^2 - y^2 \leq 3\} \\ &= \{(x, y) \in \mathbf{R}^2 \,|\, f(x, y) \leq 3\} \\ &= f^{-1}\left((-\infty, 3]\right) \end{aligned}$$

f is continuous and $(-\infty, 3]$ is closed; therefore, the set under consideration must be closed as it is a continuous inverse image of a closed set.
The set is not open. We see that $(2, 1)$ is an element of it and any open ball around $(2, 1)$ contains some (x', y') with $x' > 2$ and $y' > 1$. Clearly though, (x', y') is not an element of our set.

Set B. Open, not closed. Observe that Theorem 7 in Lecture 4 assures us that composition of continuous functions is a continuous function, i.e. if $g(x) = |x|$ and $h(x) = \sqrt{x}$, then the function $r(x) = h \circ g$ is continuous as well. Now, as above, letting $H(x, y) = r$ and $G(x, y) = y$, $H(x, y)$ and $G(x, y)$ are continuous on $\mathbf{R}^2$ and hence $f = G - H$ is continuous as well. Now consider the set we are interested in:

$$\begin{aligned} B &= \{(x, y) \in \mathbf{R}^2 \,|\, y > \sqrt{|x|}\} \\ &= \{(x, y) \in \mathbf{R}^2 \,|\, f(x, y) > 0\} \\ &= f^{-1}\left((0, \infty)\right) \end{aligned}$$

f is continuous and $(0, \infty)$ is open; therefore, the set under consideration must be open as it is a continuous inverse image of a closed set.
The set is not closed. We see that, note $(0, 0)$ is a limit point of B but it is not an element of our set.

(b) Find their closure, exterior and boundary.

Solution.
Set A.

$$\begin{aligned} \partial A &= \{(x, y) \in \mathbf{R}^2 \mid x^2 - y^2 = 3\} \\ \operatorname{ext} A &= \{(x, y) \in \mathbf{R}^2 \mid x^2 - y^2 > 3\} \\ \bar{A} &= \{(x, y) \in \mathbf{R}^2 \mid x^2 - y^2 \leq 3\} \end{aligned}$$

Set B.

$$\begin{aligned} \partial B &= \{(x, y) \in \mathbf{R}^2 \mid y = \sqrt{|x|}\} \\ \operatorname{ext} B &= \{(x, y) \in \mathbf{R}^2 \mid y < \sqrt{|x|}\} \\ \bar{B} &= \{(x, y) \in \mathbf{R}^2 \mid y \geq \sqrt{|x|}\} \end{aligned}$$

3. Let $A, B \subset X$. Suppose that $\operatorname{int} A = \operatorname{int} B = \emptyset$.

   (a) Prove that if A is closed in X, then $\operatorname{int}(A \cup B) = \emptyset$.

   Solution. Let $\operatorname{int} A = \operatorname{int} B = \emptyset$ and A is closed in X, but assume to contradiction that $\operatorname{int}(A \cup B) \neq \emptyset$. Then , there exists some point $x \in X$ with its open neighborhood U_x that is contained entirely in $\operatorname{int} A \cup B$. Lets consider the following two cases:
   Case A. Let suppose that $U_x \setminus A \neq \emptyset$. Notice that $U_x \setminus A = U_x \cap A^c$ and because intersection of two open sets is open, $U_x \setminus A$ is open. Now, if $y \in U_x \setminus A$ then $y \in \operatorname{int} B$, but if this is so, we get a contradiction because $U_x \setminus A \subset B$ and B has an empty interior.
   Case B. Let suppose now that $U_x \setminus A = \emptyset$, which means if $x \in U_x \subset A$ then $x \in \operatorname{int} A$. As before we get a contradiction that proves us the result we seek.

   (b) Give an example with $\operatorname{int}(A \cup B) \neq \emptyset$ if A isn't necessarily closed in X.

   Solution. Let $X = \mathbf{R}$, $A = \mathbf{Q}$, and $B = \mathbf{R} \setminus \mathbf{Q}$. Notice that $\operatorname{int} A = \operatorname{int} B = \emptyset$ because $\mathbf{Q}$ and $\mathbf{R} \setminus \mathbf{Q}$ are *dense* sets and for any $x \in \mathbf{R}$ and any open U_x containing x, both rational and irrational points are contained in U_x. Also, observe that neither $\mathbf{Q}$ nor $\mathbf{R} \setminus \mathbf{Q}$ are closed in $\mathbf{R}$. Finally, we have $A \cup B = \mathbf{R}$ and, definitely, $\operatorname{int} \mathbf{R} \neq \emptyset$.

4. Let f be a monotonic, increasing function from $\mathbf{R}$ to $\mathbf{R}$. Suppose that $f(0) > 0$ and $f(100) < 100$. Prove that $f(x) = x$ for some $x \in (0, 100)$.

   Solution. We provide two solutions, with the first one being more intuitive and the second one being more rigorous. Suppose that we have $f(0) > 0$ and $f(100) < 100$. Set $a_0 = 0$ and $b_0 = 100$. If $f(50) = 50$ then we stop, otherwise either $f(50) < 50$ or $f(50) > 50$. If $f(50) < 50$, set $a_1 = a_0$ and $b_1 = 50$; if $f(50) > 50$, set $a_1 = 50$ and $b_1 = b_0$. Now, lets repeat this process on (a_1, b_1), (a_2, b_2), ... This yields a sequence of open intervals $\{(a_n, b_n)\}$ such that for all $n \in \mathbf{N}$ we have

$$\begin{aligned} f(a_n) &> a_n \\ f(b_n) &< b_n \end{aligned}$$

and

$$b_n - a_n = \frac{1}{2^n} \cdot 100.$$

Also, we can immediately see that $\{a_n\}$ is an increasing sequence and $\{b_n\}$ is an decreasing sequence. By monotonicity of $f(x)$ we get that $\{f(a_n)\}$ is an increasing sequence and $\{f(b_n)\}$ is a decreasing one. Moreover, for all $n \in \mathbf{N}$ we must have $f(a_n) \leq f(b_n)$.

We claim that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x^*$$

and that

$$f(x^*) = x^*.$$

To see this, notice that both $\{a_n\}$ and $\{b_n\}$ are bounded and monotonic, thus, both $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ exist. Moreover, we must have $a = b$, because otherwise $b - a > 0$ and $\exists N$ such that for all $n > N$ we have

$$|a_n - b_n| < \frac{1}{2} \cdot 100 < b - a$$

But

$$|a_n - a| < \frac{1}{2} \cdot (b - a) \quad \text{and } |b_n - b| < \frac{1}{2} \cdot (b - a)$$

which is a contradiction and we must have $a = b = x^*$.

Also, let

$$\lim_{n\to\infty} f(a_n) = y$$
$$\lim_{n\to\infty} f(b_n) = z$$

and $f(a_n) < f(x^*) < f(b_n)$ for all $n \in \mathbf{N}$, which implies that $y \leq f(x^*) \leq z$. Notice that by construction for all $n \in \mathbf{N}$ we have $f(a_n) > a_n$ and $f(b_n) < b_n$. Thus, we get that $z \leq x^* \leq y$, or that

$$x^* \leq y \leq f(x^*) \leq z \leq x^*.$$

This means that $y = z = x^* = f(x^*)$.

Now, we provide our second proof that does not rely on the iterative process of "shrinking intervals." Lets consider a set

$$A = \{\, x \in (0, 100) \mid f(x) \geq x \}$$

Observe that A is non-empty and bounded, thus, it has a finite supremum which we will denote by x^*. We claim that $f(x^*) \geq x^*$. To see this, note that for all $x \in A$ we have $x \leq x^*$ and that by monotonicity of f we must have

$$x \leq f(x) \leq f(x^*) \quad \text{for all } x \in A,$$

from which it follows that $x^* \leq f(x^*)$ because $f(x^*)$ is also an upper bound for A and $x^* = \sup A$.

Now, if we can show that $x^* \geq f(x^*)$ we would be done. To demonstrate that, we will use again the fact that $f(x)$ is monotonically increasing. So, suppose to contradiction that $x^* < f(x^*)$. Take $x^* < x < f(x^*)$, which implies that $f(x^*) \leq f(x) \leq (f(f^*))$ since f is monotone. Putting this together, yields

$$x < f(x^*) \leq f(x) \leq f(f(x^*))$$

This, in turn, implies that $x \in A$ but $x > x^*$, contradicting our definition of x^* being a supremum of A. /par

then for all x such that $x^* < x < x^* + f(x^*) - x^*$ we have $f(x) - x \geq f(x^*) - x > f(x^*) - x^* - f(x^*) + x^* = 0$, because $f(x)$ is monotonically increasing. Thus, $f(x) > x > x^*$, but this contradicts the definition of x^* as the supremum of all x such that $x \leq f(x)$.

We are done.

5. Call a mapping of X into Y *open* if $f(V)$ is an open set in Y whenever V is an open set in X. Prove that every continuous open mapping of $\mathbf{R}$ into $\mathbf{R}$ is monotonic.

   Solution. We prove it by contradiction. Without any loss of generality, assume there are two points $x,\ z \in X$ with $x < z$ such that $f(x) = f(z)$ but f is not constant on (x, z). If it would be constant, we immediately get a contradiction as the pre-image of a closed set (point) would be open.

   Now, since f is not constant and it is continuous, it achieves on (x, z) its maximum, M, where for all $y \in [x, z] :\ f(y) \leq M$, its minimum, m, where for all $y \in [x, z] :\ f(y) \geq m$, or both. Lets suppose it achieves its maximum at some $y \in (x, z)$. So, take an open neighborhood $U_y \in (x, y)$. By our assumption $f(U_y)$ is open, but this means that $M + \epsilon \in f(U_y)$ for some arbitrarily small $\epsilon > 0$. Thus, we arrive at contradiction as M can't be maximum, or, in other words, M can't be an interior point of U_y. The case of minimum is handled similarly.

6. Suppose that $\{x_n\}$ is a convergent sequence of points while lies, together with its limit x, in a set $X \subset \mathbf{R^n}$. Suppose that $\{f_n\}$ converges on X to the function f.

   (a) Is it true that $f(x) = \lim f_n(x_n)$? Prove if true or provide counterexample.

   Solution. Consider following counterexample: $X = [0, 1],\ f_n(x) = x^n$ and $x_n = 1 - 1/n$. It is easy to see that $x_n \to 1$ as $n \to \infty$,

$$f_n(x) \to \begin{cases} 1, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

and that

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e}$$

But clearly, we have $1/e \neq f(1) = 1$. We definitely need stronger assumptions.

(b) Would you change your answer if you know that convergence of $\{f_n\}$ is uniform on X? Again, prove if true or provide counterexample.

Solution. No. Even uniform convergence does not guarantee us this result. To see this consider the following example with $f(x)$ being a Heaviside step function:

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

with $x_n = -1/n$ and $f_n(x) = f(x) + 1/n$. Clearly, $x_n \to 0$ and f_n converge uniformly to f but $\lim_{n\to\infty} f_n(x_n) = 0 \neq 1 = f(0)$. Actually, to get the result we seek, we need a uniform convergence to a continuous function f.

7. Some practice with contraction maps

(a) Let (X, d) be a space of continuous function on a closed interval $[0, \beta]$ with a supremum norm, i.e. $X = C([0, \beta])$, and $d(f, g) = \max_t |f(t) - g(t)|$, where $\beta < 1$. Define $T : X \to X$ by

$$(Tf)(t) = \int_0^t f(s)\, ds + g(t),$$

for some continuous function $g(t)$.

Show that T has a unique fixed point.[1]

Solution. Lets first show that $C([0, \beta])$ is a complete metric space, i.e. that an arbitrary Cauchy sequence $\{f_n(x)\}_{n=1}^{\infty}$ in $C([0, \beta])$ converges. In particular, we need to demonstrate that the limit of a Cauchy sequence of continuous functions on $[0, \beta]$, firstly, exists in the sense that for each $x \in [0, 1]$, $\lim f_n(x)$ exists, and secondly, pointwise limit function, $f(x)$, is continuous on $[0, \beta]$.

The first part follows immediately from the fact that Cauchy sequences of real numbers converge (to real numbers) by the completeness of $\mathbf{R}$ argument. Hence, we are guaranteed the existence of our pointwise limit function.

[1] Note that to invoke Contraction Mapping Theorem here, you need first to show that (X, d) is a complete metric space. Completeness of $\mathbf{R}$ will come to your rescue here.

Now, we just need to show continuity of $f(x)$. Fix $\epsilon > 0$ and consider $s, t \in [0,1]$. We have

$$\begin{aligned}
|f(s) - f(t)| &= |f(s) - f_n(s) + f_n(s) - f(t)| \\
&\leq |f(s) - f_n(s)| + |f_n(s) - f(t)| \\
&= |f(s) - f_n(s)| + |f_n(s) - f_n(t) + f_n(t) - f(t)| \\
&\leq |f(s) - f_n(s)| + |f_n(s) - f_n(t)| + |f_n(t) - f(t)| \quad (1)
\end{aligned}$$

From the definition of $d(f_n,\, f)$ and the fact that $d(f_n,\, f) \to 0$ as $n \to \infty$, there exists an $N(\epsilon)$ such that whenever $n > N(\epsilon)$, the first and last terms on the RHS of (1) are less than $\epsilon/3$. Furthermore, for any n, $f_n \in C([0,1])$. Fix some $n > N(\epsilon)$. (Uniform) Continuity implies that there is a $\delta_n > 0$ such that whenever $|s - t| < \delta_n \Rightarrow |f_n(s) - f_n(t)| < \epsilon/3$. Therefore, all three terms of the RHS of (*) are less than $\epsilon/3$. So we get that

$$|f(s) - f(t)| < \epsilon, \text{ whenever } |s - t| < \delta_n$$

This proves that f is (uniformly) continuous.

Now, we show that T is a contraction.

$$\begin{aligned}
d(Tf,\, Tg) &= \max_{t \in [0,\,\beta]} |Tf(t) - Tg(t)| \\
&= \max_{t \in [0,\,\beta]} \left| \int_0^t f(s)\, ds - \int_0^t g(s)\, ds \right| \\
&\leq \max_{t \in [0,\,\beta]} \int_0^t |f(s) - g(s)|\, ds \\
&\leq \int_0^\beta \max_{t \in [0,\,\beta]} \{|f(t) - g(t)|\}\, ds \\
&= \beta \cdot d(f,\, g)
\end{aligned}$$

(b) Now suppose (X, d) is an arbitrary complete metric space, but $T : X \to X$ is an *expansion,* i.e. there exists $\beta > 1$ such that $d(Tx, Ty) \geq \beta d(x, y)$ for all x, y in X, and that $T(X) = X$. Show that T has a fixed point.

Solution. We will do it in a sequence of steps. Firstly, we will verify that T is one-to-one and, thus, since it is also a surjection, the inverse must exists. Secondly, we will argue that T^{-1} is a contraction. Finally, we will invoke Contraction Mapping Theorem and show that the fixed point of T^{-1} is also a fixed point of T.

Step 1: verifying T is one-to-one, i.e. if $x_1 \neq x_2$ then $T(x_1) \neq T(x_2)$. But this is immediate since $x_1 \neq x_2$ implies directly that $d(T(x_1),\, T(x_2)) \geq$

$\beta d(x_1, x_2) > 0$. Therefore, T is one-to-one. By our assumption, T is onto, thus, there exists $T^{-1} : X \to X$ and $T \circ T^{-1} = id$.

Step 2: Show that T^{-1} is a contraction, i.e. there exists $\alpha < 1$ such that $d(T^{-1}(x_1), T^{-1}(x_2)) \leq \alpha d(x_1, x_2)$ for all $x_1, x_2 \in X$.
So, if T is one-to-one and onto, $T^{-1} : X \to X$ is also one-to-one and onto. Therefore, we can apply expansion T to any arbitrary image of T^{-1} since for any $x \in X \ : \ T^{-1}(x) \in X$

$$\begin{aligned} d(T(T^{-1}(x_1)), T(T^{-1}(x_2)) &\geq \beta d(T^{-1}(x_1), T^{-1}(x_2)) \qquad \iff \\ d(x_1, x_2) &\geq \beta d(T^{-1}(x_1), T^{-1}(x_2)) \qquad \iff \\ d(T^{-1}(x_1), T^{-1}(x_2)) &\leq \frac{1}{\beta} d(x_1, x_2) \end{aligned}$$

We have shown that there exists $\alpha = \frac{1}{\beta} < 1$ such that $d((T^{-1}(x_1), T^{-1}(x_2)) \leq \alpha d(x_1, x_2)$ for all $x_1, x_2 \in X$, i.e. T^{-1} is a contraction.
Step 3: By the Contraction Mapping Theorem, we can conclude that there exists x^*, a fixed point, such that $T^{-1}(x^*) = x^*$. Now, we show that x^* is also a fixed point of the expansion T.
Since $x^* = T^{-1}(x^*)$ is an identity, we can apply to each side any arbitrary transformation that preserve the equality. In particular, let's choose T

$$T(x^*) = T(T^{-1}(x^*))$$

which implies

$$T(x^*) = x^*$$

and we get the result we seek.