Economics 204
Fall 2011
Problem Set 2
Due Tuesday, August 2 in Lecture

1. Determine whether the following sets are open, closed, both or neither under the topology induced by the usual metric. (Hint: think about limit points of those sets.)

   (a) The interval $(0, 1)$ as a subset of $\mathbf{R}$

   (b) The interval $(0, 1)$ as a subset of $\mathbf{R}^2$, that is $\{(x, 0) \in \mathbf{R}^2 \mid x \in (0, 1)\}$

   (c) $\mathbf{R}$ as the subset of $\mathbf{R}$

   (d) $\mathbf{R}$ imbedded as a subset $\{(x, 0) \in \mathbf{R}^2 \mid x \in \mathbf{R}\}$ of $\mathbf{R}^2$

   (e) $\{1/n \mid n \in \mathbf{N}\}$ as a subset of $\mathbf{R}$

   (f) $\{1/n \mid n \in \mathbf{N}\}$ as a subset of the interval $(0, \infty)$.

2. Consider the following two sets:

$$A = \{(x, y) \in \mathbf{R}^2 \mid x^2 - y^2 \leq 3\}$$
$$B = \{(x, y) \in \mathbf{R}^2 \mid y > \sqrt{|x|}\}$$

   (a) Using the "pre-image of a closed set is closed (under a continuous function)" definition, determine whether sets A and B are open, closed, both or neither

   (b) Find their closure, exterior and boundary.

3. Let $A, B \subset X$. Suppose that $\text{int}\, A = \text{int}\, B = \emptyset$.

   (a) Prove that if A is closed in X, then $\text{int}\,(A \cup B) = \emptyset$.

   (b) Give an example with $\text{int}\,(A \cup B) \neq \emptyset$ if A isn't necessarily closed in X.

4. Let f be a monotonic, increasing function from $\mathbf{R}$ to $\mathbf{R}$. Suppose that $f(0) > 0$ and $f(100) < 100$. Prove that $f(x) = x$ for some $x \in (0, 100)$.

5. Call a mapping of X into Y *open* if $f(V)$ is an open set in Y whenever V is an open set in X. Prove that every continuous open mapping of $\mathbf{R}$ into $\mathbf{R}$ is monotonic.

6. Suppose that $\{x_n\}$ is a convergent sequence of points which lies, together with its limit x, in a set $X \subset \mathbf{R}^n$. In addition, suppose that $\{f_n\}$ converges on X to the function f.

   (a) Is it true that $f(x) = \lim f_n(x_n)$? Prove if true or provide counterexample.

   (b) Would you change your answer if you know that convergence of $\{f_n\}$ is uniform on X? Again, prove if true or provide counterexample.

7. Some practice with contraction maps

   (a) Let (X, d) be a space of continuous function on a closed interval $[0, \beta]$ with a supremum norm, i.e. $X = C([0, \beta])$, and $d(f, g) = \max_t |f(t) - g(t)|$, where $\beta < 1$. Define $T : X \to X$ by

   $$(Tf)(t) = \int_0^t f(s)\, ds + g(t),$$

   for some continuous function $g(t)$.
   Show that T has a unique fixed point.[1]

   (b) Now suppose (X, d) is an arbitrary complete metric space, but $T : X \to X$ is an *expansion,* i.e. there exists $\beta > 1$ such that $d(T(x), T(y)) > \beta d(x, y)$ for all x, y in X, and that $T(X) = X$. Show that T has a fixed point.

[1]Note that to invoke Contraction Mapping Theorem here, you need first to show that (X, d) is a complete metric space. Completeness of $\mathbf{R}$ will come to your rescue here.