Economics 204
Fall 2011
Problem Set 3 Suggested Solutions

1. For $x > 0$, define $f(x) = \frac{1}{2}(x + \frac{2}{x})$.
   (a) Show that if $X = [1, 2]$, then f is a contraction on X.
   (b) What is the fixed point of this contraction?
   (c) Show that if $X = (0, \infty)$, then f is not a contraction on X; that is, there does not exist $\beta \in (0, 1)$ such that
   $$\forall x, y \in X : |f(x) - f(y)| \leq \beta|x - y|.$$

Solution:

(a) First, X is a complete metric space as a closed subset of $\mathbb{R}$, which is complete. Then we need to show that f maps X into X. Since $f(x) = \frac{1}{2}x + \frac{1}{x}$, it suffices to show that $\frac{1}{2}x + \frac{1}{x} \geq 1$ and $\frac{1}{2}x + \frac{1}{x} \leq 2$ for all $x \in X$. The first follows from the fact that $\frac{1}{2}x \geq \frac{1}{2}$ for $x \geq 1$ and $\frac{1}{x} \geq \frac{1}{2}$ for $x \leq 2$. The second follows from the fact that $\frac{1}{2}x \leq 1$ for $x \leq 2$ and $\frac{1}{x} \leq 1$ for $x \geq 1$.
We now want to show: $|f(x) - f(y)| \leq \beta|x - y|$ for some $\beta < 1$ and all $x, y \in X$. Assuming $x \geq y$ without loss of generality, we get:
$$\begin{aligned} |f(x) - f(y)| &= |\frac{1}{2}(x + \frac{2}{x}) - \frac{1}{2}(y + \frac{2}{y})| \\ &= |\frac{1}{2}(x - y) + \frac{y - x}{xy}| \\ &= |(x - y)(\frac{1}{2} - \frac{1}{xy})| \\ &= |x - y| \cdot |\frac{1}{2} - \frac{1}{xy}| \\ &\leq \frac{1}{2}|x - y|, \end{aligned}$$
where the inequality follows from the fact that $\frac{1}{4} \leq \frac{1}{xy} \leq 1$ and therefore $\frac{1}{2} - \frac{1}{xy} \in [-\frac{1}{2}, \frac{1}{4}]$ and $|\frac{1}{2} - \frac{1}{xy}| \leq \frac{1}{2}$. So f is a contraction of modulus $\frac{1}{2}$.

(b) $\sqrt{2}$

This answer comes from solving $f(x) = x$:

$$\begin{aligned} \frac{1}{2}(x + \frac{2}{x}) &= x \\ \Leftrightarrow \frac{1}{x} &= \frac{1}{2}x \\ \Leftrightarrow 2 &= x^2 \\ \Leftrightarrow x &= \sqrt{2} \end{aligned}$$

By the Contraction Mapping Theorem, we know that this fixed point is unique.

(c) Notice that $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{2}(x + \frac{2}{x}) = \infty$. Thus the distance between $f(x)$ and $f(\sqrt{2}) = \sqrt{2}$ goes to infinity as x approaches 0, but the distance between x and $\sqrt{2}$ is bounded above by $\sqrt{2}$. If f were a contraction of modulus β, the distance between $f(x)$ and $f(\sqrt{2})$ for small x would have been bounded above by $\beta\sqrt{2}$. Therefore f is not a contraction.

2. (a) Show that boundedness and total boundedness are equivalent under the usual metric in $\mathbb{R}^n$. (In class we showed that total boundedness is a stronger condition than boundedness. Now you need to supply only the other direction.)

(b) For $x, y \in \mathbb{R}^n$, define $\rho(x, y) = \min\{d(x, y), 1\}$, where d is the usual metric. Show that $E \subset \mathbb{R}^n$ is bounded with respect to d iff E is totally bounded with respect to ρ.

Solution:

(a) We need to show that under the usual metric in $\mathbb{R}^n$, boundedness implies total boundedness. Fix $\varepsilon > 0$ and let E be a bounded set. In particular, let $||a|| < M\frac{\varepsilon}{\sqrt{n}}$ for some $M \in \mathbb{N}$ and for all $a \in E$. For ease of notation, let $\frac{\varepsilon}{\sqrt{n}} = \delta$. (See below for an explanation about why we care about this ratio) Now let:

$$X = \{(a_1, \ldots, a_n) : a_i \in \{-M\delta, \ldots, -\delta, 0, \delta, 2\delta, \ldots, (M-1)\delta, M\delta\}\}.$$

Notice that X is a finite set with $(2M + 1)^n$ elements. Now it suffices to show that the union of ε-balls centered around elements of X covers the ball $B_{M\delta}(0)$, which in turn contains E. (Notice that this uses de la Fuente's definition of total boundedness, which is equivalent to the one presented in class.) To see that, notice that

$$||a|| < M\delta \Rightarrow \sum_{i=1}^{n} |a_i|^2 < (M\varepsilon)^2 \Rightarrow \forall i : |a_i| < M\delta.$$

This implies: $E \subseteq B_{M\delta}(0) \subset [-M\delta, M\delta]^n$. Now it is enough to show:

$$[-M\delta, M\delta]^n \subset \cup\{B_\varepsilon(x)\}_{x\in X}.$$

Let $a \in [-M\delta, M\delta]^n$. By construction, there is an element x of X that is within at most $\frac{1}{2}\delta$ along all dimensions from a. Thus the distance from a to the nearest element of X cannot exceed

$$\sqrt{n(\frac{1}{2}\delta)^2} = \frac{1}{2}\delta\sqrt{n} = \frac{1}{2}\frac{\varepsilon}{\sqrt{n}}\sqrt{n} = \frac{1}{2}\varepsilon < \varepsilon.$$

This concludes the proof. The intuition behind the proof is quite simple: we constructed a finite grid of points (the set X) that is fine enough and large enough so that each element of E is within ε of an element of X. At first blush, it might seem sufficient to choose the grid so that its elements have coordinates that are multiples of ε. However, this is not enough for large n ($n > 4$), since in those cases the point $(1/2, \ldots, 1/2)$ is at a distance greater than 1 from all the vertices of the hypercube $[0, 1]^n$. That is why we needed to choose a finer grid by dividing ε by $\sqrt{n}$.

(b) Assume that E is totally bounded under ρ. Then for all $\varepsilon < 1$, there is a set $\{x_1, \ldots, x_k\} \subseteq E$ such that for all elements a of E we have: $\rho(a, x_i) < \varepsilon < 1$ for some x_i. By definition, if $\rho(a, x_i) < 1$, then $\rho(a, x_i) = d(a, x_i)$. Hence $\{x_1, \ldots, x_k\}$ is also an ε-net for E with respect to d. For $\varepsilon \geq 1$, the $\frac{1}{2}$-net can also serve as the ε-net with respect to d. Thus E is totally bounded and therefore bounded with respect to d.

Now assume that E is bounded with respect to d. Then by part (a), E is totally bounded with respect to d. Fix ε. Notice that if $\varepsilon \geq 1$, $E \subseteq B_\varepsilon(a)$ for any $a \in E$, where $B_\varepsilon(a)$ is defined with respect to the metric ρ. This gives us a trivial ε-net for E under ρ.

Let $\varepsilon < 1$ instead. Using the same argument as above, the ε-net with respect to d is also the ε-net with respect to ρ. Hence E is totally bounded under ρ.

3. Show that the set of cluster points of a bounded sequence in $\mathbb{R}^n$ is non-empty and compact.

Solution: Recall the sequential characterization of cluster points - c is a cluster point of a sequence iff the sequence has a subsequence converging to c. It will be useful in the following proof.

Non-empty. Denote the sequence by $\{x_n\}$ and let the set of its cluster points be Ω. The set Ω is non-empty by Bolzano-Weierstrass' theorem: $\{x_n\}$ is bounded and thus has a convergent subsequence. Therefore $\{x_n\}$ has at least one cluster point.

Now to show that Ω is also compact, we will show that it is closed and bounded.

Bounded. First, since $\{x_n\}$ is bounded

$$\exists M \in \mathbb{R} : \forall m, n \in \mathbb{N}, ||x_n - x_m|| \leq M.$$

Assume toward contradiction that Ω is not bounded. Then

$$\exists \omega, \omega' \in \Omega : ||\omega - \omega'|| > M + 2\varepsilon$$

for some $\varepsilon > 0$. Since ω and ω' are cluster points of $\{x_n\}$, there exist elements of the sequence x and x' such that $x \in B_\varepsilon(\omega)$ and $x' \in B_\varepsilon(\omega')$. By the triangle inequality:

$$\begin{aligned} ||\omega - \omega'|| &\leq ||\omega - x|| + ||x - x'|| + ||x' - \omega'|| \\ \Rightarrow ||x - x'|| &\geq ||\omega - \omega'|| - ||\omega - x|| - ||x' - \omega'|| \\ &> M + 2\varepsilon - \varepsilon - \varepsilon \\ &= M \end{aligned}$$

But M was chosen so that $||x_n - x_m|| \leq M$ for all n, m. Contradiction!

Closed. Let $\{\omega_n\}$ be a sequence in Ω that converges to some c. If we show $c \in \Omega$, we would be done. Assume toward contradiction that $c \notin \Omega$ and hence c is not a cluster point of $\{x_n\}$. Then there is some ε such that $A = \{n : x_n \in B_\varepsilon(c)\}$ is a finite set.

However, $\{\omega_n\} \to c$ so there is some $\omega \in B_{\varepsilon/2}(c) \cap \{\omega_n\}$. Furthermore, ω is a cluster point of $\{x_n\}$ so $B = \{n : x_n \in B_{\varepsilon/2}(\omega)\}$ is infinite. However, notice that since $||\omega - c|| < \varepsilon/2$, we have $B \subseteq A$. But this is impossible since B is infinite and A finite. Contradiction! So Ω is closed.

4. (a) For some metric space X, fix $p \in X$ and $\delta > 0$. Define A by $A = \{q \in X : d(p,q) < \delta\}$ and B by $B = \{q \in X : d(p,q) > \delta\}$. Prove that A and B are separated.

   (b) Prove that every connected metric space with at least two points is uncountable.

Solution:

(a) We want to show that A and B are separated, i.e. $\bar{A} \cap B = A \cap \bar{B} = \emptyset$. Let's show it for $A \cap \bar{B}$ first.

First of all, it is clear that if $B = \emptyset$, $A \cap \bar{B} = \emptyset$ is satisfied. Instead, let $B \neq \emptyset \neq \bar{B}$.

Assume toward contradiction that $A \cap \bar{B} \neq \emptyset$ and let $q \in A \cap \bar{B}$. Then $q \in A$ and $d(p,q) < \delta$. Fix $\varepsilon = \delta - d(p,q) > 0$. Then since $q \in \bar{B}$, $\exists x \in B \cap B_\varepsilon(q)$. But then:

$$\begin{aligned} d(p,x) &\leq d(p,q) + d(q,x) \\ &< d(p,q) + \varepsilon \\ &= d(p,q) + (\delta - d(p,q)) \\ &= \delta \\ &\Rightarrow x \in A, \end{aligned}$$

but $x \in B$ and A and B are disjoint - contradiction!

Now, let us show $\bar{A} \cap B = \emptyset$ and, analogously to the above, let $A \neq \emptyset$ and assume toward contradiction $\bar{A} \cap B \neq \emptyset$. For $q \in \bar{A} \cap B$, we have: $q \in B \Rightarrow d(p,q) > \delta$. Since $q \in \bar{A}$, for $\varepsilon = d(p,q) - \delta > 0$

we have $\exists x \in A \cap B_\varepsilon(q)$. Then:

$$\begin{aligned} d(p,q) &\leq d(p,x) + d(x,q) \\ \Rightarrow d(p,x) &\geq d(p,q) - d(x,q) \\ &> d(p,q) - \varepsilon \\ &= d(p,q) - (d(p,q) - \delta) \\ &= \delta \end{aligned}$$

This implies $x \in B$, but $x \in A$ - contradiction!

(b) Let the metric space (X, d) be connected and let $x, y \in X$ with $x \neq y$. Fix $0 < \delta < d(x, y)$. We want to show that there is some $z \in X$ such that $d(x, z) = \delta$, so assume toward contradiction that there is no such z. Then:

$$X = \{q \in X : d(x,q) < \delta\} \cup \{q \in X : d(x,q) > \delta\}.$$

But by part (a) then, these two sets are separated and X is not connected - contradiction! (It is crucial that both of these sets are non-empty: $x \in \{q \in X : d(x,q) < \delta\}$ and $y \in \{q \in X : d(x,q) > \delta\}$.)

This implies that for all $\delta < d(x,y), \exists z_\delta \in X : d(x, z_\delta) = \delta$. Since $d(x,y)$ is a real number, there are uncountably many such δ-s. Hence X is uncountable as well.

5. Let X be a compact metric space and let $\{U_i\}_{i \in I}$ be an open cover of X. Show that there exists some real number $\varepsilon > 0$ such that any closed ball in X of radius ε is entirely contained in at least one set U_i. (Hint: Assume not and take aberrant balls of radii $1, 1/2, 1/3, \ldots$ and then use the fact that X is compact.)

Solution: Assume toward contradiction that this does not hold. Then we take a sequence of balls with radii $1/n$ for $n \in \mathbb{N}$, such that:

$$\bar{B}_{1/n}(x_n) \cap (U_i)^C \neq \emptyset$$

for all $i \in I$.

Consider the sequence $\{x_1, x_2, \ldots\}$. It is a sequence in the compact metric space X so it has a convergent subsequence. Denote the convergent subsequence by $\{y_m\}$ with $\{y_m\} \to y \in X$ and let the radius corresponding to each y_m be ε_m. Now we have:

$$\bar{B}_{\varepsilon_m}(y_m) \cap (U_i)^C \neq \emptyset$$

for all $i \in I$ and all m.

The collection $\{U_i\}_{i \in I}$ is an open cover of X and $y \in X$. Therefore $y \in U_i$ for some $i \in I$. The set U_i is open, so there is some $\varepsilon > 0$ such that $B_\varepsilon(y) \subseteq U_i$. Since $\{y_m\} \to y$, there is a tail of $\{y_m\}$ contained in $B_{\varepsilon/2}(y)$. Take some y_m from that tail such that $\varepsilon_m < \varepsilon/2$ (this ε_m exists since by the Archimedean property there are only finitely many ε_m, for which $\varepsilon_m \geq \varepsilon/2$). But then:

$$\bar{B}_{\varepsilon_m}(y_m) \subset \bar{B}_\varepsilon(y) \subseteq U_i,$$

which is a contradiction!

6. Let X and Y be two non-empty sets and $\Gamma : X \to 2^Y$ a correspondence. We say that Γ is *injective* if $\Gamma(x) \cap \Gamma(x') = \emptyset$ for any distinct $x, x' \in X$, and that it is *surjective* if $\Gamma(X) = Y$, where the image of a set is defined by $\Gamma(S) = \cup\{\Gamma(x) : x \in S\}$. Finally, Γ is *bijective* if it is both injective and surjective. Prove that Γ is bijective iff $\Gamma = f^{-1}$ for some $f : Y \to X$.

   Solution: Assume that Γ is bijective. Define the correspondence $F : Y \to 2^X$ by $F(y) = \{x : y \in \Gamma(x)\}$. Notice that $F(y)$ is a singleton for all $y \in Y$, since $F(y)$ is at most a singleton for all $y \in Y$ (follows from Γ being injective), and $F(y)$ is non-empty for all $y \in Y$ (follows from Γ being surjective). Since $F(y)$ is always a singleton, we can analyze it as a function $f : Y \to X$. Now consider the inverse of f:

$$f^{-1}(x) = \{y \in Y : f(y) = x\} = \{y \in Y : y \in \Gamma(x)\} = \Gamma(x),$$

   where the second equality follows from the definition of F and f.

   Now assume that $\Gamma = f^{-1}$ for some $f : Y \to X$. We need to show that Γ is injective and surjective.

   The injective part follows from the fact that $f^{-1}(x) \cap f^{-1}(x') = \emptyset$, since otherwise we would have $f(y) = x$ *and* $f(y) = x'$. The surjective part follows from the fact that for all $y \in Y$, $y \in \Gamma(f(y))$; thus since $f(y) \in X$, $\Gamma(X) = Y$.

7. Define the correspondence $\Gamma : [0,1] \to 2^{[0,1]}$ by:

$$\Gamma(x) = \begin{cases} [0,1] \cap \mathbb{Q} & \text{if } x \in [0,1] \backslash \mathbb{Q} \\ [0,1] \backslash \mathbb{Q} & \text{if } x \in [0,1] \cap \mathbb{Q} \end{cases}.$$

Show that Γ is not continuous, but it is lhc. Is Γ uhc at any rational? At any irrational? Does this correspondence have a closed graph?

Solution: The correspondence is not uhc at any rational number q in the interval $[0,1]$. To see that consider the open set $(0,1)$, which contains $\Gamma(q) = [0,1] \backslash \mathbb{Q}$. However, any open set containing q also contains an irrational number i and $\Gamma(i) \not\subset (0,1)$ since $\{0,1\} \subset \Gamma(i)$.

The correspondence is not uhc at any irrational number i in the interval $[0,1]$ either. To see that consider the open set $(-1/2, \pi/4) \cup (\pi/4, 3/2)$, which contains $\Gamma(i) = [0,1] \cap \mathbb{Q}$. However, any open set containing i also contains a rational number q and $\Gamma(q) \not\subset (-1/2, \pi/4) \cup (\pi/4, 3/2)$, since $\pi/4 \in \Gamma(q)$.

Thus Γ is nowhere uhc and hence nowhere continuous. Now we'll show that it is lhc. Fix $q \in [0,1] \cap \mathbb{Q}$ and consider some open set $U \subset \mathbb{R}$ such that $U \cap \Gamma(q) \neq \emptyset$. Now it suffices to show that $U \cap \Gamma(x) \neq \emptyset$ for all $x \in [0,1]$. If x is rational, we have $\Gamma(x) = \Gamma(q)$ and hence $U \cap \Gamma(x) \neq \emptyset$. If x is irrational, notice that since U is an open set that contains an irrational number in the interval $[0,1]$, it must also contain a rational number in the same interval and therefore its intersection with $\Gamma(x) = [0,1] \cap \mathbb{Q}$ is non-empty. The proof for $i \in [0,1] \backslash \mathbb{Q}$ and an open U such that $U \cap \Gamma(i) \neq \emptyset$ is analogous.

The correspondence does not have a closed graph. If it did, by a theorem proven in lecture, Γ would be uhc since its codomain $[0,1]$ is compact. But we saw that Γ is nowhere uhc. Alternatively, we can use the fact that a closed graph implies that the correspondence is closed-valued. However, neither $[0,1] \cap \mathbb{Q}$ nor $[0,1] \backslash \mathbb{Q}$ are closed. Therefore, Γ cannot have a closed graph.