Economics 204
Fall 2011
Problem Set 3
Due Friday, August 5 in Lecture

1. For $x > 0$, define $f(x) = \frac{1}{2}(x + \frac{2}{x})$.
   (a) Show that if $X = [1, 2]$, then f is a contraction on X.
   (b) What is the fixed point of this contraction?
   (c) Show that if $X = (0, \infty)$, then f is not a contraction on X; that is, there does not exist $\beta \in (0, 1)$ such that[1]
   $$\forall x, y \in X : |f(x) - f(y)| \leq \beta |x - y|.$$
2. (a) Show that boundedness and total boundedness are equivalent under the usual metric in $\mathbb{R}^n$. (In class we showed that total boundedness is a stronger condition than boundedness. Now you need to supply only the other direction.)
   (b) For $x, y \in \mathbb{R}^n$, define $\rho(x, y) = \min\{d(x, y), 1\}$, where d is the usual metric. Show that $E \subset \mathbb{R}^n$ is bounded with respect to d iff E is totally bounded with respect to ρ.
3. Show that the set of cluster points of a bounded sequence in $\mathbb{R}^n$ is non-empty and compact.
4. (a) For some metric space X, fix $p \in X$ and $\delta > 0$. Define A by $A = \{q \in X : d(p, q) < \delta\}$ and B by $B = \{q \in X : d(p, q) > \delta\}$. Prove that A and B are separated.
   (b) Prove that every connected metric space with at least two points is uncountable.
5. Let X be a compact metric space and let $\{U_i\}_{i \in I}$ be an open cover of X. Show that there exists some real number $\varepsilon > 0$ such that any

[1] Strictly speaking, the definition we saw in class requires that contractions be defined on complete metric spaces. However, completeness is not necessary for the definition to make sense. For this part only, you should assume that the definition does not require completeness.

closed ball in X of radius ε is entirely contained in at least one set U_i. (Hint: Assume not and take aberrant balls of radii $1, 1/2, 1/3, \dots$ and then use the fact that X is compact.)

6. Let X and Y be two non-empty sets and $\Gamma : X \to 2^Y$ a correspondence. We say that Γ is *injective* if $\Gamma(x) \cap \Gamma(x') = \emptyset$ for any distinct $x, x' \in X$, and that it is *surjective* if $\Gamma(X) = Y$, where the image of a set is defined by $\Gamma(S) = \cup\{\Gamma(x) : x \in S\}$. Finally, Γ is *bijective* if it is both injective and surjective. Prove that Γ is bijective iff $\Gamma = f^{-1}$ for some $f : Y \to X$.

7. Define the correspondence $\Gamma : [0, 1] \to 2^{[0,1]}$ by:

$$\Gamma(x) = \begin{cases} [0,1] \cap \mathbb{Q} & \text{if } x \in [0,1] \backslash \mathbb{Q} \\ [0,1] \backslash \mathbb{Q} & \text{if } x \in [0,1] \cap \mathbb{Q} \end{cases}.$$

   Show that Γ is not continuous, but it is lhc. Is Γ uhc at any rational? At any irrational? Does this correspondence have a closed graph?