Economics 204
Fall 2011
Problem Set 4 Suggested Solutions

1. Determine whether or not each of the following sets is a vector space. In case it is, find the dimension of the space and a Hamel basis for it.

   (a) The set of solutions in $\mathbf{R}^3$ to the following system of linear equations, with vector addition and scalar multiplication defined in the usual way

$$\begin{aligned} x_1 - 5x_2 + 2x_3 &= 0 \\ 5x_1 + 3x_2 - x_3 &= 0 \end{aligned}$$

   Solution. Here we will only show it is closed under vector addition and scalar multiplication. So, for any two elements u and v from this set and scalars α and β, since both u and v satisfy both equations, it is easy to see that $\alpha u + \beta v$ also make the above equation equal to 0. $\{(1, 1, 1)\}$ is one of the Hamel basis and its dimension is 1. The checks of all other conditions are just as straightforward.

   (b) The set of $n \times n$ matrices having a trace equal to one, with matrix addition and scalar multiplication defined in the usual way [1]

   Solution. No. It is not closed under addition.

   (c) The set of $m \times n$ matrices having all their elements sum-up to zero, with matrix addition and scalar multiplication defined in a usual way

   Solution. Again, here we will only show it is closed under vector addition and scalar multiplication. So, for any two matrices A and B from this set and scalars α and β, since both A and B have all their elements sum-up to 0, it is easy to see that the sum of all elements of $\alpha A + \beta B$ is also equal to 0. One Hamel basis is given by $mn - 1$ matrices

$$\{M_{ij} : \ 1 \le i \le m, \, 1 \le j \le n\}$$

   where

$$(M_{ij})_{k\ell} = \begin{cases} 1 & \text{if } k = i, \ \ell = j \text{ and } k \neq m, \ \ell \neq n \\ -1 & \text{if } k = m, \ \ell = n \\ 0 & \text{otherwise.} \end{cases}$$

   Clearly, the dimension of the space is $mn - 1$.

[1] The trace of an $n \times n$ matrix M, denoted $tr(M)$, is the sum of the diagonal entries of M

(d) The set of 2×1 matrices with real entries, with vector addition and scalar multiplication defined as

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

Solution. No. Observe that addition defined in such a way is not commutative. Take for instance two standard basis vectors in $\mathbf{R}^2$.

(e) All strictly positive reals $\mathbf{R}_{++} = \{x \in \mathbf{R} \mid x > 0\}$, with vector addition defined as $x + y = x \cdot y$ and scalar multiplication defined as $\lambda x = x^{\lambda}$.

Solution. Again, here we will only show it is closed under vector addition and scalar multiplication. So, for any two strictly positive reals $x,\ y \in \mathbf{R}_{++}$ and scalars α and β, it is easy to see that $\alpha x + \beta y = x^{\alpha} \cdot y^{\beta}$ is a strictly positive real itself. The vector space axioms are satisfied with additive identity equal to 1 and additive inverse equal to $1/x$, they follow from the field properties of $\mathbf{R}$. One Hamel basis is $\{2\}$ and the dimension of the space is 1.

2. Let A and B be subspaces of a vector space V. Are the following assertions true? Always? Sometimes? Never?

(a) $A \cap B$ is a subspace?

Solution. Always. Take any two vectors $v,\ w \in A \cap B$ (we can always do that because the intersection is non-empty; do you see why?) and two scalars α and β . Because A and B are subspaces themselves, the linear combination $\alpha v + \beta w$ is in A and in B as well. Consequently, $\alpha v + \beta w \in A \cup B \implies A \cup B$ is closed under linear combination of two vectors. We are done.

(b) $A \cup B$ is a subspace?

Solution. Sometimes. It can only happen if one subspace is “bigger,” i.e. $A \subset B$ or $B \subset A$.

To see that the answer is not “always,” take V to be $\mathbf{R}^3$, take A to be subspace generated by the first standard basis vector, e_1, and B — generated by the second standard basis vector, e_2

$$A = \{\,\alpha e_1 :\ \alpha \in R\}$$
$$B = \{\,\alpha e_2 :\ \alpha \in R\}$$

Notice that $e_1 + e_2 \notin A \cup B$ as the sum is neither in A nor in B.

Observe that the answer is not “never,” because if $A \subset B$ or $B \subset A$ then clearly $A \cup B$ is a subspace. We prove that $A \cup B$ is a subspace only if either $A \subset B$ or $B \subset A$ by contrapositive, i.e. we assume that $A \not\subset B$ and $B \not\subset A$ to show that the union is not a subspace. The assumption that A is not a subset of B means that there is an $a \in A$ with $a \notin B$. The other assumption gives a $b \in B$ with $b \notin A$. Consider $a + b$ and note that the sum is not an element of A or else $(a + b) - a$ would be in A, which would lead to a contradiction. Similarly, the sum is not an element of B.

(c) If A is a subspace, then its complement is also a vector subspace.

Solution. Never. Note that $A^c = V \setminus A$, therefore $\vec{0} \notin A^c$ as it is contained both in V and in A.

3. Let U be a subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : \ x_2 = \frac{1}{2}x_4 \text{ and } x_1 = x_5\}.$$

Find a basis of U.

Solution. Because of the linear dependence of x_2 on x_4 and x_1 on x_5, our $U = \text{span}\{v_1, v_2, v_3\}$, where $v_1 = (0, 1, 0, 2, 0)$, $v_2 = (1, 0, 0, 0, 1)$ and $v_3 = (0, 0, 1, 0, 0)$. To see that this is true we will first check that span $\{ v_1, v_2, v_3\} \subset V$. Take $v \in U$, we have

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = (\, \alpha_2, \, \alpha_1, \, \alpha_3, \, 2\alpha_1, \, \alpha_2)$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$. By inspection, we find that $v \in U$.

Conversely, let $v = (\, x_1, \, x_2, \, x_3, \, 2x_2, \, x_1) \in U$, for some $x_1, x_2, x_3 \in \mathbf{R}$. We then obtain: $v = x_2 v_1 + x_1 v_2 + x_3 v_3$, showing that $U \subset \text{span}\{ v_1, v_2, v_3\}$.

4. Let $T : X \to Y$ be a linear transformation and U a subspace of X. Prove that the image of U under T, $T(U) = \{ T(u) \mid u \in U\}$ is a subspace of Y.

Solution. First, note that the image of T is non-empty, because U is non-empty. Now, consider $v_1, v_1, \ldots, v_n \in U$. By linearity of T we have

$$\alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots \alpha_n T(v_n) = T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n),$$

which is itself in $T(U)$ since $\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n \in U$. Thus, we have shown that $T(U)$ is closed under taking finite linear combinations, and we are done.

5. Let $T : V \to V$ be a linear transformation. Suppose that there is an $v \in V$ such that $T^n(v) = 0$ but $T^{n-1}(v) \neq 0$ for some $n > 0$. Prove that $v, \ T(v), T^2(v), \ \ldots,$ $T^{n-1}(v)$ are linearly independent.

Solution. By contradiction. Suppose that $v,\ T(v), T^2(v),\ \ldots,\ T^{n-1}(v)$ are linearly dependent, then there exist non-zero scalars, $\alpha_0, \alpha_1,\ \ldots,\ \alpha_{n-1} \in \mathbf{R}$ such that

$$\alpha_0 v + \alpha_1 T(v) + \ldots,\ \alpha_{n-1} T^{n-1}(v) = 0$$

Applying T^{n-1} to both sides and using linearity, we get that

$$\alpha_0 T^{n-1} v + \alpha_1 T^n(v) + \ldots,\ \alpha_{n-1} T^{2n-2}(v) = 0.$$

Thus, $\alpha_0 = 0$ and proceeding in a similar fashion yields $\alpha_1 = 0$, $\alpha_2 = 0$, ..., $\alpha_{n-1} = 0$. Contradiction.

6. Let $T : V \to V$ be a linear transformation. Prove that

$$\ker T \cap \mathrm{Im} T = \{0\} \implies \ker T = \ker T^2.$$

Solution. First, it is easy to see that $\ker T \subset \ker T^2$. Take any $v \in \ker T$ and apply T twice to it to get

$$T^2(v) = T(T(v)) = T(0) = 0 \implies x \in \ker T^2.$$

To show the set inclusion in the other direction, lets proceed by contradiction, and suppose that there exists $v \in \ker T^2 \setminus \ker T$. If that is the case, we must have $T(v) \neq 0$, but $T(T(v)) = 0$. But, this means $T(v) \in \ker T$ and at the same time, $T(v) \in \mathrm{Im} T$. We have a contradiction, that proves our result.

7. Let V be finite dimensional and $T : V \to W$ a linear transformation. Prove that T is surjective if and only if there exists $S \in L(W, V)$ such that TS is an identity map on W.

Solution. First, lets suppose that T is onto. Take a basis $w_1, w_2, \ldots, w_n$ of W. It is finite dimensional by Rank-Nullity Theorem. By the surjectivity of T for each j, there exists $v_j \in V$ such that $w_j = T(v_j)$. Lets define a linear transformation $S : W \to V$ as

$$S(\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n) = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

Clearly, we have then

$$\begin{aligned}(TS)(\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n) &= T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \\ &= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) = \\ &= \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n.\end{aligned}$$

Which shows that $TS \in L(W, W)$ is an identity map.

Now, lets suppose that there exists $S \in L(W, V)$ such that TS is the identity map on W. Take any $w \in W$, then $w = T(S(w))$, and therefore $w \in \mathrm{Im}\, T$. But this means that $\mathrm{Im}\, T = W$, or that T is surjective.

8. Call v a *right null* vector of a symmetric matrix A if $Av = 0$, and similarly a *left null* vector if $v^T A = 0$. Let $n \times n$ symmetric matrix A be diagonalizable and have a one-dimension null space. Prove that a non-zero left null vector of A cannot be orthogonal to a non-zero right null vector.

   Solution. Lets assume to contradiction that we have $v \cdot u = 0$ for a right null vector of v and left null vector u of a diagonalizable matrix A. The Theorem 10 in Lecture 9 guarantees us that we can choose a basis $w_1, w_2, \ldots, w_n$ of $\mathbf{R}^n$ consisting of eigenvectors of A. Notice that since v is a right null vector it must be the case that v is an eigenvector corresponding to $\lambda = 0$. Also, observe that the null space of A is assumed to be one-dimensional. The latter allows us to deduce two things: first is that one of the basis vectors $w_1, w_2, \ldots, w_n$ is a multiple of v, so lets assume without any loss of generality that $w_1 = v$. Second is that eigenvalues λ_j associated with eigenvectors w_j are nonzero for all other $j > 1$.

   Thus, for $j > 1$ we get

$$uw_j = \frac{1}{\lambda_j} u(Aw_j) = \frac{1}{\lambda_j}(uA)w_j = 0.$$

   i.e. that u is orthogonal to each of the basis vectors $w_1, w_2, \ldots, w_n$.

   We claim that u must be zero. To see this note that since $w_1, w_2, \ldots, w_n$ are basis vectors, there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zero such that

$$u = \alpha_1 \cdot w_1 + \alpha_2 \cdot w_2 + \cdots + \alpha_n \cdot w_n$$

   taking a dot product with u on both sides we get

$$\begin{aligned} ||u||^2 &= \alpha_1 \cdot (u\, w_1) + \alpha_2 \cdot (u\, w_2) + \cdots + \alpha_n \cdot (u\, w_n) \\ &= \alpha_1 \cdot 0 + \alpha_2 \cdot 0 + \cdots + \alpha_n \cdot 0 \\ &= 0 \end{aligned}$$

   Therefore, u must be a zero vector. We have arrived at contradiction, which proves the result we seek.