Economics 204
Fall 2011
Problem Set 4
Due Tuesday, August 9 in Lecture

1. Determine whether or not each of the following sets is a vector space. In case it is, find the dimension of the space and a Hamel basis for it.

   (a) The set of solutions in $\mathbf{R}^3$ to the following system of linear equations, with vector addition and scalar multiplication defined in the usual way

   $$\begin{aligned} x_1 - 5x_2 + 2x_3 &= 0 \\ 5x_1 + 3x_2 - x_3 &= 0 \end{aligned}$$

   (b) The set of $n \times n$ matrices having a trace equal to one, with matrix addition and scalar multiplication defined in the usual way [1]

   (c) The set of $m \times n$ matrices having all their elements sum-up to zero, with matrix addition and scalar multiplication defined in a usual way

   (d) The set of 2×1 matrices with real entries, with vector addition and scalar multiplication defined as

   $$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

   (e) All strictly positive reals $\mathbf{R}_{++} = \{x \in \mathbf{R} \mid x > 0\}$, with vector addition defined as $x + y = x \cdot y$ and scalar multiplication defined as $\lambda x = x^{\lambda}$.

2. Let A and B be subspaces of a vector space V. Are the following assertions true? Always? Sometimes? Never?

   (a) $A \cap B$ is a subspace?

   (b) $A \cup B$ is a subspace?

   (c) If A is a subspace, then its complement is also a vector subspace.

3. Let U be a subspace of $\mathbf{R}^5$ defined by

   $$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : \; x_2 = \frac{1}{2}x_4 \text{ and } x_1 = x_5\}.$$

   Find a basis of U.

[1] The trace of an $n \times n$ matrix M, denoted $tr(M)$, is the sum of the diagonal entries of M

4. Let $T : X \to Y$ be a linear transformation and U a subspace of X. Prove that the image of U under T, $T(U) = \{\, T(u) \mid u \in U\}$ is a subspace of Y.

5. Let $T : V \to V$ be a linear transformation. Suppose that there is an $v \in V$ such that $T^n(v) = 0$ but $T^{n-1}(v) \neq 0$ for some $n > 0$. Prove that $v,\ T(v), T^2(v),\ \ldots,$ $T^{n-1}(v)$ are linearly independent.

6. Let $T : V \to V$ be a linear transformation. Prove that

$$\ker T \cap \mathrm{Im} T = \{0\} \implies \ker T = \ker T^2.$$

7. Let V be finite dimensional and $T : V \to W$ a linear transformation. Prove that T is surjective if and only if there exists $S \in L(W, V)$ such that TS is an identity map on W.

8. Call v a *right null* vector of a symmetric matrix A if $Av = 0$, and similarly a *left null* vector if $v^T A = 0$. Let $n \times n$ symmetric matrix A be diagonalizable and have a one-dimension null space. Prove that a non-zero left null vector of A cannot be orthogonal to a non-zero right null vector.