Economics 204
Fall 2011
Problem Set 5 Suggested Solutions

1. (a) Prove that $y = h^3$ is both $o(|h|^2)$ as $h \to 0$ and $O(|h|^3)$ as $h \to 0$.
   (b) Prove that $y = sin(h)$ is not $o(|h|)$ as $h \to 0$ but is $O(|h|)$ as $h \to 0$. (You may use the fact that $|sin(h)| \leq |h|$).

   Solution:

   (a) From class we know that $y = O(|h|^{n+1})$ as $h \to 0$ implies $y = o(|h|^n)$ as $h \to 0$. So it suffices to show that h^3 is big-Oh of $|h|^3$ as $h \to 0$. But this follows trivially from the definition since for all h, we have $|h^3| = |h|^3$ (the constant K from the definition equals 1 here).

   (b) The fact that $\sin(h)$ is $O(|h|)$ as $h \to 0$ again follows directly from the definition and from the fact that $|sin(h)| \leq |h|$ (again we have $K = 1$).

   To show that $\sin(h)$ is not $o(|h|)$ as $h \to 0$, we use L'Hopital's rule since both the numerator and the denominator of $\frac{|\sin(h)|}{|h|}$ tend to 0 as h gets small:

$$\lim_{h \downarrow 0} \frac{|\sin(h)|}{|h|} = \lim_{h \downarrow 0} \frac{\sin(h)}{h} = \lim_{h \downarrow 0} \frac{\cos(h)}{1} = \frac{1}{1} = 1 \neq 0.$$

   Similarly, it can be shown that the limit is 1 for $h \uparrow 0$.

2. (a) Prove that the following identity holds for $-1 < x \leq 1$:

$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.$$

   (b) Find the second-order Taylor expansion of:

$$f(x, y) = -x^2 + 2xy + 3y^2 - 6x - 2y - 4$$

   around $(x, y) = (-\pi/4, \ln 42)$.

(c) Find the second-order Taylor expansion of $g(x, y) = y^x$ around $(x, y) = (1, 1)$.

Solution:

(a) This part of the problem turned out to be harder than intended. Apologies for any unnecessary frustration this might have caused!

Consider the n-th order term of the Taylor expansion of $\ln(1 + x)$ around $\ln(1)$; it equals $\frac{\ln^{(n)}(1)x^n}{n!}$, where $\ln^{(n)}(1)$ denotes the n-the derivative of the natural logarithm function, evaluated at 1.

Claim:$\ln^{(n)}(a) = (-1)^{n+1}(n-1)!\frac{1}{a^n}$ for all $n \in \mathbb{N}$

We can show that using induction. The base step $n = 1$ is easy: $\ln'(a) = 1/a = (-1)^{1+1}(1-1)!\frac{1}{a^1}$.

For the induction step, assume that the formula holds for some n. Consider $n + 1$ now:

$$\begin{aligned}\ln^{(n+1)}(a) &= \frac{d}{da}\ln^{(n)}(a) \\ &= \frac{d}{da}(-1)^{n+1}(n-1)!\frac{1}{a^n} \\ &= (-1)^{n+1}(n-1)!(-n)\frac{1}{a^{n+1}} \\ &= (-1)^{(n+1)+1}((n+1)-1)!\frac{1}{a^{n+1}},\end{aligned}$$

which completes the proof of the claim.

Plugging 1 into the formula, we get the expression for the n-th order term of the Taylor expansion of $\ln(1+x)$ around $\ln(1)$ to be $(-1)^{n+1}(n-1)!$. Thus, since $\ln(1 + x)$ is C^∞, the Taylor series is:

$$\begin{aligned}&\ln(1) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n-1)!x^n}{n!} \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n.\end{aligned}$$

The problem now is that a C^∞ function is not necessarily *analytic*, that is, not necessarily equal to an infinite Taylor series expansion at a point. So a function may be C^∞, hence it is possible to construct the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x+h)^n$$

but not have

$$f(x+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x+h)^n$$

for a given h (even if the power series on the right converges). The missing piece is the requirement that the remainder term go to zero as $n \to \infty$ on a neighborhood of h, and this is not guaranteed for a C^∞ function.

We can start by noticing that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \qquad (*)$$

is a power series around 0[1]. As such, there is some radius of convergence $r \geq 0$ such that the series above converges for $x \in (-r, r)$ and diverges for $|x| > r$.

The radius r can be computed in various ways. From calculus, we know that the series converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| < 1$ (this is the ratio test). For our series we have:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty} \left|\frac{nx}{n+1}\right| = |x| \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = |x|$$

So the radius of convergence of $(*)$ is 1. This means $(*)$ converges for $|x| < 1$ and diverges for $|x| > 1$. The case $x = 1$ can be handled separately; clearly $(*)$ does not equal $\ln(1+x)$ at $x = -1$ since $\ln(0)$ is not defined.

[1]Recall from calculus that a power series about c is some series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$. One of two things can happen: the series either converges only for all x, or there exists some $r \geq 0$ (the *radius of convergence*) such that the series converges if $|x-a| < r$ and diverges if $|x-a| > r$.

So we know that

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

could only possibly be valid for $-1 < x \leq 1$; otherwise the series on the right diverges.[2]

Now notice that the error term for the n^{th} order Taylor approximation of $\ln(1+x)$ around 0 has the form

$$E_n(x) = \frac{(-1)^{n+2}}{(n+1)(1+y_n)^{n+1}} x^{n+1}$$

for some y_n between x and 0 (using the mean value version of the remainder term and our induction proof from the beginning).

Only if there is a neighborhood about 0 on which $E_n(x) \to 0$ as $n \to \infty$ are we guaranteed that the "infinite Taylor series" is equal to the original function.

That is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \iff E_n(x) \to 0$$

For $x \in (0, 1]$, it is easy to see that for any $y_n \in (0, x)$, $\frac{x}{1+y_n} < 1$ so

$$E_n(x) = \frac{(-1)^{n+2}}{(n+1)} \left(\frac{x}{1+y_n} \right)^{n+1} \to 0 \text{ as } n \to \infty$$

For $x > 1$ a potential problem is that the approximating y_n changes with n. For $x > 1$, $y_n \in (0, x)$ but we cannot guarantee that

$$\frac{x}{1+y_n} \leq 1 \quad \forall n$$

[2]Knowing the series converges for $-1 < x \leq 1$ is still not enough, however. A counterexample is the function

$$f(x) = e^{-\frac{1}{x^2}} \text{ for } x > 0 \text{ and } f(x) = 0 \text{ for } x \leq 0$$

f is C^∞, the Taylor series of f at 0 of any order is uniformly 0, and converges uniformly to 0 for every x. But clearly f is not equal to 0.

or that

$$\left(\frac{x}{1+y_n}\right)^{n+1} \to 0 \text{ as } n \to \infty$$

In fact, in this case one can show that for $x > 1$, the Taylor expansions around 0 become worse approximations for $\ln(1+x)$ as n increases.

Proving that the error term vanishes as n grows when $x \in (-1, 0)$ is harder. The following argument can be used to show this and the analogous statement for $x \in (0, 1)$. Let's look at the error term directly.

$$E_n(x) = \ln(1+x) - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} x^k$$

Notice that for $|x| < 1$,

$$\begin{aligned}
|E_n(x)| &= \left| \int_0^x \left(\frac{1}{1+t} - \sum_{k=1}^{n} (-1)^{k+1} t^{k-1} \right) dt \right| \\
&= \left| \int_0^x \frac{1 - 1 + t - t + \cdots + t^{n-1} - t^{n-1} + (-1)^n t^n}{1+t} \right| \\
&= \left| \int_0^x \frac{(-1)^n t^n}{1+t} dt \right| \\
&\leq \int_0^x \left| \frac{t^n}{1+t} \right| dt \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}$$

where the first equality follows from the Fundamental Theorem of Calculus and the inequality is a property of integration. Since $|E_n(x)| \geq 0$, we have that $|E_n(x)| \to 0$, which is what we set out to show.

(b) The function f is a second-degree polynomial. Therefore its second order Taylor expansion at any point is f itself.

(c) The function evaluated at $(1,1)$ equals $g(1,1)=1$.

$$\begin{aligned}
Dg(x,y) &= \begin{pmatrix}\ln(y)y^x & xy^{x-1}\end{pmatrix}\\
\Rightarrow Dg(1,1) &= \begin{pmatrix}0 & 1\end{pmatrix}\\
D^2g(x,y) &= \begin{pmatrix}\ln^2(y)y^x & y^{x-1}+\ln(y)xy^{x-1}\\ y^{x-1}+\ln(y)xy^{x-1} & x(x-1)y^{x-2}\end{pmatrix}\\
\Rightarrow D^2g(1,1) &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}
\end{aligned}$$

The quadratic expansion is then:

$$\begin{aligned}
g(x,y) \approx g(1,1) &+ Dg(1,1)\begin{pmatrix}x-1\\ y-1\end{pmatrix}\\
&+\frac{1}{2}\begin{pmatrix}x-1\\ y-1\end{pmatrix}^T D^2g(1,1)\begin{pmatrix}x-1\\ y-1\end{pmatrix}\\
&= 1+\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}x-1\\ y-1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}x-1\\ y-1\end{pmatrix}^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x-1\\ y-1\end{pmatrix}\\
&= 1+(y-1)+(x-1)(y-1)
\end{aligned}$$

3. Define $f:\mathbb{R}^3\to\mathbb{R}$ by

$$f(x,y_1,y_2)=x^2y+e^x+z.$$

Show that there exists a differentiable function g in some neighborhood of $(1,-1)$ in $\mathbb{R}^2$, such that $g(1,-1)=0$ and

$$f(g(y,z),y,z)=0.$$

Compute $Dg(1,-1)$.

Solution: To show that such a function exists, we can invoke the Implicit Function Theorem. First, notice that $f(0,1,-1)=0$. Next we need to verify that $D_xf(0,1,-1)\neq 0$:

$$D_xf(x,y,z)=2xy+e^x\Rightarrow D_xf(0,1,-1)=1\neq 0.$$

This completes the proof of the existence of g.

Additionally, we have:

$$D_{(y,z)}f(x,y,z) = \begin{pmatrix} x^2 & 1 \end{pmatrix} \Rightarrow D_{(y,z)}f(0,1,-1) = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

So:

$$Dg(1,-1) = -[D_x f(0,1,-1)]^{-1} D_{(y,z)}f(0,1,-1) = \begin{pmatrix} 0 & -1 \end{pmatrix}.$$

4. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x,y) = (e^y \cos(x), e^y \sin(x))$.

   (a) Show that F satisfies the prerequisites of the Inverse Function Theorem for all $(x,y) \in \mathbb{R}^2$ (and is therefore locally injective everywhere) but F is not globally injective.

   (b) Compute the Jacobian of the local inverse of F and evaluate it at $F(\frac{\pi}{3}, 0)$.

   (c) Find an explicit formula for the continuous inverse of F mapping a neighborhood of $F(\frac{\pi}{3}, 0)$ into a neighborhood of $(\frac{\pi}{3}, 0)$ and verify that its Jacobian at $F(\frac{\pi}{3}, 0)$ equals the one you calculated in part (b). (You might want to look up a few basic trigonometric facts.)

Solution:

(a) The domain of F is open and F is C^∞ on it. We only need to verify that $\det(DF(x,y)) \neq 0$ for all $(x,y) \in \mathbb{R}^2$:

$$\begin{aligned} \det(DF(x,y)) &= \det \begin{pmatrix} -e^y \sin x & e^y \cos x \\ e^y \cos x & e^y \sin x \end{pmatrix} \\ &= -e^{2y} \sin^2 x - e^{2y} \cos^2 x \\ &= -e^{2y} \\ &< 0, \end{aligned}$$

where we used the fact that $\sin^2 x + \cos^2 x = 1$ for all x.

Therefore, by the Inverse Function Theorem F is locally injective. However, F is not globally injective since $F(x,y) = F(x + 2\pi, y)$ for any $(x,y) \in \mathbb{R}^2$. We mentioned in class that local injectivity (such as the one implied by the Inverse Function Theorem) need not imply global injectivity, even if it holds for all points in the function's domain. The function F is just an example of that.

(b) Denoting the local inverse by F^{-1}, directly from the Inverse Function Theorem, we have:

$$\begin{aligned} DF^{-1}(F(x,y)) &= [DF(x,y)]^{-1} \\ &= -\frac{1}{e^{2y}} \begin{pmatrix} e^y \sin x & -e^y \cos x \\ -e^y \cos x & -e^y \sin x \end{pmatrix} \\ &= \frac{1}{e^y} \begin{pmatrix} -\sin x & \cos x \\ \cos x & \sin x \end{pmatrix}. \end{aligned}$$

At $(\frac{\pi}{3}, 0)$ we have (recall that $\sin\frac{\pi}{3} = \frac{1}{2}$ and $\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}$):

$$DF^{-1}(F(\frac{\pi}{3}, 0)) = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

(c) If $F(x,y) = (a,b)$, to derive an explicit formula for the inverse of F we need to solve:

$$\begin{aligned} e^y \cos x &= a \\ e^y \sin x &= b \end{aligned}$$

for x and y. This is equivalent to:

$$\begin{aligned} \frac{b}{a} &= \frac{\sin x}{\cos x} = \tan x \\ a^2 + b^2 &= e^{2y}(\sin^2 x + \cos^2 x) = e^{2y}. \end{aligned}$$

So:

$$\begin{aligned} x &= \tan^{-1}\frac{b}{a} \\ y &= \frac{1}{2}\ln(a^2+b^2). \end{aligned}$$

This defines the function $F^{-1}(a,b) = (\tan^{-1}\frac{b}{a}, \frac{1}{2}\ln(a^2+b^2))$[3]. Then:

$$DF^{-1}(a,b) = \frac{1}{a^2+b^2} \begin{pmatrix} -b & a \\ a & b \end{pmatrix}.$$

[3] Notice that the inverse of the tangent function is not a well-defined function globally since tangent is not injective; however, locally (for a small enough neighborhood of $\frac{\pi}{3}$), $\tan^{-1}$ is well-defined. Also recall that: $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$.

At $(\frac{\pi}{3}, 0)$, we have $(a, b) = F(\frac{\pi}{3}, 0) = (e^0 \cos\frac{\pi}{3}, e^0 \sin\frac{\pi}{3}) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$. Notice that $a^2 + b^2 = 1$. Then:

$$DF^{-1}(\frac{\sqrt{3}}{2}, \frac{1}{2}) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

which is the same as the Jacobian calculated in the previous part.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on the interval (a, b), and let $a < c < d < b$.

   (a) Suppose that $f'(c) < 0 < f'(d)$. Prove that the restriction of f to $[c, d]$ does not achieve a global minimum at c or at d.

   (b) Again suppose that $f'(c) < 0 < f'(d)$. Prove that there exists some $p \in (c, d)$ such that $f'(p) = 0$. (In order to receive full credit, please prove any claims you make about the derivative at extremal points.)

   (c) Now suppose that $f'(c) < \alpha < f'(d)$. Prove that there exists some $p \in (c, d)$ such that $f'(p) = \alpha$.

Solution:

(a) We know:

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} < 0.$$

This implies that for sufficiently small positive h we have:

$$\frac{f(c+h) - f(c)}{h} < 0 \Rightarrow f(c+h) - f(c) < 0 \Rightarrow f(c+h) < f(c)$$

Hence $f(c)$ is not a global minimum of f, since $c + h \in [c, d]$ for h small enough . Similarly, for d we have:

$$\begin{aligned} f'(d) &= \lim_{h\to 0} \frac{f(d+h) - f(d)}{h} > 0 \\ &\Rightarrow \exists h < 0 : \frac{f(d+h) - f(d)}{h} > 0 \\ &\Rightarrow f(d+h) - f(d) < 0 \\ &\Rightarrow f(d+h) < f(d), \end{aligned}$$

where $c < d + h < d$ for h small enough (in absolute value).

(b) The function f is differentiable and therefore continuous. Therefore the restriction of f on $[c, d]$ achieves its global minimum on that interval and we know (from part (a)) that neither of $f(c)$ or $f(d)$ are the global minimum. Therefore there exists some interior point $p \in (c, d)$ such that $f(p) \leq f(x)$ for all $x \in [c, d]$.

Now let's examine the sign of $f'(p)$. If $f'(p) > 0$ or $f'(p) < 0$, arguments identical to the ones from part (a) can convince us that $f(p)$ cannot be a global minimum. Therefore we must have $f'(p) = 0$.

(c) Consider the function $g : (a, b) \to \mathbb{R}$ defined by $g(x) = f(x) - \alpha x$. Notice that g is differentiable everywhere on its domain and $g'(x) = f'(x) - \alpha$. Therefore:

$$g'(c) < 0 < g'(d).$$

So by part (b), there exists some $p \in (c, d)$ such that:

$$f'(p) - \alpha = g'(p) = 0 \Leftrightarrow f'(p) = \alpha.$$

Aside: It is easy to see that by inverting the argument above (examining f's global maximum, rather than minimum), we can prove that the same intermediate value property also holds whenever $f'(c) > \alpha > f'(d)$. Therefore derivative functions have an intermediate value property on any interval in their domain (even when they are not continuous!). This result is known as *Darboux's Theorem*.

6. Let $g : \mathbb{R} \to \mathbb{R}$ be C^1. Prove that there exists $\varepsilon > 0$ such that the function $f : [1, 2] \to \mathbb{R}$ given by

$$f(x) = x^3 - x^2 + \varepsilon g(x)$$

is injective. (Hint: You probably want to start by using the Extreme-Value Theorem appropriately.)

Solution: g is C^1 so its derivative g' is continuous and, by the Extreme-Value Theorem, g' is bounded on $[1, 2]$. To be specific, let $|g'(x)| < M$ for all $x \in [1, 2]$.

Now take some $x, y \in [1,2]$ such that $x > y$. In order for f to be injective we need $f(x) \neq f(y) \Leftrightarrow f(x) - f(y) \neq 0$. Consider that difference:

$$\begin{aligned} f(x) - f(y) &= x^3 - x^2 + \varepsilon g(x) - (y^3 - y^2 + \varepsilon g(y)) \\ &= (x-y)(x^2 + xy + y^2) - (x-y)(x+y) + \varepsilon(g(x) - g(y)) \\ &= (x-y)(x^2 + xy + y^2 - x - y) + \varepsilon(x-y)g'(z) \\ &= (x-y)(x^2 + xy + y^2 - x - y + \varepsilon g'(z)) \end{aligned}$$

for some $z \in (y, x) \subseteq [1,2]$, where the third equality follows from the Mean Value Theorem.

We want $f(x) - f(y) \neq 0$. Since $x \neq y$, we have $x - y \neq 0$. The function $x^2 + xy + y^2 - x - y$ is positive on $[1,2]^2$ ($a \leq a^2$ for $a \geq 1$) and achieves a minimum and a maximum value on $[1,2]$ (again from the Extreme-Value Theorem.) Let $x^2 + xy + y^2 - x - y > K > 0$ for all $(x, y) \in [1,2]^2$. Now if $\varepsilon < K/M$, we have:

$$x^2 + xy + y^2 - x - y + \varepsilon g'(z) > K - \varepsilon M > K - \frac{K}{M} M = 0.$$

Therefore for any sufficiently small ε, $f(x) - f(y) \neq 0$ for all $x, y \in [1,2]$ and f is injective on $[1,2]$.