Economics 204
Fall 2011
Problem Set 5
Due Friday, August 12 in Lecture

1. (a) Prove that $y = h^3$ is both $o(|h|^2)$ as $h \to 0$ and $O(|h|^3)$ as $h \to 0$.

   (b) Prove that $y = sin(h)$ is not $o(|h|)$ as $h \to 0$ but is $O(|h|)$ as $h \to 0$. (You can use the fact that $|sin(h)| \leq |h|$).

2. (a) Prove that the following identity holds for $-1 < x \leq 1$:

$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.$$

   (b) Find the second-order Taylor expansion of:

$$f(x, y) = -x^2 + 2xy + 3y^2 - 6x - 2y - 4$$

   around $(x, y) = (-\pi/4, \ln 42)$.

   (c) Find the second-order Taylor expansion of $g(x, y) = y^x$ around $(x, y) = (1, 1)$.

3. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = x^2 y + e^x + z.$$

   Show that there exists a differentiable function g in some neighborhood of $(1, -1)$ in $\mathbb{R}^2$, such that $g(1, -1) = 0$ and

$$f(g(y, z), y, z) = 0.$$

   Compute $Dg(1, -1)$.

4. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (e^y \cos(x), e^y \sin(x))$.

   (a) Show that F satisfies the prerequisites of the Inverse Function Theorem for all $(x, y) \in \mathbb{R}^2$ (and is therefore locally injective everywhere) but F is not globally injective.

(b) Compute the Jacobian of the local inverse of F and evaluate it at $F(\frac{\pi}{3}, 0)$.

(c) Find an explicit formula for the continuous inverse of F mapping a neighborhood of $F(\frac{\pi}{3}, 0)$ into a neighborhood of $(\frac{\pi}{3}, 0)$ and verify that its Jacobian at $F(\frac{\pi}{3}, 0)$ equals the one you calculated in part (b). (You might want to look up a few basic trigonometric facts.)

5. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on the interval (a, b), and let $a < c < d < b$.

   (a) Suppose that $f'(c) < 0 < f'(d)$. Prove that the restriction of f to $[c, d]$ does not achieve a global minimum at c or at d.

   (b) Again suppose that $f'(c) < 0 < f'(d)$. Prove that there exists some $p \in (c, d)$ such that $f'(p) = 0$. (In order to receive full credit, please prove any claims you make about the derivative at extremal points.)

   (c) Now suppose that $f'(c) < \alpha < f'(d)$. Prove that there exists some $p \in (c, d)$ such that $f'(p) = \alpha$.

6. Let $g : \mathbb{R} \to \mathbb{R}$ be C^1. Prove that there exists $\varepsilon > 0$ such that the function $f : [1, 2] \to \mathbb{R}$ given by

$$f(x) = x^3 - x^2 + \varepsilon g(x)$$

   is injective. (Hint: You probably want to start by using the Extreme-Value Theorem appropriately.)