Economics 204
Fall 2011
Problem Set 6
Due Mon, Aug 15 by 9am in Oleksa's GSI mailbox (Evans, 5th floor)

1. Consider the following quadratic forms:

$$f(x,y) = 2x^2 - 4xy + 5y^2,$$
$$g(x,y) = x^2 + 6xy + y^2,$$
$$h(x,y) = 16xy.$$

Answer the following questions for each of these forms:

(a) Find a symmetric matrix M such that the form equals $[x\ y]\ M \begin{bmatrix} x \\ y \end{bmatrix}$.

(b) Find the eigenvalues of matrix M.

(c) Find an orthonormal basis of eigenvectors.

(d) Find a unitary matrix S such that $M = S^{-1}DS$, where D is a diagonal matrix.

(e) Describe the level sets of the form and state whether the form has a local maximum, local minimum, or neither at $(0,0)$. (Level sets are solutions to $f(x,y) = c$ for some $c \in \mathbf{R}$.)

2. Suppose $\Psi_1, \Psi_2 : X \to 2^Y$ are compact-valued, upper hemicontinuous correspondences, where $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$ for some n, m. Suppose that $\Psi_1 \cap \Psi_2 \neq \emptyset$ for each $x \in X$.

(a) Show that $\Psi_1 \cap \Psi_2$ is upper hemicontinuous, where $\Psi_1 \cap \Psi_2$ is defined by

$$(\Psi_1 \cap \Psi_2)(x) = \Psi_1(x) \cap \Psi_2(x),\ \forall x \in X$$

(b) Lets now weaken our assumptions a bit: lets assume that Ψ_1 is only closed-valued, rather then compact-valued. Show that $\Psi_1 \cap \Psi_2$ is still upper hemicontinuous.

3. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be a C^1 function and define $F : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^2$ by

$$F(x,\omega) = f(x_1,x_2) - \left(5\omega_1 + \omega_1^3,\ 5\omega_2 + \omega_1(1 + 3\omega_1\omega_2)\right).$$

Show that there is a set of Lebesgue measure zero, $\Omega_0 \subset \mathbf{R}^2$, such that if $\omega \notin \Omega_0$, then for each x_0 satisfying $F(x_0,\omega_0) = 0$ there is an open set U containing x_0, an open set V containing ω_0, and a C^1 function $h : V \to U$ such that for all $\omega \in V$, $x = h(\omega)$ is the unique element of U satisfying $F(x,\omega) = 0$.

4. *The Minimax Theorem* is used for proving quite a few important results in economics, for instance, about an outcome of zero-sum games in noncooperative game theory or in analyzing Bayesian estimators in statistical decision theory. Now you have a chance to prove this Minimax Theorem yourself.

   Let X and Y be non-empty, closed, bounded and convex subsets of any two Euclidean spaces. Prove that if $f : X \times Y \to \mathbf{R}$ is continuous, and if the sets $\{ z \in X \mid f(z, y) \geq \alpha\}$ and $\{ w \in Y \mid f(x, w) \leq \alpha\}$ are convex for each $(x,\, y,\, \alpha) \in X \times Y \times \mathbf{R}$, then

   $$\max_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\max_{x\in X} f(x,y).$$

   (Hint: Start by defining two self-correspondences $\Phi(y) : Y \to 2^Y$ and $\Pi(x) : X \to 2^X$ as

   $$\Phi(y) = \underset{x\in X}{\operatorname{argmax}} f(x,y)$$
   $$\Pi(x) = \underset{y\in Y}{\operatorname{argmin}} f(x,y)$$

   Then, define self-correspondence $\Psi : X \times Y \to 2^{X\times Y}$ by

   $$\Psi(x,y) = \Pi(x) \times \Phi(y).$$

   Use Kakutani's Fixed Point theorem).

5. Show that the closure of a convex set is convex.

6. One of the most useful versions of Separating Hyperplane Theorem is the one on *strong separation* of convex sets. We say that two sets A and B are strongly separated by a hyperplane if there exists $p \in \mathbf{R}^n$ with $p \neq 0$ such that

   $$\sup p \cdot A < \inf p \cdot B$$

   (In other words, sets are strongly separated if they are contained in the closed halfspaces that are $\epsilon > 0$ away from each other. Notice that another way to show strong separation is to demonstrate existence of two constants c and d together with non-zero vector p such that $p \cdot a \leq c < d \leq p \cdot b$ for all $a \in A$ and for all $b \in B$. Please check Theorem 8 in lecture 13 to make sure you understand how strict separation is different from strong one.)

   Of course, strong separation requires a stronger initial assumptions.

   (a) Let A and B be non-empty, disjoint, convex subsets of $\mathbf{R}^n$ with A being *compact* and B closed. Show directly, without invoking Theorem 7 in Lecture 13, that A and B can be strongly separated.[1] (Hint: Look at the set $Y = B - A$. Is it compact? Closed?)

[1] Although you can't invoke theorem directly, going carefully over its proof will get you far in showing this result.

(b) Demonstrate by means of an example that the requirement that A is compact is essential, it can't be just closed. In your example, are the sets A and B strictly separated?

7. Consider the following inhomogeneous linear differential equation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

(a) Write down the corresponding homogeneous equation.

(b) Find the general solution of the homogeneous equation.

(c) Find a particular solution of the original inhomogeneous equation satisfying the initial condition $y(0) = (1,\, 1)^T$.
(Hint: The integrals can be solved by integrating by parts twice.)

(d) Find the general solution of the original inhomogeneous equation.