Economics 204 Summer/Fall 2022
Final Exam – Suggested Solutions

Answer all of the questions below. Be as complete, correct, and concise as possible. There are 6 questions for a total of 165 points possible; point values for each problem are in parentheses. For questions with subparts, each subpart is worth the same number of points. You have 180 minutes to complete the exam. Use the points as a guide to allocating your time. You may use any result from class with appropriate references unless you are specifically being asked to prove it.

1. (15) Define or state each of the following.

   (a) eigenvector of a linear transformation $T : X \to Y$ between vector spaces X and Y over the same field F

   (b) open set in a metric space (X, d)

   (c) Intermediate Value Theorem

   Solution: See notes.

2. (30) Let A and B be $n \times n$ matrices that commute, so $AB = BA$. Show that for every $k \in \mathbb{N}$ with $k \geq 2$, $A^k B = BA^k$ (where M^k is the product of k copies of the $n \times n$ matrix M).

   (**Hint:** use induction.)

   Solution: For the base case $k = 2$,

$$\begin{aligned} A^2B = A(AB) &= A(BA) && \text{since } AB = BA \\ &= (AB)A \\ &= (BA)A && \text{again since } AB = BA \\ &= BA^2 \end{aligned}$$

   So the claim holds for $k = 2$.

   Now suppose $A^k B = BA^k$ for $k \geq 2$. Then

$$\begin{aligned} A^{k+1}B = A(A^kB) &= A(BA^k) && \text{by the induction hypothesis} \\ &= (AB)A^k \\ &= (BA)A^k && \text{since } AB = BA \\ &= BA^{k+1} \end{aligned}$$

   Thus the claim holds for $k + 1$. Then by induction, $A^k B = BA^k$ for all $k \geq 2$.

3. (30) Let X and Y be vector spaces over the same field F, and let $T : X \to Y$ be a linear transformation. Let $V \subseteq X$ be linearly independent. Show that if T is one-to-one, then $T(V) \subseteq Y$ is linearly independent.

Solution: Let $y_1, \ldots, y_n \in T(V)$ and $\alpha_1, \ldots, \alpha_n \in F$ such that

$$\sum_{i=1}^{n} \alpha_i y_i = 0$$

Since $y_1, \ldots, y_n \in T(V)$, for each i there exists $v_i \in V$ such that $T(v_i) = y_i$. Then

$$\begin{aligned} 0 = \sum_{i=1}^{n} \alpha_i y_i &= \sum_{i=1}^{n} \alpha_i T(v_i) \\ &= T(\sum_{i=1}^{n} \alpha_i v_i) \quad \text{since } T \text{ is linear} \end{aligned}$$

Thus $\sum_{i=1}^{n} \alpha_i v_i \in \ker T$. Since T is linear and one-to-one, $\ker T = \{0\}$. Thus

$$\sum_{i=1}^{n} \alpha_i v_i = 0$$

But V is linearly independent and $v_i \in V$ for each $i = 1, \ldots, n$, so $\alpha_i = 0$ for each $i = 1, \ldots, n$. Thus $T(V)$ is linearly independent.

4. (30) Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a continuous function. Let $A \subseteq X$. Show that $f(\bar{A}) \subseteq \overline{f(A)}$.

Solution: Let $y \in f(\bar{A})$. Then there exists $x \in \bar{A}$ such that $f(x) = y$. Since $x \in \bar{A}$, there exists $\{x_n\} \subseteq A$ such that $x_n \to x$. Since f is continuous, $f(x_n) \to f(x) = y$. By definition, $f(x_n) \in f(A)$ for each n since $x_n \in A$ for each n, so $\{f(x_n)\} \subseteq f(A) \subseteq \overline{f(A)}$. Since $\overline{f(A)}$ is closed and $f(x_n) \to y$, $y \in \overline{f(A)}$. Thus $f(\bar{A}) \subseteq \overline{f(A)}$.

5. (30) Let $U \subseteq \mathbf{R}^n$ be open and $f : U \to \mathbf{R}$ be differentiable on U. Suppose for each $x \in U$ there exists $\varepsilon_x > 0$ and $M_x > 0$ such that $\|Df(y)\| \leq M_x$ for all $y \in B_{\varepsilon_x}(x)$.

Suppose $C \subseteq U$ is convex and compact. Show that f is Lipschitz continuous on C. (That is, show that there exists $M > 0$ such that $\|f(x) - f(y)\| \leq M\|x - y\|$ for all $x, y \in C$.)

(**Hint:** Show that there exists $M > 0$ such that $\|Df(z)\| \leq M$ for all $z \in C$.)

Solution: First claim that there exists $M > 0$ such that $\|Df(z)\| \leq M$ for all $z \in C$. To see this, note that by assumption, for each $x \in C$ there exists $\varepsilon_x > 0$ and $M_x > 0$ such that $\|Df(z)\| \leq M_x$ for each $z \in B_{\varepsilon_x}(x)$. Then for each $x \in C$, $B_{\varepsilon_x}(x)$ is an open set and $x \in B_{\varepsilon_x}(x)$, so $\{B_{\varepsilon_x}(x) : x \in C\}$ is an open cover of C. Since C is compact, there exists $x_1, \ldots, x_n \in C$ such that

$$C \subseteq B_{\varepsilon_{x_1}}(x_1) \cup \cdots \cup B_{\varepsilon_{x_n}}(x_n)$$

Let $M = \max\{M_{\varepsilon_{x_1}}, \ldots, M_{\varepsilon_{x_n}}\}$; by definition $0 < M < \infty$. Let $z \in C$. Then $z \in B_{\varepsilon_{x_i}}(x_i)$ for some $i = 1, \ldots, n$, so

$$\|Df(z)\| \leq M_{\varepsilon_{x_i}} \leq M$$

by definition of M. Therefore $\|Df(z)\| \leq M$ for all $z \in C$.

Now let $x, y \in C$. Since C is convex, $\ell(x, y) = \{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\} \subseteq C \subseteq U$. Then by the Mean Value Theorem, there exists $z \in \ell(x, y)$ such that

$$f(x) - f(y) = Df(z)(x - y)$$

Thus

$$\begin{aligned}
\|f(x) - f(y)\| &= \|Df(z)(x - y)\| \\
&\leq \|Df(z)\|\|x - y\| \\
&\leq M\|x - y\|
\end{aligned}$$

where the last inequality follows as $z \in \ell(x, y) \subseteq C$, which implies $\|Df(z)\| \leq M$. Since $x, y \in C$ were arbitrary, f is Lipschitz continuous on C.

6. (30) Let (X, d_1) be a nonempty, complete metric space and $C \subseteq \mathbf{R}^n$ be a nonempty, compact, convex set. Consider the metric space $(X \times C, d)$, where $d : X \times C \to \mathbf{R}_+$ is the metric given by $d((x, y), (z, w)) = d_1(x, z) + d_2(y, w)$ for $(x, y), (z, w) \in X \times C$, where d_1 is the metric on X and d_2 denotes the standard metric in $\mathbf{R}^n$ (you can use without proof that d is a metric on $X \times C$.)

   Let $f : X \times C \to X \times C$, and write $f(x, y) = (f_1(x, y), f_2(x, y))$, where $f_1 : X \times C \to X$ and $f_2 : X \times C \to C$. Suppose f is Lipschitz continuous, and for each $y \in C$, $f_1(\cdot, y) : X \to X$ is a contraction on (X, d_1). Show that f has a fixed point.

   Solution: By assumption, f is Lipschitz continuous, so there exists $K > 0$ such that

$$d(f(x', y'), f(x, y)) \leq Kd((x, y), (x', y')) \quad \forall\ (x, y), (x', y') \in X \times C$$

   Similarly, by assumption for each $y \in C$, $f_1(\cdot, y) : X \to X$ is a contraction on (X, d_1). Then let $y \in C$, and let $\beta_y \in (0, 1)$ such that

$$d_1(f_1(x, y), f_1(x', y)) \leq \beta_y d_1(x, x') \quad \forall\ x, x' \in X$$

   Since (X, d_1) is a nonempty complete metric space and $f_1(\cdot, y) : X \to X$ is a contraction, by the Contraction Mapping Theorem there exists a unique $x_y^* \in X$ such that

$$f_1(x_y^*, y) = x_y^*$$

Then let $x : C \to X$ denote the function such that $x(y) = x_y^*$ for each $y \in C$.

Note that $f(x,y) = (x,y) \iff f_1(x,y) = x$ and $f_2(x,y) = y$. Thus (x,y) is a fixed point of f if and only if $x = x(y)$ and $f_2(x(y),y) = y$. Then to show that f has a fixed point, it suffices to show that $g : C \to C$ has a fixed point, where g is defined by $g(y) = f_2(x(y),y)$.

To that end, first note that $x : C \to X$ is continuous. To see this, suppose $\{y_n\} \subseteq C$ and $y_n \to y$. Then $y \in C$ since C is compact, and hence closed. Moreover,

$$\begin{aligned} d_1(x(y_n),x(y)) &= d_1(f_1(x(y_n),y_n), f_1(x(y),y)) \\ &\leq d_1(f_1(x(y_n),y_n), f_1(x(y_n),y)) + d_1(f_1(x(y_n),y), f_1(x(y),y)) \\ &\leq d(f(x(y_n),y_n), f(x(y_n),y)) + \beta_y d_1(x(y_n),x(y)) \end{aligned}$$

where the first inequality follows from the triangle inequality, and the second inequality follows since $f_1(\cdot,y)$ is a contraction with modulus β_y, and

$$\begin{aligned} d_1(f_1(x(y_n),y_n), f_1(x(y_n),y)) &\leq d_1(f_1(x(y_n),y_n), f_1(x(y_n),y)) + d_2(f_2(x(y_n),y_n), f_2(x(y_n),y)) \\ &= d(f(x(y_n),y_n), f(x(y_n),y)) \end{aligned}$$

Then using the Lipschitz continuity of f,

$$\begin{aligned} d_1(x(y_n),x(y)) &\leq d(f(x(y_n),y_n), f(x(y_n),y)) + \beta_y d_1(x(y_n),x(y)) \\ &\leq Kd((x(y_n),y_n),(x(y_n),y)) + \beta_y d_1(x(y_n),x(y)) \\ &= Kd_2(y_n,y) + \beta_y d_1(x(y_n),x(y)) \end{aligned}$$

This implies $(1-\beta_y)d_1(x(y_n),x(y)) \leq Kd_2(y_n,y)$. Since $\beta_y \in (0,1)$ and $y_n \to y$, this implies $x(y_n) \to x(y)$. Thus $x : C \to X$ is a continuous function.

Now since f_2 is continuous, $g : C \to C$ is continuous, as $g(y) = f_2(x(y),y)$ for each $y \in C$. Since $C \subseteq \mathbf{R}^n$ is nonempty, compact, and convex, by Brouwer's Fixed Point Theorem there exists $y^* \in C$ such that $g(y^*) = y^*$. Then let $x^* = x(y^*)$. Note that (x^*,y^*) is a fixed point of f, as

$$f_1(x^*,y^*) = f_1(x(y^*),y^*) = x(y^*) = x^* \text{ and } f_2(x^*,y^*) = f_2(x(y^*),y^*) = g(y^*) = y^*$$

Thus $f(x^*,y^*) = (f_1(x^*,y^*), f_2(x^*,y^*)) = (x^*,y^*)$.